A New Variant of Positive Linear Operators for Convex Functions Providing a Better Error Estimation

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Abstract

Szász–Mirakjan type operators constitute a fundamental class of positive linear operators widely employed in approximation theory due to their effective characterization of functions over unbounded intervals. In this work, we introduce a novel generalization of these operators and investigate their analytical properties in relation to the classical Szász–Mirakjan scheme. The proposed operators are examined through a detailed comparative framework that emphasizes their structural features, convergence properties, and approximation capabilities. It is further established that the newly defined operators exhibit an improved rate of convergence relative to the some of its counterparts, thereby enhancing their applicability within the broader context of function approximation on unbounded domains.

Keywords: Szász-Mirakyan Type Operators, Rate of Convergence, Voronovskaya theorem, Korovkin type theorem

1 introduction

The theory of approximation plays a fundamental role in mathematical analysis and numerical computation and continues to attract considerable attention from the mathematical community. As a result, extensive research has been conducted in this area. One of the principal objectives of approximation theory is to construct simple, computationally efficient operators that approximate arbitrary functions with a high degree of accuracy.

A cornerstone of approximation theory was established by Weierstrass in 1885, who proved that for any continuous function defined on a closed interval $[a, b]$, there exists a sequence of polynomials that converges uniformly to that function on the same interval [1]. Although this celebrated theorem guarantees the existence of such approximating polynomials, it does not provide explicit constructions or describe their structural properties. This limitation motivated further investigations aimed at identifying concrete polynomial operators and determining the conditions under which effective approximation can be achieved.

In this direction, positive linear operators have emerged as a powerful and systematic tool for studying convergence in approximation theory. Early developments in this area were primarily confined to bounded intervals. In particular, Bernstein introduced a sequence of polynomial operators that provided a constructive proof of the Weierstrass theorem and laid the foundation for approximation on compact intervals [2].

Subsequently, Korovkin established a fundamental criterion for approximation by positive linear operators. The well-known Korovkin-type approximation theorem has since become one of the central results in approximation theory. According to this theorem, for a function $f \in C[a,b]$, it suffices to verify the convergence of a sequence of positive linear operators $(L_n)_{n\geq 1}$ on the test functions

$$e_s(x) = x^s, \quad s = 0, 1, 2,$$

corresponding to 1, x, and x^2, in order to guarantee uniform convergence to f on $[a,b]$.

Although Bernstein operators constitute a cornerstone of approximation theory on compact intervals, their applicability is limited when dealing with functions defined on unbounded domains. Since many problems in analysis and applied mathematics naturally arise on unbounded intervals, extending approximation processes beyond compact sets is of substantial importance. Motivated by this need, Szász and Mirakyan independently introduced in the 1950s a sequence of positive linear operators that generalize the Bernstein operators to the interval $[0,\infty)$. These operators, now known as the *Szász–Mirakyan operators*, are defined by

$$S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \quad x \geq 0. \tag{1}$$

An important feature of the Szász–Mirakyan operators is their intrinsic connection with the Poisson distribution. Indeed, the weights

$$\frac{(nx)^k}{k!} e^{-nx}, \quad k = 0, 1, 2, \ldots,$$

form the probability mass function of a Poisson random variable with parameter nx. This probabilistic structure guarantees positivity and facilitates the computation of moments, which plays a crucial role in verifying Korovkin-type conditions and studying convergence behavior. Using these operators, uniform approximation of continuous functions on unbounded intervals was rigorously established, which in turn stimulated the development of numerous generalizations and modifications of the Szász–Mirakyan operators (see, for example, [3, 4, 5, 6, 7, 11, 12, 13, 14, 15]).

Furthermore, motivated by the Poisson-type structure of the Szász–Mirakyan operators, Jain [8] introduced a new class of positive linear operators depending on a parameter $\mu \in [0,1)$, defined as

$$J_n^{[\mu]}(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{nx\,(nx+k\mu)^{k-1}}{k!} f\left(\frac{k}{n}\right).$$

It can be clearly observed that, for $\mu = 0$, the Jain operators reduce to the classical Szász–Mirakyan operators defined in (1). The introduction of the parameter μ provides additional flexibility in controlling the approximation process and has led to a rich body of literature devoted to modified and generalized operator families with enhanced approximation properties.

Motivated by these developments, the present work introduces a new generalization of Szász–Mirakyan type operators and investigates their approximation behavior in detail. In particular, we analyze their convergence properties, establish quantitative error estimates, and demonstrate that the proposed operators achieve an improved rate of convergence compared to certain existing operators. These results contribute to the ongoing development of efficient approximation schemes on unbounded intervals.

1.1 Background

The mathematical foundations of approximation theory were established by the German mathematician Weierstrass in 1885, who proved that any continuous function defined on a closed interval can be uniformly approximated by a sequence of polynomials. Since then, the field has developed substantially, with significant contributions by Bernstein, Szász, and Mirakyan, among others, who introduced and analyzed classes of positive linear operators to provide constructive approaches to approximation of arbitrary functions.

In recent years, further developments in approximation theory have opened new directions for the study of positive linear operators. In particular, Szász–Mirakyan operators have emerged as powerful tools for investigating approximation processes on unbounded intervals. Despite the considerable progress achieved so far, several fundamental problems remain open, including the refinement of convergence rates, the construction of more flexible operator families, and the analysis of approximation properties under weaker assumptions.

1.2 Main Contributions

The primary contributions of this work can be summarized as follows:

1. **Theoretical Framework:** In this work, we introduce a new class of positive linear operators inspired by the classical Szász–Mirakyan operators and develop a unified analytical framework that integrates key concepts from approximation theory, providing a coherent foundation for the systematic investigation of Szász–Mirakyan type operators.

2. **New Theorems:** We prove several new results that extend classical theorems in theory of approximation, including voronovskaya-type theorem and also rate of convergence.

3. **Computational Methods:** We design efficient computation by using the rate of convergence for comparing the our newly-constructed operators with other operators given in literature .

2 Constructing the Framework of the Newly-Defined Operators

In order to refine the approximation performance of the classical Szász–Mirakjan operators, we introduce a modified framework that retains their essential properties of positivity and linearity while exhibiting enhanced convergence behaviour. The construction of the proposed operators is formulated so that their moments and associated approximation estimates provide more accurate and sharper results than those obtained in the classical setting. In what follows, we present the rigorous definition of the newly defined operators and lay the theoretical foundation for their subsequent analytical examination.

Definition 2.1. Let $C(R^+)$ be the space of all continuous functions defined on positive real axis. For any $f \in C(R^+)$ and $x \in (0, \infty)$, the construction of the proposed operator is defined as follows:

$$\mathscr{S}_n(f;x) = \frac{4e^{-nx}}{4nx+1} \sum_{k=0}^{\infty} \left(k - nx - \frac{1}{2}\right)^2 \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right) \tag{2}$$

Lemma 2.1. Considering $f \in C(\mathbb{R}^+)$ and $x \in (0, \infty)$, the result established in Definition (2.1) can equivalently be expressed within the setting of the classical Szász–Mirakyan type operators given in (1), thereby preserving the structure of the underlying approximation scheme. So, we can write;

$$\mathscr{S}_n(e_m;x) = \frac{4}{4nx+1}\left[n^2 S_n(e_{m+2};x) - 2n\left(nx + \frac{1}{2}\right) S_n(e_{m+1};x) + \left(nx + \frac{1}{2}\right)^2 S_n(e_m;x)\right]$$

Proof. To establish the relationship between the newly introduced operators and the classical ones, let us denote $e_m = t^m$ for $m \in \mathbb{N}$. Accordingly, we arrive at the following conclusion.

$$\begin{aligned}
\mathscr{S}_n(e_m;x) &= \frac{4e^{-nx}}{4nx+1} \sum_{k=0}^{\infty} \left(k - nx - \frac{1}{2}\right)^2 \frac{(nx)^k}{k!} \left(\frac{k}{n}\right)^m \\
&= \frac{4e^{-nx}}{4nx+1} \sum_{k=0}^{\infty} \left[k^2 - 2k\left(nx + \frac{1}{2}\right) + \left(nx + \frac{1}{2}\right)^2\right] \frac{(nx)^k}{k!} \left(\frac{k}{n}\right)^m \\
&= \frac{4e^{-nx}}{4nx+1} \left[\sum_{k=0}^{\infty} k^2 \frac{(nx)^k}{k!} \left(\frac{k}{n}\right)^m - 2\left(nx + \frac{1}{2}\right) \sum_{k=0}^{\infty} k \frac{(nx)^k}{k!} \left(\frac{k}{n}\right)^m \right. \\
&\quad \left. + \left(nx + \frac{1}{2}\right)^2 \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \left(\frac{k}{n}\right)^m\right] \\
&= \frac{4e^{-nx}}{4nx+1} \left[n^2 \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \left(\frac{k}{n}\right)^{m+2} - 2n\left(nx + \frac{1}{2}\right) \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \left(\frac{k}{n}\right)^{m+1} \right. \\
&\quad \left. + \left(nx + \frac{1}{2}\right)^2 \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \left(\frac{k}{n}\right)^m\right]
\end{aligned}$$

By taking the classical Szász-Mirakyan operators in (1) into consideration, we obtain the following result:

$$\mathscr{S}_n(e_m;x) = \frac{4}{4nx+1}\left[n^2 S_n(e_{m+2};x) - 2n\left(nx + \frac{1}{2}\right) S_n(e_{m+1};x) + \left(nx + \frac{1}{2}\right)^2 S_n(e_m;x)\right]$$

Hence, the proof is completed. $\square$

3 Auxiliary Results

This section is devoted to the derivation of several auxiliary results that will play a crucial role in the proofs of the main theorems. The presented lemmas and technical estimates form the analytical basis for the investigation of the approximation behaviour of the newly introduced operators.

Lemma 3.1. Considering the operators defined in (2), the following result holds for all $x \in (0, \infty)$ and $n \in \mathbb{N}$.

1. $\mathscr{S}_n(e_0; x) = 1$
2. $\mathscr{S}_n(e_1; x) = x$
3. $\mathscr{S}_n(e_2; x) = x^2 + \dfrac{12x^2n + x}{4xn^2 + n}$
4. $\mathscr{S}_n(e_3; x) = x^3 + \dfrac{x(1 + nx(31 + 36nx))}{n^2(1 + 4nx)}$
5. $\mathscr{S}_n(e_4; x) = x^4 + \dfrac{x(1 + nx(67 + 2nx(89 + 36nx)))}{n^3(1 + 4nx)}$

In order to investigate the central moments of the operators introduced in (2), we consider and construct their new equivalent representation given below:

$$M_n^j(x) = \mathscr{S}_n\left((e_1 - e_0x)^j; x\right) \tag{3}$$

Observe that $j \in \mathbb{N}$. The identities established in Lemma (3.1) provide the foundation for the subsequent lemma, from which further analytical conclusions can be drawn.

Lemma 3.2. The following results can be found for the sequence of operators M_n^j.

1. $M_n^0(x) = 1$
2. $M_n^1(x) = 0$
3. $M_n^2(x) = \dfrac{12x^2n + x}{4xn^2 + n}$
4. $M_n^3(x) = \dfrac{x(1 + 28nx)}{n^2(1 + 4nx)}$
5. $M_n^4(x) = \dfrac{x(1 + 3nx(21 + 20nx))}{n^3(1 + 4nx)}$

Proof. The proof of the preceding lemma can be quickly derived by invoking Lemma (3.1) and employing the operator given in (3). Hence, the proof is omitted as it follows straightforwardly from the preceding arguments. □

Theorem 3.1. $\lim_{n\to\infty} n^2 M_n^4(x) = 15x^2$ uniformly for all $x \in [0, A]$, where $A > 0$.

Proof. The proof of the theorem follows from straightforward calculations based on the limit, and is therefore omitted here. □

4 Approximation Properties in Weighted Spaces

In this part of the paper, we study the approximation behaviour of the newly constructed operators $\mathscr{S}_n$ within the framework of a Korovkin-type theorem in weighted function spaces. We adopt the weight function $\theta(x) = 1 + x^2$ to regulate the growth of functions under consideration. Let K_f denote a positive constant that depends solely on the function f.

We introduce the weighted space where the associated weighted norm is defined by

$$\|f\|_\theta = \sup_{x\geq 0} \frac{|f(x)|}{\theta(x)}.$$

By employing the norm introduced above, we define the associated weighted spaces by

$$\begin{aligned} A_\theta &= \left\{ f : \mathbb{R}^+ \to \mathbb{R} \ : \ |f(x)| \leq K_f \theta(x), \ x \geq 0 \right\}, \\ B_\theta &= C(\mathbb{R}^+) \cap A_\theta(\mathbb{R}^+), \\ C_\theta^\mu &= \left\{ f \in B_\theta(\mathbb{R}^+) : \lim_{x\to\infty} \frac{f(x)}{\theta(x)} = \mu_f < \infty \right\}, \end{aligned}$$

Functions belonging to C_θ^μ exhibit at most quadratic growth and remain bounded when normalized by the weight function $\theta(x)$. Within this framework, we analyse the convergence of $\mathscr{S}_n(f;x)$ to $f(x)$ and establish sufficient conditions under which such convergence holds in the weighted sense.

Lemma 4.1. [9] Let $(L_n)_{n\geq 0}$ be a sequence of positive linear operators. Then, the operators L_n map the space B_θ into A_θ if and only if there exists a positive constant H_n, depending on n, such that the inequality

$$|L_n(\theta;x)| \leq H_n\, \theta(x), \qquad x \geq 0,$$

holds, where $\theta(x)$ denotes the associated weight function.

Theorem 4.1. Let $(L_n)_{n\geq 0}$ be a sequence of positive linear operators. If the operators L_n map the space B_θ into A_θ, then the sequence (L_n) satisfies the following condition.

$$\lim_{n\to\infty} \|L_n(e_m) - e_m\|_\theta = 0 \qquad m = 0, 1, 2.$$

Consequently for the sequence of positive linear operators and $f \in B_\theta$

$$\lim_{n\to\infty} \|L_n(f) - f\|_\theta = 0$$

is obtained.

In view of the above considerations, we can now formulate the following theorem.

Theorem 4.2. For a function $f \in B_\mu$ and the sequence of positive linear operators $\mathscr{S}_n$ defined in (2) the upcoming result

$$\lim_{n\to\infty} \|\mathscr{S}_n(f) - f\|_\theta = 0$$

is gathered.

Proof. To begin the proof, we first consider the case $m = 0$. Then, we obtain

$$\lim_{n\to\infty} \|\mathscr{S}_n(e_0) - e_0\|_\theta = 0$$

Similarly, for $m = 1$, we have

$$\lim_{n\to\infty} \|\mathscr{S}_n(e_1) - e_1\|_\theta = 0.$$

Finally, for $m = 2$, we derive

$$\begin{aligned} \lim_{n\to\infty} \|\mathscr{S}_n(e_2) - e_2\|_\theta &= \sup_{x\geq 0} \frac{|\mathscr{S}_n(e_2) - e_2|}{\theta(x)} \\ &= \sup_{x\geq 0} \frac{12x^2n + x}{4xn^2 + n} \sup_{x\geq 0} \frac{1}{1+x^2}. \end{aligned}$$

Under the limit condition as $n \to \infty$, one can obtain

$$\lim_{n\to\infty} \|\mathscr{S}_n(e_2) - e_2\|_\theta = 0$$

Hence, the required conditions of the Korovkin-type theorem are satisfied for $m = 0, 1, 2$, which completes the proof. □

5 Voronovskaya-Type Theorem

The Voronovskaya-type theorem stands as one of the central results in approximation theory, providing a deeper understanding of how sequences of positive linear operators approximate a given function beyond the mere pointwise or uniform convergence. It establishes a second-order asymptotic expansion that captures the rate and nature of convergence by relating the deviation of the operator from the target function to the higher-order derivatives of the function. This asymptotic characterization not only measures the operator's approximation precision but also reveals its structural efficiency in reproducing smooth functions. In the context of the operator defined in (2), we now formulate its corresponding Voronovskaya-type estimate, which elucidates its limiting behavior and offers valuable analytical insight into its convergence dynamics.

Theorem 5.1. Let $f \in C^2(\mathbb{R}^+)$ be a function whose first and second derivatives are twice continuously differentiable on $\mathbb{R}^+$. Then, the following equality is obtained:

$$\lim_{n\to\infty} n\left[\mathscr{S}_n(f;x) - f(x)\right] = \frac{3}{2} x f''(x)$$

Proof. From the well-known Taylor's formula, we can express $f(t)$ as

$$f(t) = f(x) - f'(x)(t-x) + \frac{1}{2} f''(x)(t-x)^2 + \alpha(t,x)(t-x)^2,$$

where $x \in (0, \infty)$. Here, η denotes a point lying between x and t such that

$$\alpha(t,x) := \frac{f''(\eta) - f''(x)}{2}.$$

It is worth noting that $\alpha(t, x)$ is continuous and tends to zero as $t \to x$.
The above representation corresponds to the Peano form of the remainder. Hence, by employing the newly introduced operator defined in (2) to the both sides of the equation , following equality can be obtained.

$$\mathscr{S}_n(f; x) = f(x) + f'(x)\mathscr{S}_n((t-x); x) + \frac{1}{2}f''(x)\mathscr{S}_n((t-x)^2; x) + \mathscr{S}_n(\alpha(t,x)(t-x)^2; x).$$

Multiplying the both sides wit n we get;

$$n\left[\mathscr{S}_n(f; x) - f(x)\right] = nf'(x)\mathscr{S}_n((t-x); x) + \frac{1}{2}nf''(x)\mathscr{S}_n((t-x)^2; x) + n\mathscr{S}_n(\alpha(t,x)(t-x)^2; x).$$

At this stage, we make use of Lemma (3.2), which provides the explicit expressions for the first and second central moments of the operator. Substituting these relations into the above equation yields

$$n\left[\mathscr{S}_n(f; x) - f(x)\right] = nf'(x)M_n^1 + \frac{1}{2}nf''(x)M_n^2 + n\mathscr{S}_n(\alpha(t,x)(t-x)^2; x).$$

Upon taking the limit as $n \to \infty$ on both sides of the equality, we arrive at;

$$\lim_{n\to\infty} n\left[\mathscr{S}_n(f; x) - f(x)\right] = \lim_{n\to\infty} nf'(x)M_n^1 + \lim_{n\to\infty} \frac{1}{2}nf''(x)M_n^2 + \lim_{n\to\infty} n\mathscr{S}_n(\alpha(t,x)(t-x)^2; x).$$

Utilizing the results obtained in Lemma (3.2), we arrive at the following relation.

$$\lim_{n\to\infty} n\left[\mathscr{S}_n(f; x) - f(x)\right] = \frac{3}{2}xf''(x) + \lim_{n\to\infty} n\mathscr{S}_n(\alpha(t,x)(t-x)^2; x) \tag{4}$$

Then applying the Cauchy-Schwarsz inequality to the right side of the equation (4) we obtain

$$\lim_{n\to\infty} n\mathscr{S}_n(\alpha(t,x)(t-x)^2; x) \le \left(n^2\mathscr{S}_n(\alpha^2(t,x); x)\right)^{1/2}\left(\mathscr{S}_n\left((e_1 - e_0x)^4; x); x\right)\right)^{1/2}.$$

Consequently, applying the Lemma (3.2)-(v) we come into following conclusion

$$\lim_{n\to\infty} n\mathscr{S}_n(\alpha(t,x)(t-x)^2; x) \le \left(n^2\mathscr{S}_n(\alpha^2(t,x); x)\right)^{1/2}\left(M_n^4\right)^{1/2}. \tag{5}$$

It follows from the Korovkin Theorem that for $\alpha(.; x) \in C(\mathbb{R}^+)$

$$\lim_{n\to\infty} \mathscr{S}_n(\alpha^2(t,x); x) = \alpha^2(x,x) = 0$$

Finally applying the theorem (3.1) to the inequality in (5) we obtained the desired result. □

6 Rate of Convergence

In this chapter, we provide a brief discussion on the rate of convergence, an essential concept in approximation theory and numerical analysis. The rate of convergence characterizes the speed at which a sequence of approximations approaches its limiting value as the parameter n (or any relevant index) increases. A higher rate signifies that the approximation method attains greater accuracy with fewer computational steps, reflecting both efficiency and precision. This notion is particularly important when comparing different operators or numerical schemes, as it provides a theoretical measure of their effectiveness and practical reliability.

Theorem 6.1. Let $\omega_f(\delta)$ denote the modulus of continuity of a function $f \in B_\theta$. Then, for all $x \in (0,\infty)$ and $n \in \mathbb{N}$, the following inequality holds:

$$\|\mathscr{S}_n(f;x) - f(x)\| \leq 2\,\omega_f(\eta_x),$$

where $\eta_x = \sqrt{\frac{12x^2n+x}{4xn^2+n}}$.

Proof. Consider the sequence of positive linear operators $(\mathscr{S}_n)_{n\geq 1}$ defined in (2). For any $f \in B_\theta$ and $x \in (0,\infty)$, we have

$$\begin{aligned} |\mathscr{S}_n(f;x) - f(x)| &\leq |\mathscr{S}_n(f;x) - \mathscr{S}_n(f(x);x)| \\ &\leq \mathscr{S}_n(|f(t) - f(x)|;x). \end{aligned}$$

By employing the well-known property of the modulus of continuity, we can write

$$\begin{aligned} \mathscr{S}_n(|f(t) - f(x)|;x) &\leq \mathscr{S}_n\left(1 + \frac{|t-x|}{\delta};x\right)\omega_f(\delta) \\ &= \left[\mathscr{S}_n(1;x) + \frac{1}{\delta}\mathscr{S}_n(|t-x|;x)\right]\omega_f(\delta). \end{aligned}$$

Applying the Cauchy–Schwarz inequality together with Lemma (3.2), we obtain

$$\begin{aligned} \mathscr{S}_n(|t-x|;x) &\leq (\mathscr{S}_n((t-x)^2;x))^{1/2}(\mathscr{S}_n(1;x))^{1/2} \\ &= (M_n^2)^{1/2}. \end{aligned}$$

Combining these estimates yields

$$|\mathscr{S}_n(f;x) - f(x)| \leq \left(1 + \frac{1}{\delta}(M_n^2)^{1/2}\right)\omega_f(\delta).$$

Finally, choosing $\delta = \eta_x$ gives the desired inequality, which completes the proof. □

We now proceed to compare our proposed operators with those introduced by Mahmudov in [10]. Before undertaking the comparison, we briefly recall the convergence properties and approximation behavior of the operators in [10].

Remark 6.1. Operators introduced in [10] have following relation for $f \in C(\mathbb{R}^+)$ and $x \in (0,\infty)$

$$|M_n^*(f;x) - f(x)| \leq 2\omega_f(\psi_x)$$

Where $\psi_x = \sqrt{\dfrac{3xn+1}{n^2}}$

Theorem 6.2. For sequence of positive linear operators introduced in (2) have better convergence rate than the operators introduced in [10] for $f \in C(\mathbb{R}^+)$ and $x \in (0,\infty)$.

Proof. Let $f \in C^2(\mathbb{R}^+)$. In order for the newly defined operators to provide a better approximation than those introduced in [10], the following inequality must be satisfied:

$$\eta_x \leq \psi_x$$

To establish this inequality, we observe that

$$\eta_x = \frac{12x^2n + x}{4xn^2 + n} < \frac{12x^2n + x}{4xn^2} < \frac{12x^2n + 4x}{4xn^2} = \frac{3xn + 1}{n^2} = \psi_x$$

Hence, the desired inequality holds, and the proof is thereby completed. □

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