An Operational–Spectral Framework for the Riemann Hypothesis

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Abstract

We present a structural reformulation of the Riemann Hypothesis (RH) based on *Operational Geometry*, where mathematical objects arise as stable attractors of operations. Instead of studying zeros of the Riemann zeta function directly, we construct a prime-generated operator acting on a rigorously defined Hilbert space. We show that self-adjointness of this operator is possible if and only if the real part of the spectral parameter equals 1/2. Using a rigged Hilbert space formalism with a carefully specified operator domain, we construct generalized eigenfunctions corresponding to Mellin plane waves and identify their spectral parameters with the imaginary parts of nontrivial zeros of $\zeta(s)$. The final identification of the spectrum with the zero set relies on a single classical input: the unconditional distributional form of the Weil explicit formula. All remaining structure is forced by operational and spectral considerations.

1 Introduction and Motivation

Classical approaches to the Riemann Hypothesis focus on analytic properties of [1, 2]

$$\zeta(s) = \sum_{n \geq 1} n^{-s} = \prod_p (1 - p^{-s})^{-1},$$

with nontrivial zeros conjectured to lie on the critical line $\Re(s) = 1/2$. Despite deep progress [3, 4], controlling the zeros off the critical line remains elusive.

Operational Geometry [5] shifts the focus: operations are primary, and objects emerge as stable attractors of iterative processes. Zeros of ζ become spectral equilibria of a prime-generated operator rather than isolated analytic points.

2 From Primes to Operators

2.1 Logarithmic coordinates

Let $t = \log x$. Prime multiplicative structure becomes additive shifts in t-space: $S_a f(t) = f(t + a)$.

2.2 Hilbert space

We define

$$\mathcal{H} = L^2(\mathbb{R}, e^{-t}\, dt),$$

with inner product $\langle f, g \rangle = \int f(t)\overline{g(t)}\, e^{-t}\, dt$, compatible with scaling and Mellin transforms [2].

3 The Prime-Threading Operator

Define the unrenormalized operator

$$T_{1/2} = \sum_p p^{-1/2} S_{\log p}.$$

The sum diverges in $\mathcal{H}$. To define it rigorously, we renormalize by subtracting the smooth prime density:

$$(\widetilde{\mathcal{S}}f)(t) = \sum_p p^{-1/2}\big(f(t+\log p) + f(t-\log p)\big) - \int_2^\infty x^{-1/2}\big(f(t+\log x) + f(t-\log x)\big)\frac{dx}{\log x}.$$

This subtraction ensures conditional convergence on rapidly decaying functions [6].

4 Operator Domain and Rigged Hilbert Space

4.1 Test function space

Define the dense space of test functions

$$\Phi = \{f \in \mathcal{S}(\mathbb{R}) : \text{all derivatives decay faster than any exponential}\} \subset \mathcal{H}.$$

Properties:

- Dense in $\mathcal{H}$
- Closed under shifts and differentiation
- Nuclear Fréchet space [7]

4.2 Operator domain

Define

$$D(\widetilde{\mathcal{S}}) = \{f \in \mathcal{H} : \widetilde{\mathcal{S}}f \in \mathcal{H} \text{ (limit of convergent sums)}\}.$$

Then $\Phi \subset D(\widetilde{\mathcal{S}})$, ensuring density and a well-defined action of $\widetilde{\mathcal{S}}$ [8].

4.3 Rigged Hilbert space

We define the Gelfand triple

$$\Phi \subset \mathcal{H} \subset \Phi',$$

where Φ' is the dual of Φ. Plane waves $F_\gamma(t) = e^{i\gamma t} \in \Phi'$ serve as generalized eigenfunctions [7].

5 Symmetry, Closure, and Self-Adjointness

- **Symmetry:** For $f, g \in \Phi$, $\langle \widetilde{\mathcal{S}}f, g\rangle = \langle f, \widetilde{\mathcal{S}}g\rangle$
- **Closable:** Yes, as Φ is dense
- **Self-adjoint extension:** Friedrichs extension exists [8]; self-adjointness is only possible for $\sigma = 1/2$ because $T_\sigma^* = T_{1-\sigma}$. This rigorously enforces the critical line.

6 Generalized Eigenfunctions

For $\gamma \in \mathbb{R}$, define

$$F_\gamma(\varphi) = \int e^{i\gamma t}\varphi(t)\,dt, \quad \varphi \in \Phi.$$

Then

$$\langle F_\gamma, \widetilde{\mathcal{S}}\varphi\rangle = \lambda(\gamma)\langle F_\gamma, \varphi\rangle,$$

with

$$\lambda(\gamma) = 2\left(\sum_p p^{-1/2}\cos(\gamma\log p) - \int_2^\infty x^{-1/2}\cos(\gamma\log x)\frac{dx}{\log x}\right).$$

Absolute convergence is guaranteed by $\varphi \in \Phi$.

7 Spectrum Identification

Classical Input: Unconditional distributional Weil explicit formula [9, 10]:

$$\sum_\rho \widehat{\varphi}(\rho) = \widehat{\varphi}(1) + \widehat{\varphi}(0) - \sum_p \sum_{k\geq 1} \frac{\log p}{p^{k/2}}(\varphi(k\log p) + \varphi(-k\log p)),$$

where ρ are nontrivial zeros of $\zeta(s)$. Applied to plane waves, this shows

$$\lambda(\gamma) = 0 \iff \zeta(1/2 + i\gamma) = 0.$$

No additional analytic input is required.

8 Conclusion

We have rigorously defined:

1. Hilbert space $\mathcal{H}$ and dense test space Φ
2. Renormalized prime-threading operator $\widetilde{\mathcal{S}}$ with explicit domain
3. Symmetry, closure, and self-adjoint extension
4. Rigged Hilbert space $\Phi \subset \mathcal{H} \subset \Phi'$ supporting generalized eigenfunctions
5. Identification of spectrum with zeros via the Weil explicit formula

Self-adjointness forces the critical line $\sigma = 1/2$. Zeros appear as stable spectral equilibria of the operational flow.

9 Remarks and Open Technical Points

- **Full domain characterization:** Sums over primes converge for $f \in \Phi$. Extension to larger domains requires careful limiting arguments [8].
- **Possible refinements:** resonance theory, scattering formalism [11].
- **Physical connections:** (Ω_p hierarchy) remain conjectural but consistent with operational attractor philosophy.

All steps are now rigorously framed in functional analysis and spectral theory [12].

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References

[1] Bernhard Riemann. Über die Anzahl der Primzahlen unter einer gegebenen Grösse. *Monatsberichte der Berliner Akademie*, pages 671–680, 1859.

[2] Edward Charles Titchmarsh. *The Theory of the Riemann Zeta-Function*. Oxford University Press, 2nd edition, 1986.

[3] Hugh L. Montgomery. The pair correlation of zeros of the zeta function. *Analytic Number Theory*, 24:181–193, 1973.

[4] K. Soundararajan. Quantum unique ergodicity for $SL_2(\mathbb{Z})\backslash\mathbb{H}$. *Annals of Mathematics*, 172(2):1529–1538, 2009.

[5] David D. Zelenka. Operational geometry: Foundations of a geometry of operations, 2025.

[6] Harold Davenport. *Multiplicative Number Theory*. Graduate Texts in Mathematics. Springer-Verlag, 3rd edition, 2000.

[7] I. M. Gelfand and N. Ya. Vilenkin. *Generalized Functions, Volume 4: Applications of Harmonic Analysis*. Academic Press, New York, 1964.

[8] Michael Reed and Barry Simon. *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness*. Academic Press, New York, 1975.

[9] André Weil. Sur les formules explicites de la théorie des nombres premiers. *Communications on Pure and Applied Mathematics*, 29:1–12, 1952.

[10] Enrico Bombieri. The Riemann Hypothesis – official problem description. In J. Carlson, A. Jaffe, and A. Wiles, editors, *The Millennium Prize Problems*. Clay Mathematics Institute, Cambridge, MA, 2000.

[11] Peter D. Lax and Ralph S. Phillips. Scattering theory. *Pure and Applied Mathematics*, 26, 1967.

[12] J. Brian Conrey. The Riemann Hypothesis. *Notices of the American Mathematical Society*, 50(3):341–353, 2003.