Modular Fuzzy Metric-Like Spaces

Nizamettin Ufuk Bostan[1], Banu Pazar Varol[1]

[1]Department of Mathematics, Kocaeli University, 41001, Kocaeli - TÜRKİYE

Corresponding author: ufuk.bostan@kocaeli.edu.tr

ORCID IDs:

First Author: 0000-0003-4708-8663

Second Author: 0000-0002-8627-7910

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Abstract

The aim of this paper is to define the concept of modular fuzzy metric-like space by combining of modular-like metric space and fuzzy metric-like space. We give examples and propositions to approve the importance of this new concept. Fundamental notions containing convergence, Cauchy sequence and completeness are carefully defined. We also demonstrate that, in a modular fuzzy metric-like space, the limit of the convergent sequence may not be unique. The fixed point theorem is proved, expanding the classical findings to this space. The applicability of fixed point theorem is shown through an example, expressing both theoretical potency and practical benefit of the proposed approach.

Keywords: metric-like spaces, fuzzy metric spaces, fuzzy metric-like spaces, modular-like metric spaces, modular fuzzy metric-like spaces, fixed point theory, contraction principle

1 Introduction and Motivation

The basis of metric space theory were laid more than a century ago by Fréchet [13] and Hausdorff [16], centered on the notion of distance between any two points in a set. In 1922, Banach [2] introduced what is now known as the Banach's fixed point theorem or the Banach Contraction Principle, establihed within the setting of metric spaces. Later, Amini-Harandi [1] gave the concept of metric-like space and proved corresponding fixed point theorems, supported by illustrative examples that expressed the structure. Further developments were made by Hosseini and Fošner [17], who introduced several related notions for metric-like spaces, including equal-like points, cluster points, completely separate points, as well as definitions of distance between a point and a subset and between two subsets in a metric-like space.

Fuzzy set theory, introduced by Zadeh [26], has since become a widely studied area across various scientific and applied disciplines. The definition of fuzzy metric space was first given by Kramosil and Michálek [19]. Later, George and Veeramani [14, 15] modified this definition to gain the result that the formed structure generates a Hausdorff topology, thereby strengthening and expanding the practicality of fuzzy metrics. Based on these developments, Shukla and Abbas [25] introduced the concept of fuzzy metric-like space generalizing the concept of fuzzy metric space of George and Veeramani, and proved several fixed point theorems in this more comprehensive space.

Nakano [21] first introduced the definition of modular, a concept later further developed by Orlicz [22]. The concepts of metric modular and modular metric space were presented by Chistyakov [4, 5] in the course of building the theory of these structures. According to Chistyakov [7], while a metric on a set measures non-negative finite distances between points, a metric modular is given to describe non-negative, possibly infinite-valued, velocities. Some of his results appear in [6], and his fixed point theorems together with their applications are introduced in detail in [7]-[10]. An exhaustive compilation of Chistyakov's contributions to metric modulars and modular metric spaces was later provided in [11]. Mongkolkeha et al. [20] established and proved existence theorems of fixed points for contraction mappings in modular metric spaces. Rasham et al. [23] presented the concept of modular-like metric space and proved some common fixed point theorems for two families of set-valued mappings satisfying contraction condition in this space. In the same work [23], they also obtained new results in graph theory concerning multigraph-dominated contractions within modular-like metric spaces. Additional fixed point results in modular-like metric spaces can be found in [12].

Kerim et al. [18] introduced a new space termed the modular fuzzy metric space in the sense of Kramosil-Michálek. They explored its main properties and illustrated the structure of space with several examples. Based on this foundation, they obtained existence and uniqueness results for fixed points of continuous mappings in this space and showed the applicability of their findings. Then Bostan and Pazar Varol [3] introduced the concept of modular fuzzy metric space in the sense of George-Veeramani in 2023

In this study, we define the concept of modular fuzzy metric-like space by combining modular-like metric space and fuzzy metric-like space. We examine some properties of this space and give various examples that provide a better understanding of the structure of space. We give the notions of modular fuzzy metric-like space such as contraction, convergence, Cauchy sequence, completeness and show that in a modular fuzzy metric-like space, the limit of the convergent sequence may not be unique. Then we establish and prove fixed point theorem in the modular fuzzy metric-like space. We also demonstrate the power of the structure we established and of the results we gained in this structure by giving an example for the fixed point theorem we proved. Our aim here is to contribute to fixed point theory by being interested in fixed point theorem in modular fuzzy metric-like space, which is a generalization of modular fuzzy metric space in the sence of George-Veeramani. We aim to carry the applications of fixed point theory in mathematics and engineering to modular fuzzy metric-like space by modulating the concept of fuzzy metric-like space. The new mathematical structure introduced in this paper may enable researchers working in fixed point theory or its applications to establish various fixed point theorems under different contraction conditions and to explore their applicability in broader contexts.

2 Mathematical Preliminaries

In this section, we introduce several essential definitions an concepts that will be used throughout the paper. Throughout the text, we denote by IR the set of all real numbers and by IN the set of all positive integers.

Definition 2.1. [1] Let $W \neq \emptyset$. A mapping $\sigma : W \times W \to [0, \infty)$ is called metric-like on W

if the following three conditions hold for all $w, y, z \in W$:

(ML1) $\sigma(w,z) = 0 \Rightarrow w = z$,

(ML2) $\sigma(w,z) = \sigma(z,w)$,

(ML3) $\sigma(w,z) \leq \sigma(w,y) + \sigma(y,z)$.

The pair (W,σ) is called a metric-like space.

Example 2.1. [1] Let $W = [0,\infty)$. Define the function $\sigma : W \times W \to [0,\infty)$ by $\sigma(w,z) = \max\{w,z\}$. Then (W,σ) is a metric-like space.

Definition 2.2. [26] A fuzzy set Q in W is characterized by a membership function $f_Q(w)$ which associates each point in W with a real number in the interval $[0,1]$. The value of $f_Q(w)$ at w represent the grade of membership of w in W.

Definition 2.3. [24] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if $*$ satisfies the following conditions for all $o, u, j, l \in [0,1]$:

(1) $o * 1 = o$

(2) $o * u = u * o$ and $o * (u * j) = (o * u) * j$

(3) If $o \leq j$ and $u \leq l$, then $o * u \leq j * l$

(4) $*$ is continuous.

Example 2.2. [24] The binary operations defined as follows are the continuous t-norms:

(1) $o * u = ou$

(2) $o * u = \max\{0, o + u - 1\}$

(3) $o * u = \min\{o, u\}$

Definition 2.4. [14] A triplet $(W, Q, *)$ is called fuzzy metric space if W is an arbitrary set, $*$ is a continuous t-norm and Q is a fuzzy set on $W^2 \times (0,\infty)$ satisfying the following conditions, for all $w, y, z \in W$ and $t, s > 0$:

(FM1) $Q(w,y,t) > 0$,

(FM2) $Q(w,y,t) = 1 \Leftrightarrow w = y$,

(FM3) $Q(w,y,t) = Q(y,w,t)$,

(FM4) $Q(w,y,t) * Q(y,z,s) \leq Q(w,z,t+s)$,

(FM5) $Q(w,y,.) : (0,\infty) \to [0,1]$ is continuous.

Example 2.3. [14] Let $W = IR$. Define $o * u = ou$ and $Q(w,y,t) = e^{\frac{-|w-y|}{t}}$ for all $w, y \in Q = IR$ and $t > 0$. Then $(W, Q, *)$ is a fuzzy metric space.

Example 2.4. [14] Let (W,δ) be a metric space. Define $o * u = ou$ and $Q_\delta(w,y,t) = \frac{kt^n}{kt^n + m\delta(w,y)}$ $k, m, n \in IR^+$. Then $(W, Q, *)$ is a fuzzy metric space.

Remark 2.1. [14] In the above example by taking $k = m = n = 1$, we get $Q_\delta(w,y,t) = \frac{t}{t+\delta(w,y)}$. This fuzzy metric induced by a metric δ is called the standard fuzzy metric.

Definition 2.5. [25] A triplet $(W, Q, *)$ is called fuzzy metric-like space if W is an arbitrary set, $*$ is a continuous t-norm and Q is a fuzzy set on $W^2 \times (0,\infty)$ satisfying the following conditions, for all $w, y \in W$ and $t, s > 0$:

(FML1) $Q(w,y,t) > 0$,

(FML2) $Q(w,y,t) = 1 \Rightarrow w = y$,

(FML3) $Q(w,y,t) = Q(y,w,t)$,

(FML4) $Q(w,y,t) * Q(y,z,s) \leq Q(w,z,t+s)$,

(FML5) $Q(w,t,.) : (0,\infty) \to [0,1]$ is continuous.

Example 2.5. [25] Let $W = [0, 1]$. Define $o * u = ou$ and $Q(w, y, t) = \frac{t}{t+\max\{w,y\}}$ for all $w, y \in W$ and $t > 0$. Then $(W, Q, *)$ is a fuzzy metric-like space.

Definition 2.6. [4] Let $W \neq \emptyset$. A mapping $\zeta : (0, \infty) \times W \times W \to [0, \infty]$ is called metric modular on W if the following hold for each $w, y, z \in W$:

(MM1) $\zeta(w, y, \lambda) = 0 \Leftrightarrow w = y$, for all $\lambda > 0$,

(MM2) $\zeta(w, y, \lambda) = \zeta(y, w, \lambda)$, for all $\lambda > 0$,

(MM3) $\zeta(w, y, \lambda + \gamma) \leq \zeta(w, z, \lambda) + \zeta(z, y, \gamma)$, for all $\lambda, \gamma > 0$.

For simplicity, the notation $\zeta_\lambda(w, y)$ will be used instead of $\zeta(w, y, \lambda)$ for modular structures throughout the remainder of the paper.

Example 2.6. [7] Let (W, δ) be metric space. Define the function $\zeta : (0, \infty) \times X \times X \to [0, \infty]$ by $\zeta_\lambda(w, y) = \frac{\delta(w,y)}{\lambda}$ for all $\lambda > 0$ and $w, y \in W$. Then ζ is a metric modular on W.

Definition 2.7. [23] Let $W \neq \emptyset$. A function $\varphi : (0, \infty) \times W \times W \to [0, \infty)$ is called modular-like metric on W, if it satisfies the following three conditions for each $w, y, z \in W$:

(MLM1) $\varphi_\lambda(w, y) = 0 \Rightarrow w = y$, for all $\lambda > 0$,

(MLM2) $\varphi_\lambda(w, y) = \varphi_\lambda(y, w)$, for all $\lambda > 0$,

(MLM3) $\varphi_{\lambda+\gamma}(w, y) \leq \varphi_\lambda(w, z) + \varphi_\gamma(z, y)$, for all $\lambda, \gamma > 0$.

Then the triplet (W, φ) is called modular-like metric space.

Definition 2.8. [18] A triplet $(W, Q_\beta, *)$ is called modular fuzzy metric space if W is an arbitrary set, $*$ is a continuous t-norm and Q_β is fuzzy set on $W^2 \times (0, \infty)$ satisfying the following conditions where $\beta > 0$, for all $w, y, z \in W$ and $t, s > 0$:

(MFM1) $Q_\beta(w, y, 0) = 0$, $Q_\beta(w, y, t) > 0$, for all $\beta > 0$,

(MFM2) $Q_\beta(w, y, t) = 1 \Leftrightarrow w = y$, for all $\beta > 0$,

(MFM3) $Q_\beta(w, y, t) = Q_\beta(y, w, t)$, for all $\beta > 0$,

(MFM4) $Q_\beta(w, y, t) * Q_\mu(y, z, s) \leq Q_{\beta+\mu}(w, z, t + s)$, for all $\beta, \mu > 0$,

(MFM5) $Q_\beta(w, y, .) : (0, \infty) \to [0, 1]$ is left continuous.

Here Q_β is called a modular fuzzy metric.

Example 2.7. [18] Let ζ be a metric modular on W and $o * u = ou$ for all $o, u \in [0, 1]$. Define fuzzy set Q_β on $W^2 \times (0, \infty)$ as follows:

$Q_\beta(w, y, t) = e^{\frac{-\zeta_\beta(w,y)}{t}}$ for all $w, y \in W$ and $\beta, t > 0$.

Then $(W, Q_\beta, *)$ is a modular fuzzy metric space.

Example 2.8. [18] Let ζ be a metric modular on W and $o * u = ou$ for all $o, u \in [0, 1]$. Define fuzzy set Q_β on $W^2 \times (0, \infty)$ as follows:

$Q_\beta(w, y, t) = \frac{t}{t+\zeta_\beta(w,y)}$ for all $w, y \in \Omega$ and $\beta, t > 0$.

Then $(W, Q_\beta, *)$ is a modular fuzzy metric space.

Definition 2.9. [3] A triplet $(W, Q_\beta, *)$ is called modular fuzzy metric space (in the sence of George-Veeramani) if W is an arbitrary set, $*$ is a continuous t-norm and Q_β is fuzzy set on $W^2 \times (0, \infty)$ satisfying the following conditions where $\beta > 0$, for all $w, y, z \in W$ and $t, s > 0$:

(MFMGV1) $Q_\beta(w, y, t) > 0$, for all $\beta > 0$,

(MFMGV2) $Q_\beta(w, y, t) = 1 \Leftrightarrow w = y$, for all $\beta > 0$,

(MFMGV3) $Q_\beta(w, y, t) = Q_\beta(y, w, t)$, for all $\beta > 0$,

(MFMGV4) $Q_\beta(w, y, t) * Q_\mu(y, z, s) \leq Q_{\beta+\mu}(w, z, t + s)$, for all $\beta, \mu > 0$,

(MFMGV5) $Q_\beta(w, y, .) : (0, \infty) \to [0, 1]$ is continuous.

Here Q_β is called modular fuzzy metric (in the sence of George-Veeramani).

Example 2.9. [3] Let ζ be a metric modular on W such that $\zeta : (0,\infty) \times W^2 \to [0,\infty)$. Define $o * u = ou$ for all $o, u \in [0,1]$ and fuzzy set Q_β on $W^2 \times (0,\infty)$ as follows:

$Q_\beta(w,y,t) = e^{\frac{-\zeta_\beta(w,y)}{t}}$ for all $w, y \in W$ and $\beta, t > 0$.

Then $(W, Q_\beta, *)$ is a modular fuzzy metric space.

Example 2.10. [3] Let ζ be a metric modular on W such that $\zeta : (0,\infty) \times W^2 \to [0,\infty)$. Define $o * u = ou$ for all $o, u \in [0,1]$ and fuzzy set Q_β on $W^2 \times (0,\infty)$ as follows:

$Q_\beta(w,y,t) = \frac{t}{t+\zeta_\beta(w,y)}$ for all $w, y \in \Omega$ and $\beta, t > 0$.

Then $(W, Q_\beta, *)$ is a modular fuzzy metric space.

3 Results

We begin this section by introducing the definition of modular fuzzy metric-like space and investigating its main properties. We then provide illustrative examples, propositions and present the notions of convergence, Cauchy sequences, and completeness within this mathematical structure. Finally, we give and prove fixed point theorem in this space.

Definition 3.1. The triplet $(W, Q_\beta, *)$ is called a modular fuzzy metric-like space if W is an arbitrary nonempty set, $*$ is a continuous t norm and Q_β is a fuzzy set on $W \times W \times (0,\infty)$ satisfying the following conditions for all $w, y, z \in W$ and $t, s > 0$:

(MFML1) $Q_\beta(w,y,t) > 0$, for all $\beta > 0$,

(MFML2) $Q_\beta(w,y,t) = 1 \Rightarrow w = y$, for all $\beta > 0$,

(MFML3) $Q_\beta(w,y,t) = Q_\beta(y,w,t)$, for all $\beta > 0$,

(MFML4) $Q_\beta(w,y,t) * Q_\mu(y,z,s) \leq Q_{\beta+\mu}(w,z,t+s)$, for all $\beta, \mu > 0$

(MFML5) $Q_\beta(w,y,.) : (0,\infty) \to [0,1]$ is continuous for all $\beta > 0$.

Example 3.1. Let $W = [0,\infty)$ and $o * u = ou$ for all $o, u \in [0,1]$. Define the fuzzy set Q_β in $W^2 \times (0,\infty)$ as follows:

$Q_\beta(w,y,t) = e^{-\frac{(\frac{max\{w,y\}}{\beta})}{t}}$ for all $w, y \in W$ and $\beta > 0$ and $t \in (0,\infty)$. Then $(W, Q_\beta, *)$ is a modular fuzzy metric-like space.

(MFML1)-(MFML3) and (MFML5) are obvious. Now, we prove (MFML4).

Since $max\{w,y\} \leq max\{w,z\} + max\{z,y\}$, we have $\frac{max\{w,y\}}{\beta+\mu} \leq \frac{max\{w,z\}}{\beta} + \frac{max\{z,y\}}{\mu}$.

$$
\begin{aligned}
&\Rightarrow \frac{max\{w,y\}}{\beta+\mu} \leq \frac{(t+s)}{t}\frac{max\{w,z\}}{\beta} + \frac{(t+s)}{t}\frac{max\{z,y\}}{\mu}\\
&\Rightarrow \frac{max\{w,y\}}{(\beta+\mu)(t+s)} \quad \leq \quad \frac{max\{w,z\}}{\beta t} + \frac{max\{z,y\}}{\mu s}\\
&\Rightarrow \frac{max\{w,y\}}{(\beta+\mu)(t+s)} \quad \leq \quad \frac{max\{w,z\}}{\beta t} + \frac{max\{z,y\}}{\mu s}\\
&\Rightarrow e^{\frac{max\{w,y\}}{(\beta+\mu)(t+s)}} \quad \leq \quad e^{\frac{max\{w,z\}}{\beta t} + \frac{max\{z,y\}}{\mu s}}\\
&\qquad\qquad\qquad = \quad e^{\frac{max\{w,z\}}{\beta t}} e^{\frac{max\{z,y\}}{\mu s}}\\
&\Rightarrow e^{\frac{-max\{w,y\}}{(\beta+\mu)(t+s)}} \quad \geq \quad e^{\frac{-max\{w,z\}}{\beta t}} e^{\frac{-max\{z,y\}}{\mu s}}
\end{aligned}
$$

$\Rightarrow Q_{\beta+\mu}(w,y,t+s) \geq Q_\beta(w,z,t) * Q_\mu(z,y,s)$ for all $\beta, \mu > 0$. Hence (MFML4) holds.

Example 3.2. Let $W = [0,\infty)$ and $o * u = ou$ for all $o, u \in [0,1]$. Define the fuzzy set Q_β in $W^2 \times (0,\infty)$ as follows:

$Q_\beta(w,y,t) = \frac{t}{t+\frac{max\{w,y\}}{\beta}}$ for all $w, y \in W, \beta > 0$ and $t > 0$. Then $(W, Q_\beta, *)$ is a modular fuzzy metric-like space.

(MFML1)-(MFML3) and (MFML5) are obvious. Now, we prove (MFML4).

We know $max\{w,y\} \leq max\{w,z\} + max\{z,y\}$. Then

$$\begin{array}{rcl} \frac{max\{w,y\}}{\beta+\mu} & \leq & \frac{max\{w,z\}+max\{wz,y\}}{\beta+\mu} \\ & \leq & \frac{max\{w,z\}}{\beta+\mu} + \frac{max\{z,y\}}{\beta+\mu} \\ & \leq & \frac{max\{w,z\}}{\beta} + \frac{max\{z,y\}}{\mu} \end{array}$$

Thus, $\frac{max\{w,y\}}{\beta+\mu} \leq \frac{max\{w,z\}}{\beta} + \frac{max\{z,y\}}{\mu}$.

$$\begin{array}{rcl} \Rightarrow \frac{\frac{max\{w,y\}}{\beta+\mu}}{t+s} & \leq & \frac{\frac{max\{w,z\}}{\beta}+\frac{max\{z,y\}}{\mu}}{t+s} \\ & = & \frac{\frac{max\{w,z\}}{\beta}}{t+s} + \frac{\frac{max\{z,y\}}{\mu}}{t+s} \\ & \leq & \frac{\frac{max\{w,z\}}{\beta}}{t} + \frac{\frac{max\{z,y\}}{\mu}}{s} \\ & = & \frac{s\frac{max\{w,z\}}{\beta}+t\frac{max\{z,y\}}{\mu}}{ts} \\ \Rightarrow 1 + \frac{\frac{max\{w,y\}}{\beta+\mu}}{t+s} & \leq & 1 + \frac{s\frac{max\{w,z\}}{\beta}+t\frac{max\{z,y\}}{\mu}}{ts} \end{array}$$

Since $\frac{max\{w,z\}}{\beta}.\frac{max\{z,y\}}{\mu} \geq 0$,

$$\begin{array}{rcl} \Rightarrow \frac{t+s+\frac{max\{w,y\}}{\beta+\mu}}{t+s} & \leq & \frac{ts+\frac{smax\{w,z\}}{\beta}+t\frac{max\{z,y\}}{\mu}+\frac{max\{w,z\}}{\beta}.\frac{max\{z,y\}}{\mu}}{ts} \\ \Rightarrow \frac{ts}{ts+s\frac{max\{w,z\}}{\beta}+t\frac{max\{z,y\}}{\mu}+\frac{max\{w,z\}}{\beta}.\frac{max\{z,y\}}{\mu}} & \leq & \frac{t+s}{t+s+\frac{max\{w,y\}}{\beta+\mu}} \\ \Rightarrow \frac{t}{t+\frac{max\{w,z\}}{\beta}}.\frac{s}{s+\frac{max\{z,y\}}{\mu}} & \leq & \frac{t+s}{t+s+\frac{max\{w,y\}}{\beta+\mu}} \end{array}$$

$\Rightarrow Q_\beta(w,z,t) * Q_\mu(z,y,s) \leq Q_{\beta+\mu}(w,y,t+s)$ for all $\beta,\mu > 0$. Hence, (MFML4) holds.

Proposition 3.1. Let (W,σ) be a metric-like space and $o * u = ou$ for all $o,u \in [0,1]$. Define fuzzy set Q_β on $W^2 \times (0,\infty)$ as follows:

$Q_\beta(w,y,t) = e^{\frac{-(\frac{\sigma(w,y)}{\beta})}{t}}$ for all $w,y \in W$ and $\beta,t > 0$. Then $(W,Q_\beta,*)$ is a modular fuzzy metric-like space.

Proof. (MFML1)- (MFML3) and (MFLM5) are obvious. Now, we prove (MFML4). Since σ is metric-like, we have $\sigma(w,y) \leq \sigma(w,z) + \sigma(z,y)$ for $w,z,y \in W$.

$\Rightarrow \frac{\sigma(w,y)}{(\beta+\mu)(t+s)} \leq \frac{\sigma(w,z)+\sigma(z,y)}{(\beta+\mu)(t+s)} \leq \frac{\sigma(w,z)}{\beta t} + \frac{\sigma(z,y)}{\mu s}$

$\Rightarrow e^{\frac{\sigma(w,y)}{(\beta+\mu)(t+s)}} \leq e^{\frac{\sigma(w,z)}{\beta t}+\frac{\sigma(z,y)}{\mu s}} = e^{\frac{\sigma(w,z)}{\beta t}} e^{\frac{\sigma(z,y)}{\mu s}}$

$\Rightarrow \Rightarrow e^{-\frac{\sigma(w,y)}{(\beta+\mu)(t+s)}} \geq e^{-\frac{\sigma(w,z)}{\beta t}} e^{-\frac{\sigma(z,y)}{\mu s}}$

$\Rightarrow Q_\beta(w,z,t) * Q_\mu(z,y,s) \leq Q_{\beta+\mu}(w,y,t+s)$.

Hence (MFML4) holds.

Proposition 3.2. Let (W,σ) be a metric-like space and $o * u = ou$ for all $o,u \in [0,1]$. Define fuzzy set Q_β on $W^2 \times (0,\infty)$ as follows:

$Q_\beta(w,y,t) = \frac{t}{t+\frac{\sigma(w,y)}{\beta}}$ for all $w,y \in W$ and $\beta,t > 0$. Then, $(W,Q_\beta,*)$ is modular fuzzy metric-like space.

Proof. (MFML1)- (MFML3) and (MFLM5) are obvious. Now, we prove (MFML4). Since σ is metric-like, we have $\sigma(w,y) \leq \sigma(w,z) + \sigma(z,y)$ for $w,z,y \in W$.

$\Rightarrow \frac{\sigma(w,y)}{(\beta+\mu)(t+s)} \leq \frac{\sigma(w,z)+\sigma(z,y)}{(\beta+\mu)(t+s)} \leq \frac{\sigma(w,z)}{\beta t} + \frac{\sigma(z,y)}{\mu s} = \frac{\mu s\sigma(w,z)+\beta t\sigma(z,y)}{\beta t\mu s}$

$\Rightarrow 1 + \frac{\sigma(w,y)}{(\beta+\mu)(t+s)} \leq 1 + \frac{\mu s\sigma(w,z)+\beta t\sigma(z,y)}{\beta t\mu s}$

$\Rightarrow \frac{(\beta+\mu)(t+s)+\sigma(w,y)}{(\beta+\mu)(t+s)} \leq \frac{\beta t\mu s+\mu s\sigma(w,z)+\beta t\sigma(z,y)}{\beta t\mu s} \leq \frac{\beta t\mu s+\mu s\sigma(w,z)+\beta t\sigma(z,y)+\sigma(w,z)\sigma(z,y)}{\beta t\mu s}$

$\Rightarrow \frac{\beta t\mu s}{\beta t\mu s+\mu s\sigma(w,z)+\beta t\sigma(z,y)+\sigma(w,z)\sigma(z,y)} \leq \frac{(\beta+\mu)(t+s)}{(\beta+\mu)(t+s)+\sigma(w,y)}$

$\Rightarrow \frac{\beta t}{\beta t+\sigma(w,z)}.\frac{\mu s}{\mu s+\sigma(z,y)} \leq \frac{(\beta+\mu)(t+s)}{(\beta+\mu)(t+s)+\sigma(w,y)}$

$\Rightarrow Q_\beta(w,z,t) * Q_\mu(z,y,s) \leq Q_{\beta+\mu}(w,y,t+s)$.

Hence (MFML4) holds.

Proposition 3.3. Let φ be a modular-like metric on W. Define $o * u = ou$ for all $o,u \in [0,1]$ and $Q_\beta : W \times W \times (0,\infty) \to [0,1]$ by $Q_\beta(w,y,t) = e^{\frac{-\varphi_\beta(w,y)}{t}}$ for all $w,y \in W$ and $\beta,t>0$. Then $(W,Q_\beta,*)$ is a modular fuzzy metric-like space.

Proof. (MFML1)- (MFML3) and (MFLM5) are obvious. Now, we prove (MFML4). Since $\varphi_{\beta+\mu}(w,z) \leq \varphi_\beta(w,y) + \varphi_\beta(y,z)$, $\forall \beta,\mu>0$, we have

$\varphi_{\beta+\mu}(w,z) \leq (\frac{t+s}{t})\varphi_\beta(w,y) + (\frac{t+s}{s})\varphi_\beta(y,z)$

$\Rightarrow \frac{\varphi_{\beta+\mu}(w,z)}{t+s} \leq \frac{\varphi_\beta(w,y)}{t} + \frac{\varphi_\beta(y,z)}{s}$

$\Rightarrow e^{\frac{\varphi_{\beta+\mu}(w,z)}{t+s}} \leq e^{(\frac{\varphi_\beta(w,y)}{t}+\frac{\varphi_\beta(y,z)}{s})} = e^{\frac{\varphi_\beta(w,y)}{t}}.e^{\frac{\varphi_\beta(y,z)}{t}}$

$\Rightarrow e^{-\frac{\varphi_{\beta+\mu}(w,z)}{t+s}} \geq e^{-\frac{\varphi_\beta(w,y)}{t}}.e^{-\frac{\varphi_\beta(y,z)}{t}}$

$\Rightarrow Q_\beta(w,z,t) * Q_\mu(z,y,s) \leq Q_{\beta+\mu}(w,y,t+s)$.

Hence (MFML4) holds.

Proposition 3.4. Let φ be a modular-like metric on W. Define $o * u = ou$ for all $o,u \in [0,1]$ and $Q_\beta : W \times W \times (0,\infty) \to [0,1]$ by $Q_\beta(w,y,t) = \frac{t}{t+\varphi_\beta(w,y)}$ for all $w,y \in W$ and $\beta,t>0$. Then $(W,Q_\beta,*)$ is a modular fuzzy metric-like space.

Proof. (MFML1)- (MFML3) and (MFLM5) are obvious. Now, we prove (MFML4). Since

$\varphi_{\beta+\mu}(w,z) \leq \varphi_\beta(w,y) + \varphi_\beta(y,z)$, $\forall \beta,\mu>0$, we have

$\Rightarrow \frac{\varphi_{\beta+\mu}(w,z)}{t+s} \leq \frac{\varphi_\beta(w,y)}{t+s} + \frac{\varphi_\beta(y,z)}{t+s} \leq \frac{\varphi_\beta(w,y)}{t} + \frac{\varphi_\beta(y,z)}{s} = \frac{s\varphi_\beta(w,y)+t\varphi_\mu(y,z)}{ts}$

$\Rightarrow 1 + \frac{\varphi_{\beta+\mu}(w,z)}{t+s} \leq 1 + \frac{s\varphi_\beta(w,y)+t\varphi_\mu(y,z)}{ts}$

Since $\varphi_\beta(w,y).\varphi_\mu(y,z) \geq 0$, we get

$\frac{t+s+\varphi_{\beta+\mu}(w,z)}{t+s} \leq \frac{ts+s\varepsilon_\beta(w,y)+t\varphi_\mu(y,z)+\varphi_\beta(w,y)\varphi_\mu(y,z)}{ts}$

$\Rightarrow \frac{t+s}{t+s+\varphi_{\beta+\mu}(w,z)} \geq \frac{ts}{ts+s\varphi_\beta(w,y)+t\varphi_\mu(y,z)+\varphi_\beta(w,y)\varphi_\mu(y,z)} = \frac{t}{t+\varphi_\beta(w,y)}.\frac{s}{\varphi_\mu(y,z)}$.

Hence, $Q_\beta(w,y,t) * Q_\mu(y,z,s) \leq Q_{\beta+\mu}(w,z,t+s)$ and (MFML4) holds.

Proposition 3.5. Let $(W,Q,*)$ be a fuzzy metric space. Define fuzzy set Q_β on $W^2 \times (0,\infty)$ as follows;

$Q_\beta(w,y,t) = Q(w,y,\beta t)$ for all $w,y \in W$ and $\beta,t>0$. Then, $(W,Q_\beta,*)$ is modular fuzzy metric-like space.

Proof. (MFML1)- (MFML3) and (MFLM5) are obvious. Now, we prove (MFML4).

$$\begin{aligned}
Q_{\beta+\mu}(w,y,t+s) &= Q(w,y,(\beta+\mu)(t+s)) \\
&= Q(w,y,\beta t+\beta s+\mu t+\mu s) \\
&\geq Q(w,y,(\beta t+\mu s)) \\
&\geq Q(w,z,\beta t) * Q(z,y,\mu s) \\
&= Q_\beta(w,z,t) * Q_\mu(z,y,s)
\end{aligned}$$

$\Rightarrow Q_{\beta+\mu}(w,y,t+s) \geq Q_\beta(w,z,t) * Q_\mu(z,y,s)$.

Definition 3.2. Let $(W,Q_\beta,*)$ be a modular fuzzy metric-like space.

(a) Sequence $\{w_n\}_{n\in N}$ in W is called convergent to $w \in W$ if $\lim_{n\to\infty} Q_\beta(w_n,w,t) = Q_\beta(w,w,t)$ for all $\beta,t>0$.

(b) Sequence $\{w_n\}_{n\in N}$ in W is called a Cauchy sequence if $\lim_{n\to\infty} Q_\beta(w_{n+p},w_n,t)$ exists and is finite for all $\beta,t>0$, $p\geq 1$.

(c) A modular fuzzy metric-like space is called complete if every Cauchy sequence $\{w_n\}_{n\in N}$ in W converges to some $w \in W$ such that

$\lim_{n\to\infty} Q_\beta(w_n, w, t) = Q_\beta(w, w, t) = \lim_{n\to\infty} Q_\beta(w_{n+p}, w_n, t)$ for all $\beta, t > 0, p \geq 1$.

Remark 3.1. In a modular fuzzy metric-like space, the limit of a convergent sequence may not be unique. Consider Example 3.2. Define the sequence $\{w_n\}_{n\in N}$ in W by $\{w_n\} = \{1 + \frac{1}{n}\}$ for all $n \in IN$.

If $w \geq 2$, then

$\lim_{n\to\infty} Q_\beta(w_n, w, t) = \lim_{n\to\infty} \frac{\beta t}{\beta t + max\{w_n, w\}} = \lim_{n\to\infty} \frac{\beta t}{\beta t + w} = \frac{\beta t}{\beta t + w} = \frac{\beta t}{\beta t + max\{w, w\}} = Q_\beta(w, w, t)$.

Hence, the sequence $\{w_n\}_{n\in N}$ converges to all $w \in W$ with $w \geq 2$.

Definition 3.3. Let $(W, Q_\beta, *)$ be a modular fuzzy metric-like space. We will say that the mapping $T : W \to W$ is a modular fuzzy contractive mapping if there exists $k \in (0, 1)$ such that $\frac{1}{Q_\beta(T(w),T(y),t)} - 1 \leq k[\frac{1}{Q_\beta(w,y,t)} - 1]$ for all $\beta, t > 0$ and $w, y \in W$.

Theorem 3.1. Let $(W, Q_\beta, \star)$ be a complete modular fuzzy metric-like space and $T : W \to W$ be a modular fuzzy contractive mapping with contractive constant k. Then T has a unique fixed point $w \in W$ and $Q_\beta(w, w, t) = 1$ for all $\beta, t > 0$.

Proof. Let $(W, Q_\beta, \star)$ be a complete modular fuzzy metric-like space. For an arbitrary $w_0 \in W$, define a sequence $\{w_n\}$ in W by $w_n = T(w_{n-1})$ for all $n \in IN$. If $w_n = w_{n-1}$ for some $n \in IN$, then w_n is a fixed point of T since $w_n = T(w_{n-1}) = T(w_n)$. Therefore, we assume that $w_n \neq w_{n-1}$ for all $n \in IN$. For any $n \in IN$ and $\beta, t > 0$, from Definition 3.3 we obtain the following,

$\frac{1}{Q_\beta(w_n,w_{n+1},t)} - 1 = \frac{1}{Q_\beta(T(w_{n-1}),T(w_n),t)} - 1 \leq k[\frac{1}{Q_\beta(w_{n-1},w_n,t)} - 1] = \frac{k}{Q_\beta(w_{n-1},w_n,t)} - k.$

Let $Q_\beta(w_n, w_{n+1}, t) = Q_\beta^{(n)}(t)$ and $1 - k = h$.

Since $1 - k = h \in (0, 1)$, $\frac{1}{Q_\beta^{(n)}(t)} - 1 \leq \frac{k}{Q_\beta(w_{n-1},w_n,t)} - k = \frac{k}{Q_\beta^{(n-1)}(t)} - k.$

Then, $\frac{1}{Q_\beta^{(n)}(t)} \leq \frac{k}{Q_\beta^{(n-1)}(t)} + h.$

Let this approach be maintained; then

$\frac{1}{Q_\beta(w_{n-1},w_n,t)} - 1 = \frac{1}{Q_\beta(T(w_{n-2}),T(w_{n-1}),t)} - 1 \leq k[\frac{1}{Q_\beta(w_{n-2},w_{n-1},t)} - 1] = \frac{k}{Q_\beta(w_{n-2},w_{n-1},t)} - k$

$\Rightarrow \frac{1}{Q_\beta^{(n-1)}(t)} - 1 \leq \frac{k}{Q_\beta^{(n-2)}(t)} - k$

$\Rightarrow \frac{1}{Q_\beta^{(n-1)}(t)} \leq \frac{k}{Q_\beta^{(n-2)}(t)} - k + 1 = \frac{k}{Q_\beta^{(n-2)}(t)} + h.$

The others can be shown similarly. Then,

$$\begin{aligned}
\frac{1}{Q_\beta^{(n)}(t)} &\leq \frac{k}{Q_\beta^{(n-1)}(t)} + h \\
&= k\frac{1}{Q_\beta^{(n-1)}(t)}) + h \\
&\leq k[\frac{k}{Q_\beta^{(n-2)}(t)} + h] + h \\
&= k^2\frac{1}{Q_\beta^{(n-2)}(t)} + kh + h \\
&\leq k^2[\frac{k}{Q_\beta^{(n-3)}(t)} + h] + kh + h \\
&= k^3[\frac{k}{Q_\beta^{(n-3)}(t)} + h] + k^2h + kh + h
\end{aligned}$$

If we continue like this, we get that

$$\begin{aligned}
\frac{1}{Q_{\beta}^{(n)}(t)} &\leq k^n \frac{1}{Q_{\beta}^{(0)}(t)} + k^{n-1}h + k^{n-2}h + ... + kh + h \\
&= \frac{k^n}{Q_{\beta}^{(0)}(t)} + (k^{n-1} + k^{n-2} + ... + k + 1)h \\
&= \frac{k^n}{Q_{\beta}^{(0)}(t)} + \frac{h(1-k^n)}{1-k} \\
&= \frac{k^n}{Q_{\beta}^{(0)}(t)} + (1-k^n)
\end{aligned}$$

Hence, we have $\frac{1}{\frac{k^n}{Q_{\beta}^{(0)}(t)}+1-k^n} \leq Q_{\beta}^{(n)}(t)$, for all $\beta, t > 0, n \in IN$. (1)

Now, $p \geq 1, n \in IN$, for all $\beta, t > 0$,

$$\begin{aligned}
Q_{\beta}(w_n, w_{n+p}, t) &\geq Q_{\frac{\beta}{p}}(w_n, w_{n+1}, \tfrac{t}{p}) * Q_{\frac{p\beta-\beta}{p}}(w_{n+1}, w_{n+p}, \tfrac{pt-t}{p}) \\
&\geq Q_{\frac{\beta}{p}}(w_n, w_{n+1}, \tfrac{t}{p}) * Q_{\frac{\beta}{p}}(w_{n+1}, w_{n+2}, \tfrac{t}{p}) * Q_{\frac{p\beta-2\beta}{p}}(w_{n+2}, w_{n+p}, \tfrac{pt-2t}{p}) \\
&\geq Q_{\frac{\beta}{p}}(w_n, x_{n+1}, \tfrac{t}{p}) * Q_{\frac{\beta}{p}}(w_{n+1}, w_{n+2}, \tfrac{t}{p}) * Q_{\frac{\beta}{p}}(w_{n+2}, w_{n+3}, \tfrac{t}{p}) \\
&\qquad * Q_{\frac{p\beta-3\beta}{p}}(w_{n+3}, w_{n+p}, \tfrac{pt-3t}{p}) \\
&\geq Q_{\frac{\beta}{p}}(w_n, w_{n+1}, \tfrac{t}{p}) * Q_{\frac{\beta}{p}}(w_{n+1}, w_{n+2}, \tfrac{t}{p}) * Q_{\frac{\beta}{p}}(w_{n+2}, w_{n+3}, \tfrac{t}{p}) * ... \\
&\qquad * Q_{\frac{p\beta-(p-2)\beta}{p}}(w_{n+p-2}, w_{n+p}, \tfrac{pt-(p-2)t}{p}) \\
&\geq Q_{\frac{\beta}{p}}(w_n, w_{n+1}, \tfrac{t}{p}) * Q_{\frac{\beta}{p}}(w_{n+1}, w_{n+2}, \tfrac{t}{p}) * ... * Q_{\frac{\beta}{p}}(w_{n+p-2}, w_{n+p-1}, \tfrac{t}{p}) \\
&\qquad * Q_{\frac{p\beta-(p-2)\beta-\beta}{p}}(w_{n+p-1}, w_{n+p}, \tfrac{pt-(p-2)t-t}{p}) \\
&\geq Q_{\frac{\beta}{p}}(w_n, w_{n+1}, \tfrac{t}{p}) * Q_{\frac{\beta}{p}}(w_{n+1}, w_{n+2}, \tfrac{t}{p}) * ... * Q_{\frac{\beta}{p}}(w_{n+p-2}, w_{n+p-1}, \tfrac{t}{p}) \\
&\qquad * Q_{\frac{\beta}{p}}(w_{n+p-1}, w_{n+p}, \tfrac{t}{p}) \\
&= Q_{\frac{\beta}{p}}^{(n)}(\tfrac{t}{p}) * Q_{\frac{\beta}{p}}^{(n+1)}(\tfrac{t}{p}) * ... * Q_{\frac{\beta}{p}}^{(n+p-2)}(\tfrac{t}{p}) * Q_{\frac{\beta}{p}}^{(n+p-1)}(\tfrac{t}{p}).
\end{aligned}$$

From (1),

$$\begin{aligned}
Q_{\beta}(w_n, w_{n+p}, t) &\geq \frac{1}{\frac{k^n}{Q_{\frac{\beta}{p}}^{(0)}(\frac{t}{p})}+1-k^n} * \frac{1}{\frac{k^{n+1}}{Q_{\frac{\beta}{p}}^{(0)}(\frac{t}{p})}+1-k^{n+1}} * ... * \frac{1}{\frac{k^{n+p-1}}{Q_{\frac{\beta}{p}}^{(0)}(\frac{t}{p})}+1-k^{n+p-1}} \\
&\geq \frac{1}{\frac{k^n}{Q_{\frac{\beta}{p}}^{(0)}(\frac{t}{p})}+1} * \frac{1}{\frac{k^{n+1}}{Q_{\frac{\beta}{p}}^{(0)}(\frac{t}{p})}+1} * ... * \frac{1}{\frac{k^{n+p-1}}{Q_{\frac{\beta}{p}}^{(0)}(\frac{t}{p})}+1}
\end{aligned}$$

Taking the limit as $n \to \infty$, since $k \in (0,1)$, $\lim_{n\to\infty} k^n = 0$ and we get

$\lim_{n\to\infty} Q_{\beta}(w_{n+p}, w_n, t) \geq 1 * 1 * ... * 1 = 1$. Moreover, $1 \geq \lim_{n\to\infty} Q_{\beta}(w_{n+p,w_n,t}) \geq 1$ and then

$\lim_{n\to\infty} Q_{\beta}(w_{n+p}, x_n, t) = 1$, for all $\beta, t > 0$ and $p \geq 1$.

Hence $\{w_n\}$ is a Cauchy sequence on $(W, Q_{\beta}, *)$. Since $(W, Q_{\beta}, *)$ is complete, there exists $w \in W$ such that $\lim_{n\to\infty} Q_{\beta}(w_n, w, t) = Q_{\beta}(w, w, t) = \lim_{n\to\infty} Q_{\beta}(w_{n+p}, w_n, t)$. (2)

Now, let me show you that w is a fixed point of T:

We know that $\frac{1}{Q_{\beta}(T(w_n), T(w), t)} - 1 \leq k[\frac{1}{Q_{\beta}(w_n, w, t)} - 1] = \frac{k}{Q_{\beta}}(w_n, w, t) - k$,

$\frac{1}{\frac{k}{Q_{\beta}(w_n, w, t)}+1-k} \leq Q_{\beta}(T(w_n), T(w), t)$.

Using the above inequalities, we get

$$\begin{aligned}
Q_{\beta}(w, T(w), t) \geq Q_{\frac{\beta}{2}}(w, w_{n+1}, \tfrac{t}{2}) &= Q_{\frac{\beta}{2}}(w, w_{n+1}, \tfrac{t}{2}) * Q_{\frac{\beta}{2}}(T(w_n), T(w), \tfrac{t}{2}) \\
&\geq Q_{\frac{\beta}{2}}(w, w_{n+1}, \tfrac{t}{2}) * \frac{1}{\frac{k}{Q_{\frac{\beta}{2}}(w_n, w, \frac{t}{2})}+1-k}
\end{aligned}$$

Taking the limit as $n \to \infty$, from (2), $\lim_{n\to\infty} Q_{\beta}(w_n, w, t) = 1$.

Then, $1 \geq \lim_{n\to\infty} Q_{\beta}(w, T(w), t) \geq 1 * 1$ and $\lim_{n\to\infty} Q_{\beta}(w, T(w), t) = Q_{\beta}(w, T(w), t) = 1$, for all $\beta, t > 0$.

Then we have $w = T(w)$ from the condition MFML2. Thus w is a fixed point of T. Also, $Q_\beta(w, w, t) = 1$ since $T(w) = w$ and $Q_\beta(w, T(w), t) = 1$ for all $\beta, t > 0$.

To show the uniqueness of fixed point, let y be another fixed point of T such that $Q_{\beta_0}(w, y, t_0) < 1$ for some $\beta_0, t_0 > 0$.

It follows from Definition 3.3. that
$\frac{1}{Q_{\beta_0}(w,y,t_0)} - 1 = \frac{1}{Q_{\beta_0}(T(w),T(y),t_0)} - 1 \leq k[\frac{1}{Q_{\beta_0}(w,y,t_0)} - 1] < \frac{1}{Q_{\beta_0}(w,y,t_0)} - 1$, a contradiction.
Hence we must have $Q_\beta(w, y, t) = 1$ for all $\beta, t > 0$ and therefore $w = y$.

Example 3.3. Let $W = [0, 1]$ and $o * u = ou$. Define fuzzy set Q_β on $W^2 \times (0, \infty)$ by
$Q_\beta(w, y, t) = \frac{t}{t + \frac{max\{w,y\}}{\beta}}$ for all $w, y \in W$ and $\beta, t > 0$. Then $(W, Q_\beta, *)$ is a complete modular fuzzy metric-like space. If $T : W \to W$ is given by

$$T(w) = \begin{cases} 0, & w = 1 \\ \frac{w}{2}, & \text{otherwise} \end{cases}$$

then T is a modular fuzzy contractive mapping with modular contractive constant $k = \frac{1}{2}$. Note that all conditions of the Theorem 3.1. are satisfied. Thus T has a unique fixed point $0 \in W$ and $Q_\beta(0, 0, t) = 1$ for all $\beta, t > 0$.

4 Conclusion

In this study, we obtained fixed point results within the newly introduced modular fuzzy metric-like space and verified their applicability through illustrative examples. The extension of fixed point theory to this enriched mathematical structure provides a basis for improving more general contraction principles and for addressing a wider class of theoretical and applied problems. Thanks to the flexibility of this new space, future research may apply novel contractive mappings and analytical techniques capable of tackling problems in areas such as analysis, differential and integral equations, systems of linear equations, and graph theory. Overall, the results presented here are expected to contribute to the progress of the theoretical foundations and to encourage innovative applications across various scientific disciplines.

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