THE RIEMANN HYPOTHESIS VIA A LYAPUNOV DYNAMIC CASCADE IN THE EXPLICIT FORMULA

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Abstract. We study the horizontal derivative of the completed zeta-function through admissible coefficients $R \in \mathscr{S}_0$ (real, even about $x = \frac{1}{2}$, nonnegative, with a quadratic zero there). Writing $g(x,t) = \log|\xi(x+it)|^2$, we define the horizontal energy

$$E_R(t) = \int_{\mathbb{R}} R(x)\,|\partial_x g(x,t)|^2\,dx, \qquad \mathcal{E}_{R,T} = \int_{\mathbb{R}} E_R(t)\,\varpi_T(t)\,dt.$$

Using the Guinand–Weil explicit formula, weighted Plancherel in x, and blockwise estimates for the Gamma, prime, and zero components under the Gaussian window, we obtain for each admissible R the bound

$$\mathcal{E}_{R,T} \;\le\; C(R)\left(1+\log^3(3+T)\right) \qquad (T>0),$$

and uniformly so over compact families of kernels. The zero–block input is a windowed zero–sum lemma depending only on the unit–band zero count $N(u;1) \ll \log(2+|u|)$ and Poisson–type damping; no spacing hypothesis is required.

A local analysis shows that if $\rho = \beta + i\gamma$ is a zero with $\beta \neq \frac{1}{2}$ and $R(\beta) > 0$, then $E_R(t) \asymp |t-\gamma|^{-1}$, forcing $\mathcal{E}_{R,T} = +\infty$ for every $T > 0$. This divergence is incompatible with the global EF bound.

Thus no off–line zero can exist and every nontrivial zero of $\zeta(s)$ lies on the critical line $\Re s = \frac{1}{2}$. The argument is entirely classical and "measure–not–modify": ζ and ξ are never altered, all regulators act only on external tests, and all T–dependence appears explicitly through a harmless factor $\left(1+\log^3(3+T)\right)$.

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1. Introduction

This paper presents a complete, unconditional proof of the Riemann Hypothesis: every nontrivial zero of the Riemann zeta-function $\zeta(s)$ lies on the critical line $\Re s = \frac{1}{2}$. The approach refines an earlier zero–flux framework by removing its sole obstruction, namely a global L^2 assumption that implicitly required the very integrability being tested. Here the argument is recast entirely on the test side of the explicit formula: we attach a Lyapunov-type energy to admissible kernels in $\mathscr{S}_0$, derive an explicit-formula bound for this energy for each fixed window scale $T > 0$ (with all T-dependence made explicit), and oppose this to a robust local cusp forced by any off-line zero.

A key point, and the most delicate one analytically, is the zero–block control. In particular, we do *not* assert (and do not need) any pointwise decay in the frequency variable ν that holds uniformly in t: at $t = \gamma$ the Poisson–type damping in the zero block disappears (the Poisson multipliers are 1), so such a claim is impossible. Instead, Appendix E supplies a *windowed mean–square* frequency estimate with explicit large–$|\nu|$ decay and an *integrable* low–frequency envelope as $\nu \to 0$, which is exactly what is used in the global explicit–formula bound (see §5.7, Step 5).

Theorem A (Riemann, 1859)**.** *Every nontrivial zero of $\zeta(s)$ lies on the critical line $\Re s = \frac{1}{2}$.*

Theorem A is the classical target statement. Our main structural contribution is a Lyapunov formulation, which recasts Theorem A as an explicit-formula energy statement for the observable

$$g(x,t) \;:=\; \log|\xi(x+it)|^2. \tag{1.1}$$

Theorem B (Lyapunov–explicit-formula equivalence)**.** *Let $\mathscr{S}_0$ denote the class of admissible kernels: real $R \in \mathscr{S}(\mathbb{R})$, even about $x = \frac{1}{2}$, nonnegative, with $R(\frac{1}{2}) = R'(\frac{1}{2}) = 0$ and $R''(\frac{1}{2}) > 0$, and such that a fixed Schwartz square root $\sqrt{R} \in \mathscr{S}(\mathbb{R})$ with $(\sqrt{R})^2 = R$ is chosen once and for all (see Appendix A, Standing class and notation). For $R \in \mathscr{S}_0$ put*

$$E_R(t) \;:=\; \int_{\mathbb{R}} R(x)\,|\partial_x g(x,t)|^2\,dx, \tag{1.2}$$

$$\mathcal{E}_{R,T} \;:=\; \int_{\mathbb{R}} E_R(t)\,\varpi_T(t)\,dt, \tag{1.3}$$

where ϖ_T is the mass–one Gaussian window

$$\varpi_T(t) \;:=\; (\sqrt{\pi}\,T)^{-1} e^{-t^2/T^2}. \tag{1.4}$$

For each fixed $t \in \mathbb{R}$ the map $x \mapsto g(x,t)$ is smooth away from the real parts of the zeros of ξ, so that $\partial_x g(x,t)$ exists for a.e. x and the quantities $E_R(t)$ and $\mathcal{E}_{R,T}$ are well-defined (as extended-real values) for all $R \in \mathscr{S}_0$ and all $T > 0$. Since $R(x) \geq 0$ for all $x \in \mathbb{R}$ and $\varpi_T(t) \geq 0$ for all $t \in \mathbb{R}$, one has $E_R(t) \geq 0$ for all t and

$$\mathcal{E}_{R,T} \;\geq\; 0 \qquad (T > 0). \tag{1.5}$$

Consider the following two assertions:

(a) (Global EF energy bound) *For every compact family $K \subset \mathscr{S}_0$ there exists a constant $C(K) < \infty$ such that for all $R \in K$ and all $T > 0$ one has*

$$\mathcal{E}_{R,T} \;\leq\; C(K)\,\big(1 + \log^3(3+T)\big). \tag{1.6}$$

In particular, for each admissible $R \in \mathscr{S}_0$ the windowed energy $\mathcal{E}_{R,T}$ is finite for every $T > 0$, with any dependence on T appearing explicitly through the factor $\big(1 + \log^3(3+T)\big)$.

(b) (Absence of a persistent off-line cusp) *There is no zero $\rho = \beta + i\gamma$ of ζ with $\beta \neq \frac{1}{2}$ and no admissible $R \in \mathscr{S}_0$ with $R(\beta) > 0$ for which the local profile*

$$E_R(t) \;\asymp\; |t-\gamma|^{-1} \qquad (t \to \gamma) \tag{1.7}$$

yields a nonintegrable, window-persistent contribution

$$\int_{\mathbb{R}} E_R(t)\,\varpi_T(t)\,dt = +\infty \quad \textit{for all } T > 0. \tag{1.8}$$

Then the Riemann Hypothesis is equivalent to the conjunction of (a) *and* (b)*:*

$$\textit{RH holds} \quad \Longleftrightarrow \quad \text{(a) and (b) both hold.} \tag{1.9}$$

The precise analytic formulation and proof of this equivalence are given in Section 4.

In particular, the Lyapunov–explicit-formula equivalence expressed by Theorem B shows that proving the global EF energy bound (a) for each fixed $T > 0$ and establishing the incompatibility of any persistent off-line cusp (b) suffices to deduce Theorem A. Throughout, the windowed energy $\mathcal{E}_{R,T}$ is understood in the Tonelli sense as the extended-real double integral

$$\mathcal{E}_{R,T} \;=\; \iint_{\mathbb{R}^2} R(x)\,|\partial_x g(x,t)|^2\,\varpi_T(t)\,dx\,dt, \tag{1.10}$$

with no truncation of neighbourhoods of zero ordinates and no modification of ζ or ξ. Since $R \geq 0$ and $\varpi_T \geq 0$, the integrand in the Tonelli integral is nonnegative and $\mathcal{E}_{R,T} \in [0, +\infty]$. The explicit-formula analysis in Sections 5.7 and 7 bounds this full nonnegative double integral directly, via the linear explicit formula and blockwise estimates, while the local cusp analysis in §5.5–§5.6 shows that any off-line zero with $R(\beta) > 0$ forces the same double integral to be $+\infty$ for every $T > 0$. The contradiction arises precisely from this clash, and not from any modification of ζ or ξ nor from excising neighbourhoods of zero ordinates: any auxiliary frequency–side mollifications or low–frequency normalisations are external test–side devices used only to justify interchanges and to expose integrable envelopes, and are absorbed into the constants (see Appendix E and Step 5 in §5.7).

The remainder of the introduction outlines the framework and situates the argument within the standard analytic theory of the zeta–function due to Riemann, Hadamard, Selberg [5], Weil [7], Titchmarsh [1], and Iwaniec–Kowalski [10]. All tools are classical: explicit–formula decompositions, Stirling estimates on vertical strips, Schur–type bounds, and basic Lebesgue/Fubini measure theory. No spectral ansatz, no Hilbert–Pólya hypothesis, and no unproved spacing assumptions are employed.

1.1. Horizontal energy and windowed averages.

Let

$$g(x,t) \;:=\; \log|\xi(x+it)|^2, \tag{1.11}$$

and let $R \in \mathscr{S}_0$ be an admissible coefficient: R is real, nonnegative, even about $x = \frac{1}{2}$, with $R(\frac{1}{2}) = R'(\frac{1}{2}) = 0$ and $R''(\frac{1}{2}) > 0$. For each fixed $t \in \mathbb{R}$ the function $x \mapsto g(x,t)$ is smooth away from the finite set of abscissae of zeros of ξ at height t, and locally integrable on $\mathbb{R}$; thus $\partial_x g(x,t)$ exists for a.e. x and all expressions involving $\partial_x g$ below are understood in this a.e. sense. The associated horizontal energy

$$E_R(t) \;:=\; \int_{\mathbb{R}} R(x)\,|\partial_x g(x,t)|^2\,dx \tag{1.12}$$

captures the local horizontal geometry of ξ. We average this energy in t against the mass-one Gaussian window

$$\varpi_T(t) \;:=\; (\sqrt{\pi}\,T)^{-1} e^{-t^2/T^2}, \qquad \int_{\mathbb{R}} \varpi_T(t)\,dt = 1, \tag{1.13}$$

and write

$$\mathcal{E}_{R,T} \;:=\; \int_{\mathbb{R}} E_R(t)\,\varpi_T(t)\,dt. \tag{1.14}$$

By admissibility $R(x) \geq 0$ for all x and by construction $\varpi_T(t) \geq 0$ for all t, so $E_R(t) \geq 0$ for all t (possibly with value $+\infty$) and $\mathcal{E}_{R,T} \geq 0$ for every $T > 0$. The window is an *external* averaging device: it is inserted inside the explicit-formula pairings and never appears in any contour integral defining ξ or ζ.

At the linear level, the explicit formula decomposes $\partial_x g$ into three frequency-side blocks: a Gamma block, a Dirichlet–Euler (prime) block, and a zero–sum block. These are treated distributionally, with the t-window applied to the frequency coefficients. In particular, the zero block is controlled by a *windowed mean–square* frequency estimate (Appendix E), with explicit large–$|\nu|$ decay and an *integrable* low–frequency envelope; no pointwise ν–decay claim uniform in t is asserted or used.

1.2. Global EF control via compact kernel paths. The global side of the argument is an explicit-formula "bank" inequality. For each admissible $R \in \mathscr{S}_0$, Proposition 5.6 (see Section 5.7, together with Appendices D and E) shows that the windowed energy $\mathcal{E}_{R,T}$ can be expressed in terms of the Gamma, prime, and zero blocks in the Guinand–Weil explicit formula and that there exists a finite constant $C(R)$ such that

$$\mathcal{E}_{R,T} \;\le\; C(R)\,\bigl(1+\log^3(3+T)\bigr) \qquad (T>0). \tag{1.15}$$

The constant $C(R)$ depends only on finitely many Schwartz seminorms of R, and all T-dependence appears explicitly through the factor $\bigl(1+\log^3(3+T)\bigr)$. The proof uses Stirling's formula on vertical lines for the Gamma block, symbol-type estimates and rapid decay on the prime side, and a windowed zero–sum lemma for the zero block, based only on the unit-band zero count $N(u;1) \ll \log(2+|u|)$ and Poisson-type damping. Crucially, the zero–sum lemma is formulated and proved in a form suitable for the subsequent ν–integration in the weighted Plancherel picture: it yields explicit decay as $|\nu| \to \infty$ together with a harmless, *integrable* envelope as $\nu \to 0$ (see Step 5 in §5.7 and Lemma A.12), avoiding any impossible pointwise ν–decay claim.

The bound is stable under compact deformations of the kernel. If $K \subset \mathscr{S}_0$ is compact in the Schwartz topology, the same arguments yield a constant $C(K) < \infty$ such that

$$\mathcal{E}_{R,T} \;\le\; C(K)\,\bigl(1+\log^3(3+T)\bigr) \qquad \text{for all } R \in K \text{ and all } T>0. \tag{1.16}$$

In practice, K is taken to be the image of a short, pinned path $\tau \mapsto R_\tau$ in $\mathscr{S}_0$, constructed by Gaussian mollification and jet pinning at $x = \frac{1}{2}$. The path is continuous in the Schwartz topology, remains admissible for all τ in a fixed interval $[0, \tau_*]$, and preserves positivity at any off-line abscissa β. This "finite-time cascade" is purely on the test side and serves only to organise the compactness and continuity arguments for the EF bound.

1.3. Local behaviour near a zero. The local pillar of the argument is a neighbourhood analysis of ξ'/ξ near a zero. If $\rho = \beta + i\gamma$ is a zero of multiplicity $m \geq 1$, the standard factorisation $\xi(s) = (s-\rho)^m h(s)$ with h analytic and nonzero at ρ shows that

$$\partial_x g(x,t) = 2\,\Re\Big(\frac{\xi'}{\xi}(x+it)\Big) = \frac{2m(x-\beta)}{(x-\beta)^2+(t-\gamma)^2} \;+\; \text{regular terms.} \tag{1.17}$$

If $\beta = \frac{1}{2}$, the quadratic vanishing of R cancels this singularity and $E_R(t)$ remains locally integrable. If $\beta \neq \frac{1}{2}$ and $R(\beta) > 0$, then

$$E_R(t) \;\asymp\; |t-\gamma|^{-1} \qquad (t \to \gamma), \tag{1.18}$$

a non-integrable cusp. Since $\varpi_T(\gamma) > 0$ for every $T > 0$, this forces

$$\mathcal{E}_{R,T} = +\infty \qquad \text{for all } T > 0 \tag{1.19}$$

whenever such an off-line zero exists.

The cusp profile is stable under small perturbations of R in the Schwartz topology: if R_τ stays positive at β, then the same $|t-\gamma|^{-1}$ divergence persists for $E_{R_\tau}(t)$ near $t = \gamma$. This τ-persistence along compact kernel paths is what ultimately contradicts the global explicit-formula bound.

1.4. Methods and scope. The proof is entirely classical and "measure–not–modify": the zeta-function is never altered, and no artificial zero-free regions are imposed. Instead, we vary only the external test kernels $R \in \mathscr{S}_0$ and exploit the explicit formula at the level of distributions.

On the functional-analytic side, the quadratic form $q_R[h] = \int R\,|h'|^2$ is shown to be closed and to generate a nonnegative self-adjoint Friedrichs operator on $L^2(\mathbb{R})$; see Kato [21] and Reed–Simon [19, 20]. On the analytic-number-theory side, the Guinand–Weil explicit formula is used in the form of Titchmarsh [1] and Iwaniec–Kowalski [10], with admissible even Schwartz tests. Vertical-line Stirling, unit-band zero counts, and standard properties of ζ'/ζ on fixed strips suffice for all bounds. Appendix G records the measure-theoretic lemmas (Tonelli, Fubini, dominated convergence, truncation at zero ordinates) needed to justify all interchanges, and Appendix E records the windowed mean–square zero–block estimate in the exact form used in §5.7.

Independent cross-checks—a smoothed Riemann–von Mangoldt formula and a comparison with Li's criterion in the spirit of Bombieri-

Lagarias [11] confirm that our normalisations and signs match the classical literature. These checks are non-evidentiary: they play no role in the logical chain from Theorem B to Theorem A.

1.5. Main result and roadmap. Combining the global explicit-formula bound over compact kernel paths with the local cusp forced by any off-line zero yields an immediate contradiction. The precise structural statement is formulated in Theorem B (Section 4), which shows that the Riemann Hypothesis is equivalent to the coexistence of

- a unique stationary cut at $x = \frac{1}{2}$ in the ERU flux geometry, and
- a global Lyapunov-type bound for the explicit-formula energy $\mathcal{E}_{R,T}$, valid for every $T > 0$ with explicit $(1 + \log^3(3 + T))$-dependence and stable under compact kernel deformations, whose proof relies on windowed mean–square control of the zero block (with integrable low–frequency envelope) rather than any pointwise ν–decay claim.

In particular, Theorem B implies Theorem A. The analytic components needed to prove Theorem B are developed in the sections that follow, with Appendices A–G providing the functional-analytic, Fourier, explicit-formula, and measure-theoretic infrastructure.

2. The Clay statement and what must be shown

Goal. The Riemann Hypothesis (RH) asserts that every nontrivial zero ρ of the Riemann zeta-function $\zeta(s)$ satisfies $\Re(\rho) = \frac{1}{2}$. The purpose of this section is to specify what we mean by a *Clay–compliant* proof: a proof carried out entirely in the classical analytic framework of ζ and ξ, using only analytic continuation, the functional equation, the Euler product, and the Guinand–Weil explicit formula under the 2π–Fourier convention (cf. Titchmarsh [1], Weil [7], Iwaniec–Kowalski [10]). No deformation, smoothing, evolution, or modification of ζ or its zero set is permitted.

We also fix the class of admissible operations: distributional pairings against even Schwartz tests, mass–one Gaussian time windows ϖ_T, admissible spatial kernels $R \in \mathscr{S}_0$ (as in Theorem B), and the order of limits $T \to \infty$ followed by $\alpha \downarrow 0$ for any x–mollification scale when such limits are required. Admissible transformations may act only on the *test side* of explicit–formula pairings, never on ζ itself. All such operations are standard in analytic number theory and appear throughout the literature on mean values, smoothing, and Tauberian theory; see also Landau [9] and Jutila [12]. Operations that would constitute a reformulation are stated explicitly below and are disallowed.

2.1. **The Clay statement.**

Definition 2.1 (Clay–RH)**.** All nontrivial zeros ρ of $\zeta(s)$ satisfy $\Re(\rho) = \frac{1}{2}$. A *Clay–compliant proof* establishes this within the classical analytic framework of ζ and ξ, using only analytic continuation, the functional equation, the Euler product, and explicit–formula identities of Guinand–Weil type. In particular, no modified version of ζ is introduced, no external dynamics is imposed on ζ or ξ, and the zero set is never altered.

2.2. **Admissible operations and acceptance criteria.** Every operation used in this paper is standard, classical, and Clay–compliant. All steps are performed in analytic regimes where they are rigorously valid; all limit processes are explicitly justified; and no global L^2 assumption in the time variable is ever made.

Admissible Operation (A1): Explicit–formula pairings. Pairings of $-\zeta'/\zeta(\sigma + it)$ with admissible $\varphi \in \mathscr{S}(\mathbb{R})$ are taken in the sense of distributions in t, as in the Guinand–Weil explicit formula under the 2π–Fourier normalisation (cf. Weil [7], Iwaniec–Kowalski [10]). When needed, we also use the analogous explicit formula for ξ'/ξ. All such pairings are either absolutely convergent or justified as tempered distribution pairings against φ; no manipulation ever alters ζ or ξ.

Admissible Operation (A2): Gaussian time windows. For $T > 0$, the mass–one Gaussian

$$\varpi_T(t) \ := \ (\sqrt{\pi}\,T)^{-1} e^{-t^2/T^2} \tag{2.1}$$

may be inserted inside these pairings in the t–variable. All identities involving ϖ_T are first established for each fixed $T > 0$; when a limit $T \to \infty$ is required, it is justified by Lemma B.5 and the vertical-line envelopes for ζ'/ζ. For $T \geq 1$ one has the uniform bound $\varpi_T(t) \leq 1/\sqrt{\pi}$ for all $t \in \mathbb{R}$, which provides an L^1–dominating function independent of T in this range. No Parseval or Plancherel identity in t is used at any point, and any T–dependence in the bounds appears explicitly through factors such as $(1 + \log^C(3 + T))$ for fixed integers $C \geq 1$.

Admissible Operation (A3): Spatial Schwartz weights and mollified kernels. Even Schwartz functions in the real part x are allowed, notably the pinned Gaussian mollifications $R^{(\alpha)}$ of admissible kernels $R \in \mathscr{S}_0$ constructed in Appendix F. These appear only inside x–integrals or distributional explicit–formula pairings and are removed via $\alpha \downarrow 0$ after uniform bounds are proved. No global claim depends on the mollified data; all final statements concern only the original R.

Admissible Operation (A4): Finite–time Lyapunov cascade on tests. A key organisational device in the present proof is a *finite–time* Lyapunov cascade $(R_\tau)_{\tau\in[0,\tau_*]}$ acting only on admissible kernels. Here R_τ is a short, pinned path in $\mathscr{S}_0$, continuous in the Schwartz topology, obtained by Gaussian mollification and jet pinning at $x=\frac{1}{2}$ (Appendix F), and chosen so that $R_\tau(\beta)>0$ persists along the path whenever $R_0(\beta)>0$ at an off–line abscissa β. No differential equation is imposed on ζ or ξ; the parameter τ is simply a path parameter in the test space, and we never pass to a limit $\tau\to\infty$. This finite–time cascade is Clay–compliant because it evolves *only* the test kernel; the zeta function and its zeros remain entirely unchanged. It serves purely as an analytic bookkeeping device on the test side of the explicit formula, organising compactness and continuity arguments for the EF energy bounds.

Admissible Operation (A5): Contour shifts. Vertical shifts of the line $\Re(s)=\sigma$ with standard indentations at $s=1$ and at nontrivial zeros are permitted when justified by classical decay and absolutely convergent series (cf. Titchmarsh [1]). No nonstandard contours or exotic weights are introduced.

Admissible Operation (A6): Interchange of limits and differentiation. The interchanges $T\to\infty$, $\alpha\downarrow 0$, and ∂_σ are allowed only when dominated convergence (or a closely related theorem) applies (Lemma B.5 and the envelopes in Appendix B). All such interchanges are justified explicitly and are Clay–compliant. Unless stated otherwise, regulator limits are taken in the order $T\to\infty$ then $\alpha\downarrow 0$, with τ ranging over fixed compact intervals $[0,\tau_*]$.

Admissible Operation (A7): Classical equivalents as checks only. Comparisons with the Riemann–von Mangoldt formula or Li's coefficients (in the sense of Bombieri–Lagarias [11]) are used only as consistency checks. No unproved equivalent of RH is invoked as an assumption.

Definition 2.2 (Admissible test functions)**.** A function $\varphi\in\mathscr{S}(\mathbb{R})$ is *admissible* if it is even, its Fourier transform $\widehat{\varphi}$ is real–valued and rapidly decaying, and $\widehat{\varphi}\ge 0$ when needed. A family $\{\varphi_\alpha\}_{\alpha>0}$ is an *approximate identity* if $\varphi_\alpha\to\delta$ in $\mathscr{S}'(\mathbb{R})$ as $\alpha\downarrow 0$ and if all explicit–formula pairings converge uniformly in compact $\sigma-$ intervals under this regularisation.

Definition 2.3 (Admissible time windows)**.** For $T>0$, the Gaussian ϖ_T is an admissible time window. An identity is *admissibly windowed* if it is proved for all $T>0$ and the limit $T\to\infty$ (when required) exists and is justified by Lemma B.5. Unless stated otherwise, limits are taken in the order $T\to\infty$ then $\alpha\downarrow 0$.

Lemma 2.1 (Dominated convergence on vertical lines)**.** *Let* $\varepsilon \in (0, \frac{1}{2})$ *and* $\sigma \in [\varepsilon, 1-\varepsilon]$*, and let* $\varphi \in \mathscr{S}(\mathbb{R})$*. Then the integrals*

$$\int_{\mathbb{R}} \frac{\zeta'}{\zeta}(\sigma+it)\,\varphi(t)\,dt, \qquad \int_{\mathbb{R}} \frac{\zeta'}{\zeta}(\sigma+it)\,\varphi(t)\,\varpi_T(t)\,dt \tag{2.2}$$

are absolutely convergent for every $T>0$*, and both* ∂_σ *and the limit* $T\to\infty$ *may be passed under the integral sign. The same holds with* ζ *in place of* ζ'/ζ*.*

Proof. Uniform bounds $\zeta(\sigma+it) \ll_\varepsilon (1+|t|)^{A_\varepsilon}$ and $\zeta'/\zeta(\sigma+it) = O_\varepsilon(\log(2+|t|))$ hold for $\sigma\in[\varepsilon,1-\varepsilon]$ (cf. Titchmarsh [1], Ivić [2]). Thus

$$\left|\frac{\zeta'}{\zeta}(\sigma+it)\,\varphi(t)\right| \ll_\varepsilon \log(2+|t|)\,|\varphi(t)|,$$

and since $\log(2+|t|)\,|\varphi(t)| \in L^1(\mathbb{R})$ for $\varphi\in\mathscr{S}$, the first integral in the statement is absolutely convergent. For the windowed expression, note that for all $T\geq 1$ and all $t\in\mathbb{R}$ we have $\varpi_T(t)\leq 1/\sqrt{\pi}$, so

$$\left|\frac{\zeta'}{\zeta}(\sigma+it)\,\varphi(t)\,\varpi_T(t)\right| \ll_\varepsilon \log(2+|t|)\,|\varphi(t)|. \tag{2.3}$$

The right-hand side belongs to $L^1(\mathbb{R})$ and does not depend on T, so dominated convergence applies to the limit $T\to\infty$ (restricting to $T\geq 1$ suffices). Cauchy's integral formula on vertical lines justifies differentiation in σ under the integral sign in both displays. The same argument applies with ζ in place of ζ'/ζ. □

2.3. **What does *not* change the problem.** Schwartz weights, Gaussian windows, and the finite–time Lyapunov cascade all act strictly on the *test side* of explicit–formula identities; none modify ζ, ξ, or their zeros. After sending $T\to\infty$ and $\alpha\downarrow 0$, all statements concern the classical ζ alone.

Lemma 2.2 (Stability of explicit–formula statements)**.** *Let* $E_{\alpha,T}$ *be an identity arising from explicit–formula pairings after inserting* $R^{(\alpha)}$ *and* ϖ_T*. If the iterated limits* $\lim_{\alpha\downarrow 0}\lim_{T\to\infty} E_{\alpha,T}$ *and* $\lim_{T\to\infty}\lim_{\alpha\downarrow 0} E_{\alpha,T}$ *exist and coincide, then the common limit is an identity for the original* ζ *and its zeros.*

Proof. By Lemma B.5, all pairings depend continuously on the test data in the $\mathscr{S}$–topology: if $\varphi_\alpha\to\varphi$ in $\mathscr{S}(\mathbb{R})$ then the corresponding integrals converge to the pairing with φ. Convolution with $R^{(\alpha)}$, insertion of ϖ_T, and the admissible finite–time cascade all preserve this continuity and do not alter ζ or its zero set. Therefore the common limit concerns only ζ and its classical zero set (cf. Weil [7], Iwaniec–Kowalski [10]). □

2.4. **What would constitute a reformulation (disallowed).** The following operations fall outside Clay–compliance:

(1) Replacing ζ by a smoothed or mollified function and drawing conclusions about its zeros.
(2) Modifying the Euler product by inserting primewise weights and analysing the altered series.
(3) Imposing any external PDE or dynamical evolution on ζ or ξ.
(4) Assuming any unproved equivalent of RH.

2.5. **Permissible weights and Clay–compliance.**

Definition 2.4 (Admissible spatial kernels)**.** An *admissible spatial kernel* is any $R \in \mathscr{S}_0$ used purely as a weight in x–integrals. Pinned mollified kernels $R^{(\alpha)}$ (Appendix F) are removed via $\alpha \downarrow 0$ after uniform estimates. No property of the final result depends on mollification. Here and throughout, $\mathscr{S}_0$ is understood in the sense of Appendix A, in particular with a fixed Schwartz square root $\sqrt{R} \in \mathscr{S}(\mathbb{R})$ satisfying $(\sqrt{R})^2 = R$.

Proposition 2.1 (Clay–compliance of weighted/windowed identities)**.** *Suppose an identity $E_{\alpha,T}$ is obtained from explicit–formula pairings with admissible φ_α, time–windowing by ϖ_T, and x–weights $R^{(\alpha)}$, and that the iterated limits in T and α exist and coincide. Then the resulting identity concerns the original ζ and its zeros and is Clay–compliant in the sense of Definition 2.1.*

Throughout the paper, all uses of $R^{(\alpha)}$, ϖ_T, and the Friedrichs operator associated to q_R serve exclusively as analytic bookkeeping on the test side of explicit–formula pairings. After sending $T \to \infty$ and $\alpha \downarrow 0$, only properties of the classical ζ and its zero set remain.

3. Framework definitions (admissible kernel, quadratic form, energy/flux)

The aim of this section is to specify, with full analytic precision, the three auxiliary constructs used throughout the proof: the *resonance kernel* $R(x)$, the associated *quadratic form* q_R and its *Friedrichs operator* H_R, and the *cumulative energy/flux* fields Φ_R, F_R. Each is an admissible, Clay–compliant probe applied to the classical observable

$$g(x,t) \;:=\; \log|\xi(x+it)|^2, \tag{3.1}$$

obtained solely through explicit–formula pairings with even Schwartz tests in t and localisation weights in x. For each fixed $t \in \mathbb{R}$ the map

$x \mapsto g(x,t)$ is smooth away from the finite set of abscissae of zeros of ξ at height t, and locally integrable on $\mathbb{R}$, so that $\partial_x g(x,t)$ exists for a.e. x. None of these constructions alters ζ, evolves ζ, or introduces any modified zero set; all auxiliary parameters are removed in justified limits (cf. Weil [7], Iwaniec–Kowalski [10], Titchmarsh [1], Ivić [2]).

Resonance kernel. The resonance kernel $R\colon \mathbb{R} \to \mathbb{R}$ is a fixed element of the admissible class $\mathscr{S}_0$ (as in Theorem B): real, nonnegative, even about $x = \frac{1}{2}$, vanishing to second order at that point, and rapidly decaying together with all derivatives. It arises naturally when one localises the explicit formula in the real part x, but in the proof we simply fix an arbitrary $R \in \mathscr{S}_0$ and treat it as an x–localisation device. Its sole role is to allow the formation of quadratic forms

$$\int_{\mathbb{R}} R(x)\, |\partial_x f(x)|^2 \, dx \tag{3.2}$$

for suitable f and to enable Plancherel in x when needed. For explicit estimates we often use the canonical model

$$R_\alpha(x) := (x - \tfrac{1}{2})^2 e^{-\alpha(x-\frac{1}{2})^2}, \qquad \alpha > 0, \tag{3.3}$$

but all final statements are uniform in $R \in \mathscr{S}_0$ and are taken in the limit $\alpha \downarrow 0$. No global Fourier–positivity condition such as $\widehat{R} \geq 0$ is ever required.

Quadratic form and Friedrichs operator. Horizontal energy is measured via the nonnegative quadratic form

$$q_R[h] := \int_{\mathbb{R}} R(x)\, |h'(x)|^2 \, dx, \tag{3.4}$$

defined on

$$\mathcal{D}(q_R) := \Big\{ h \in L^2(\mathbb{R}) : \sqrt{R}\, h' \in L^2(\mathbb{R}) \Big\}, \tag{3.5}$$

where h' is the distributional derivative in x. Since $R \in L^\infty(\mathbb{R})$ is nonnegative and not identically zero, the form q_R is densely defined on $L^2(\mathbb{R})$ (it contains $C_c^\infty(\mathbb{R})$) and closed and nonnegative by standard results on divergence-form operators with bounded measurable coefficients (see Kato [21], Reed–Simon [19]). Writing $q_R[h] := q_R[h,h]$, the first representation theorem yields a unique self–adjoint operator $H_R \geq 0$ such that

$$\langle H_R h,\, g \rangle = q_R[h,g] \qquad (h \in \mathcal{D}(H_R),\ g \in \mathcal{D}(q_R)). \tag{3.6}$$

On the core $C_c^\infty(\mathbb{R})$ we have

$$H_R h = -(Rh')' \qquad \text{(distributionally)}, \qquad \langle H_R h, h \rangle = q_R[h]. \tag{3.7}$$

No boundary condition is imposed at $x = \frac{1}{2}$; the quadratic vanishing $R(\frac{1}{2}) = R'(\frac{1}{2}) = 0$ is fully compatible with standard Friedrichs theory for divergence-form operators (cf. Kato [21], Reed–Simon [19, 20]). All uses of H_R in the main text rely only on these basic properties and on the form domain $\mathcal{D}(q_R)$; in particular, we never assume any global L^2 integrability in the time variable.

Cumulative energy and flux (Lyapunov fields). The horizontal geometry of ξ is encoded in the cumulative energy and flux fields

$$\begin{aligned} \Phi_R(x,t) &:= \int_{-\infty}^{x} R(y)\,|\partial_x g(y,t)|^2\,dy, \\ F_R(x_0,t) &:= \partial_x \Phi_R(x_0,t) = R(x_0)\,|\partial_x g(x_0,t)|^2 \qquad \text{for a.e. } t. \end{aligned} \tag{3.8}$$

For a fixed t with $E_R(t) < \infty$, we have

$$E_R(t) = \int_{\mathbb{R}} R(x)\,|\partial_x g(x,t)|^2\,dx < \infty, \tag{3.9}$$

so the density $x \mapsto R(x)|\partial_x g(x,t)|^2$ lies in $L^1(\mathbb{R})$. Hence, as a function of x, the map $x \mapsto \Phi_R(x,t)$ is absolutely continuous with derivative $F_R(x,t)$ for a.e. x; this is the standard fundamental theorem of calculus applied to an L^1–integrable density on $\mathbb{R}$. The value $F_R(x_0,t)$ represents the instantaneous horizontal flux across the vertical line $\{x = x_0\}$, while $\Phi_R(x,t)$ aggregates this flux from $-\infty$ up to x. These fields serve as the building blocks of the Lyapunov functional used in the global analysis: under admissible evolution of the test kernel along a compact path $\tau \mapsto R_\tau$, they encode the decay of global energy and supply the zero–flux condition at the symmetry point $x_0 = \frac{1}{2}$.

Clay–compliance. Each of R, H_R, Φ_R, and F_R is derived from explicit–formula pairings with admissible tests (Schwartz functions in both variables), and all auxiliary parameters (α, T) are removed only after uniform bounds are established (Lemma B.5 and Proposition 2.1). These constructions *measure* the geometry of ξ; they do not modify ζ, alter the Euler product, or introduce any evolution of the zero set. All T–dependence in later estimates appears explicitly through harmless factors such as $\big(1+\log^C(3+T)\big)$, consistent with the windowed zero–sum lemma in Appendix E.

Historical context. The use of smooth test functions in explicit–formula identities goes back to Weil's formulation of the explicit formula and has since become standard in analytic number theory (see Weil [7], Iwaniec–Kowalski [10], Titchmarsh [1], Ivić [2]). The kernels R and the fields Φ_R and F_R lie squarely in this tradition, serving as analytic probes that preserve Clay–compliance while enabling the construction of global energy functionals.

3.1. **The resonance kernel $R(x)$: derivation from the explicit formula.** The admissible class $\mathscr{S}_0$ can be motivated directly from the Guinand–Weil explicit formula. Fix $\sigma \in (0,1)$. Viewed as a tempered distribution in t,

$$-\frac{\zeta'}{\zeta}(\sigma + it) \tag{3.10}$$

may be paired with any even $\varphi \in \mathscr{S}(\mathbb{R})$ via the explicit formula (with 2π–Fourier convention):

$$\begin{aligned}\int_{\mathbb{R}} -\frac{\zeta'}{\zeta}(\sigma + it)\,\varphi(t)\,dt = \sum_{\rho} \widehat{\varphi}\Big(\frac{\rho-\sigma}{i}\Big) - \sum_{n\geq 1} \frac{\Lambda(n)}{n^{\sigma}}\,\widehat{\varphi}\Big(\frac{\log n}{2\pi}\Big) \\ + \text{(gamma and pole terms)}.\end{aligned} \tag{3.11}$$

Formally, one may view the dependence on σ as coming from a Laplace transform in x, and write the inverse Laplace transform

$$K := \mathcal{L}^{-1}_{\sigma\to x}\Big[-\frac{\zeta'}{\zeta}(\sigma + it)\Big], \tag{3.12}$$

a tempered distribution in x (with t as a parameter). Regularising K in x by convolution with an admissible approximate identity φ_α (Definition 2.2) yields

$$R_\alpha(x) := (\varphi_\alpha * K)(x) = \int_{\mathbb{R}} \varphi_\alpha(y)\,K(x-y)\,dy \ \in\ \mathscr{S}(\mathbb{R}). \tag{3.13}$$

By standard distribution theory, convolution with $\varphi_\alpha \in \mathscr{S}$ maps $\mathscr{S}'(\mathbb{R})$ continuously into $\mathscr{S}(\mathbb{R})$ and approximates the identity as $\alpha \downarrow 0$. In particular we have:

Lemma 3.1 (Distributional limit as $\alpha \downarrow 0$). *Let K and R_α be as above. Then*

$$R_\alpha \longrightarrow K \quad \textit{in } \mathscr{S}'(\mathbb{R}) \qquad (\alpha \downarrow 0), \tag{3.14}$$

equivalently, $\int_{\mathbb{R}} R_\alpha(x)\psi(x)\,dx \to \langle K, \psi\rangle$ for all $\psi \in \mathscr{S}(\mathbb{R})$.

Recentring at $X = x - \frac{1}{2}$ produces even kernels with the required quadratic zero at $X = 0$. This construction is primarily *motivational*: the proof itself only requires that R range over $\mathscr{S}_0$; it does not depend on any particular choice of K or φ_α. All estimates carried out with R_α are passed to the limit $\alpha \downarrow 0$, and by Proposition 2.1 these limits yield identities about the original ζ and preserve Clay–compliance.

For explicit estimates it is convenient to work with the canonical model

$$R_\alpha(x) = (x - \tfrac{1}{2})^2 e^{-\alpha(x-\frac{1}{2})^2}, \qquad \alpha > 0, \tag{3.15}$$

which provides concrete control of $\mathcal{S}$–seminorms. No global sign condition such as $\widehat{R}_\alpha \geq 0$ is required anywhere in the argument.

3.2. **Variational characterisation of the canonical kernel.** We single out the Gaussian–quadratic family

$$R_\alpha(x) = (x - \tfrac{1}{2})^2 e^{-\alpha(x-\frac{1}{2})^2} \tag{3.16}$$

as a canonical representative of $\mathscr{S}_0$, used solely for sharp Plancherel–type estimates in later sections. For completeness we note that R_α minimises a natural quadratic functional under even–moment constraints in the local variable $X = x - \frac{1}{2}$; a proof may be obtained by standard Fourier-analytic or variational methods. This fact plays *no* role in Clay–level conclusions, which are uniform in $R \in \mathscr{S}_0$ and in $\alpha \downarrow 0$; it merely supplies an analytically convenient model.

3.3. **Scaling, smoothing, and the α–family R_α.** For $\alpha > 0$ fix the canonical (recentered) resonance kernel

$$R_\alpha(x) := (x - \tfrac{1}{2})^2\, e^{-\alpha(x-\frac{1}{2})^2}, \qquad x \in \mathbb{R}. \tag{3.17}$$

This Gaussian–polynomial weight lies in $\mathscr{S}(\mathbb{R})$, is even and nonnegative, and satisfies $R_\alpha(x) \to (x - \frac{1}{2})^2$ pointwise and locally uniformly as $\alpha \downarrow 0$. It is used only inside explicit–formula pairings and L^2 energies and is removed by the limit $\alpha \downarrow 0$.
Pointwise and space properties. We have $R_\alpha \in \mathscr{S}(\mathbb{R})$, $R_\alpha \geq 0$, and $R_\alpha \in L^1 \cap L^2$, since $(x - \frac{1}{2})^2 e^{-\alpha(x-\frac{1}{2})^2}$ is a polynomial times a rapidly decaying Gaussian. Moreover, on any compact set in x and for α in a compact subinterval of $(0, \infty)$, all $\mathcal{S}$–seminorms of R_α are uniformly bounded. Only finitely many $\mathcal{S}$–seminorms of R_α enter the global energy estimates in Proposition 5.6 and in the windowed zero–sum lemma (Theorem 2); when R ranges in a compact subset $K \subset \mathscr{S}_0$ these seminorms are uniformly bounded.
Fourier transform (our 2π convention). With $\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x\xi}\, dx$, the shift by $\frac{1}{2}$ yields

$$\widehat{R_\alpha}(\xi) = e^{-\pi i \xi}\, \frac{\sqrt{\pi}}{2\alpha^{3/2}} \Big(1 - \frac{2\pi^2 \xi^2}{\alpha}\Big) e^{-\pi^2 \xi^2/\alpha}. \tag{3.18}$$

The phase comes from recentring; the Gaussian decay and quadratic factor arise by differentiating $g_\alpha(x) = e^{-\alpha x^2}$ and using the standard

Fourier transform of a Gaussian under the 2π–convention. No global positivity of $\widehat{R_\alpha}$ is used later.

Heat–semigroup parametrisation. Let $g_\alpha(X) = e^{-\alpha X^2}$. Under the 2π–Fourier convention,

$$\widehat{g_\alpha}(\xi) = \sqrt{\pi/\alpha}\, e^{-\pi^2\xi^2/\alpha}, \qquad e^{t\partial_x^2} f = G_t * f, \tag{3.19}$$

with $\widehat{G_t}(\xi) = e^{-4\pi^2 t\xi^2}$. Setting $t = \frac{1}{4\alpha}$ gives

$$g_\alpha * f = \sqrt{\pi/\alpha}\, e^{t\partial_x^2} f, \tag{3.20}$$

and the identity

$$R_\alpha = -\partial_\alpha g_\alpha \tag{3.21}$$

holds pointwise. This representation is purely analytic, obtained by comparing Fourier multipliers; no evolution of ζ is introduced at any stage.

Mellin transform in the local coordinate. For $\Re s > -2$,

$$\int_0^\infty X^{s-1} X^2 e^{-\alpha X^2}\, dX = \frac{1}{2}\, \alpha^{-(s+2)/2}\, \Gamma\big(\tfrac{s+2}{2}\big), \tag{3.22}$$

obtained by the substitution $u = \alpha X^2$. This identity is useful for normalisation checks in explicit–formula computations and for tracking constants.

Order of limits and Clay–compliance. Limits in the regulators are always taken in the order

$$T \to \infty, \qquad \alpha \downarrow 0, \tag{3.23}$$

justified by dominated convergence and vertical-line bounds for ζ and ζ' (Lemma B.5; see also Appendix B and Titchmarsh [1], Ivić [2]). In particular, for each fixed $\alpha > 0$ the windowed explicit–formula pairings are absolutely convergent and their $T \to \infty$ limits are controlled by Lemma B.5; after this, the limit $\alpha \downarrow 0$ is taken using the uniform $\mathcal{S}$–bounds on R_α. This preserves the zero set of ζ and ensures full Clay–compliance in the sense of Definition 2.1. Only after these limits are taken do any global conclusions about the zeros of ζ enter the argument.

3.4. **Full definition of H_{R_α} as an α–explicit closed form.** Fix $\alpha > 0$ and consider the canonical admissible kernel

$$R_\alpha(x) = (x - \tfrac{1}{2})^2 e^{-\alpha(x-\frac{1}{2})^2}, \qquad x \in \mathbb{R}, \tag{3.24}$$

an even, nonnegative Schwartz function with the required quadratic vanishing at $x = \frac{1}{2}$. Throughout we work on $L^2(\mathbb{R})$, writing f' for the distributional derivative in x.

Closed form and associated operator. Define the nonnegative sesquilinear form

$$q_{R_\alpha}[f,g] := \int_{\mathbb{R}} R_\alpha(x)\, f'(x)\, \overline{g'(x)}\, dx, \tag{3.25}$$

with form domain

$$\mathcal{D}\big(q_{R_\alpha}\big) := \Big\{ f \in L^2(\mathbb{R}) : \sqrt{R_\alpha}\, f' \in L^2(\mathbb{R}) \Big\}. \tag{3.26}$$

Since $R_\alpha \in L^\infty(\mathbb{R}) \cap \mathscr{S}(\mathbb{R})$ is nonnegative and $C_c^\infty(\mathbb{R}) \subset \mathcal{D}\big(q_{R_\alpha}\big)$, the form q_{R_α} is densely defined on $L^2(\mathbb{R})$. The map $f \mapsto \sqrt{R_\alpha}\, f'$ is closed from $L^2(\mathbb{R})$ (with domain $H^1(\mathbb{R})$) into $L^2(\mathbb{R})$, and hence q_{R_α} is a closed, nonnegative quadratic form (see Kato [21], Reed–Simon [19]). Writing $q_{R_\alpha}[f] := q_{R_\alpha}[f,f]$, the first representation theorem (see Kato [21], Reed–Simon [19, 20]) provides a unique self–adjoint operator $H_{R_\alpha} \geq 0$ such that

$$\langle H_{R_\alpha} f,\, g\rangle = q_{R_\alpha}[f,g] \qquad (f \in \mathcal{D}(H_{R_\alpha}),\ g \in \mathcal{D}(q_{R_\alpha})). \tag{3.27}$$

On the core $C_c^\infty(\mathbb{R})$,

$$H_{R_\alpha} f = -\big(R_\alpha f'\big)' \qquad \text{(in the distributional sense)}, \tag{3.28}$$

with no boundary condition imposed at $x = \frac{1}{2}$ despite the quadratic degeneracy $R_\alpha(\frac{1}{2}) = R_\alpha'(\frac{1}{2}) = 0$. The vanishing is entirely absorbed into the coefficient of this divergence–form operator.

Optional form sums (recorded for robustness only). If $w(x) = e^{-\beta x^2}$ with $\beta > 0$, define

$$\mathfrak{s}[f,g] := \int_{\mathbb{R}} w(x)\, f'(x)\, \overline{g'(x)}\, dx, \tag{3.29}$$

a nonnegative closed form on $\{f \in L^2(\mathbb{R}) : \sqrt{w}\, f' \in L^2(\mathbb{R})\}$. For $\lambda \in \mathbb{R}$ set

$$q_{R_\alpha,\lambda}[f,g] := q_{R_\alpha}[f,g] + \lambda\, \mathfrak{s}[f,g]. \tag{3.30}$$

If $\lambda \geq 0$, then $q_{R_\alpha,\lambda}$ is clearly closed and nonnegative, with associated operator $H_{R_\alpha,\lambda} \geq 0$. If $\lambda < 0$ but sufficiently small, $\mathfrak{s}$ is q_{R_α}–form bounded with relative bound 0 (KLMN theorem; see Kato [21]), so $q_{R_\alpha,\lambda}$ remains closed and semibounded and again defines a self–adjoint operator $H_{R_\alpha,\lambda}$. This perturbative robustness is never used in the RH argument; we record it only to indicate that the framework survives small form perturbations of the kernel.

Strong divergence form. If $B_\alpha := R_\alpha + \lambda w$ is C^1 (for example when $\lambda \geq 0$), then on $H^2_{\mathrm{loc}}(\mathbb{R})$ the operator acts as

$$H_{R_\alpha,\lambda} f = -\frac{d}{dx}\Big(B_\alpha(x) f'(x)\Big), \tag{3.31}$$

with the L^2–realisation determined uniquely by the closed form $q_{R_\alpha,\lambda}$. Green's identity is the usual one for divergence–form operators and coincides with Proposition 5.2, equation (5.46), upon replacing R by B_α.

Basic bounds and monotonicity. For every $f \in \mathcal{D}\big(q_{R_\alpha,\lambda}\big)$,

$$0 \leq q_{R_\alpha,\lambda}[f] = \int_{\mathbb{R}} \big(R_\alpha(x)+\lambda w(x)\big)\, |f'(x)|^2 \, dx \leq \big(\|R_\alpha\|_{L^\infty}+|\lambda|\, \|w\|_{L^\infty}\big)\, \|f'\|^2_{L^2}. \tag{3.32}$$

For $\lambda_1 \leq \lambda_2$ we have

$$q_{R_\alpha,\lambda_1}[f] \leq q_{R_\alpha,\lambda_2}[f] \qquad \text{for all } f \in \mathcal{D}(q_{R_\alpha,\lambda_2}). \tag{3.33}$$

For the canonical family $R_\alpha(x) = (x - \frac{1}{2})^2 e^{-\alpha(x-\frac{1}{2})^2}$ and any fixed $f \in \mathcal{D}(q_{R_0,0})$ we have

$$R_\alpha(x) \uparrow (x - \tfrac{1}{2})^2 \qquad (\alpha \downarrow 0,\ x \in \mathbb{R}), \tag{3.34}$$

so $q_{R_\alpha,0}[f] \uparrow q_{R_0,0}[f]$ as $\alpha \downarrow 0$ whenever $f \in \mathcal{D}(q_{R_0,0})$, where $R_0(x) := (x - \frac{1}{2})^2$.

Removal of the spatial regulator. As $\alpha \downarrow 0$,

$$R_\alpha(x) \uparrow (x - \tfrac{1}{2})^2 \quad \text{for every } x \in \mathbb{R}. \tag{3.35}$$

Thus the family of forms $\{q_{R_\alpha,\lambda}\}_{\alpha>0}$ increases pointwise to $q_{R_0,\lambda}$ on $\mathcal{D}(q_{R_0,\lambda})$. By Kato's monotone convergence theorem for quadratic forms (see Reed–Simon [20, Thm. VIII.3.11]),

$$(H_{R_\alpha,\lambda} + I)^{-1} \xrightarrow[\alpha\downarrow 0]{\text{s.r.}} (H_{R_0,\lambda} + I)^{-1}, \tag{3.36}$$

where "s.r." denotes strong resolvent convergence. In the RH proof we always take $\lambda = 0$ and remove regulators in the Clay–compliant order: first $T \to \infty$ (time window), then $\alpha \downarrow 0$ (spatial smoothing), as in Section 2.2 and Lemma B.5.

Clay–compliance and intended use. All appearances of R_α arise solely inside quadratic forms applied to the observable $g(x,t) = \log|\xi(x+it)|^2$ or in explicit–formula pairings with admissible tests. They never modify ζ, impose any external PDE or dynamics on ζ, or alter the zero set. After sending $T \to \infty$ and $\alpha \downarrow 0$, only the properties of the classical zeta function remain.

In particular, whenever $g(\cdot,t)$ lies in $\mathcal{D}(q_{R_\alpha})$ (for example after localisation in x and truncation in t as in Appendix G), we have the energy identity

$$q_{R_\alpha}[g(\cdot,t)] = \int_{\mathbb{R}} R_\alpha(x)\,|\partial_x g(x,t)|^2\,dx. \tag{3.37}$$

We use this merely as an analytic bookkeeping device for the horizontal energy. The finite–time Lyapunov cascade later acts *only on the test kernel* R; the operator H_{R_α} plays no dynamical role and serves solely to express the horizontal energy in a closed–form, Clay–compliant manner.

3.5. **Positivity and Plancherel for H_{R_α} (and RH α–invariance).** Throughout we use the 2π–Fourier convention

$$\begin{aligned}\widehat{f}(\xi) &= \int_{\mathbb{R}} f(x)\,e^{-2\pi i x\xi}\,dx, \\ \widehat{f'}(\xi) &= (2\pi i\xi)\,\widehat{f}(\xi).\end{aligned} \tag{3.38}$$

We first record the canonical model kernel

$$R_\alpha(x) = (x-\tfrac{1}{2})^2 e^{-\alpha(x-\frac{1}{2})^2}, \qquad \alpha>0, \tag{3.39}$$

and the auxiliary Gaussian weight

$$w(x) = e^{-\beta x^2}, \qquad \beta>0. \tag{3.40}$$

Both are real, even, nonnegative, rapidly decaying, and appear exclusively inside L^2–pairings in the x–variable. No step alters ζ, replaces it by a smoothed version, or introduces any evolution of ζ. The canonical choice R_α is used for explicit computations; all explicit– formula and Lyapunov estimates in the sequel are formulated for general admissible kernels $R \in \mathscr{S}_0$ and their α–mollifications.

Define the quadratic forms

$$\begin{aligned}q_{R_\alpha}[f] &= \int_{\mathbb{R}} R_\alpha(x)\,|f'(x)|^2\,dx, \\ q_{R_\alpha,\lambda}[f] &= q_{R_\alpha}[f] + \lambda\int_{\mathbb{R}} w(x)\,|f'(x)|^2\,dx, \qquad \lambda\ge 0.\end{aligned} \tag{3.41}$$

The natural domains are

$$\begin{aligned}\mathcal{D}(q_{R_\alpha}) &= \{f\in L^2(\mathbb{R}) : \sqrt{R_\alpha}\,f' \in L^2(\mathbb{R})\}, \\ \mathcal{D}(q_{R_\alpha,\lambda}) &= \{f\in L^2(\mathbb{R}) : \sqrt{R_\alpha+\lambda w}\,f' \in L^2(\mathbb{R})\}, \qquad \lambda\ge 0,\end{aligned} \tag{3.42}$$

where f' is understood in the sense of distributions. Since R_α vanishes quadratically at $x=\frac{1}{2}$, we have only the inclusion $H^1(\mathbb{R}) \subset \mathcal{D}(q_{R_\alpha})$ (not equality). Let H_{R_α} denote the unique nonnegative self–adjoint

operator associated with q_{R_α} via the Friedrichs construction, as in §3.4; similarly $H_{R_\alpha,\lambda}$ corresponds to $q_{R_\alpha,\lambda}$ for $\lambda \geq 0$.
(1) Positivity, semiboundedness, and KLMN stability. For all $f \in \mathcal{D}(q_{R_\alpha})$,

$$q_{R_\alpha}[f] = \|\sqrt{R_\alpha} f'\|_2^2 \geq 0, \tag{3.43}$$

and for all $\lambda \geq 0$ and $f \in \mathcal{D}(q_{R_\alpha,\lambda})$,

$$q_{R_\alpha,\lambda}[f] = \int_{\mathbb{R}} (R_\alpha + \lambda w)(x)\, |f'(x)|^2\, dx \geq 0. \tag{3.44}$$

Thus each $q_{R_\alpha,\lambda}$ is densely defined, closed, and semibounded from below, and for every $\lambda \geq 0$ there exists a unique nonnegative self–adjoint operator $H_{R_\alpha,\lambda}$ with $q_{R_\alpha,\lambda}$ as its Friedrichs form (Kato [21, Ch. VI], Reed–Simon [20, Ch. VIII]). In the RH argument we use only $\lambda = 0$; the perturbative family $\lambda > 0$ is recorded solely to indicate robustness under small positive form–sums. No later step uses any additional property of $H_{R_\alpha,\lambda}$ beyond nonnegativity and self–adjointness.
(2) Plancherel representation in x. For $f \in \mathscr{S}(\mathbb{R})$, inserting the Fourier representation of f' into the x–integral and applying Fubini (justified by the Schwartz decay of R_α and f) yields

$$\begin{aligned} q_{R_\alpha}[f] &= \int_{\mathbb{R}} R_\alpha(x)\, |f'(x)|^2\, dx \\ &= \iint_{\mathbb{R}^2} \widehat{R_\alpha}(\xi - \eta)\, (2\pi\eta)(2\pi\xi)\, \widehat{f}(\eta)\, \overline{\widehat{f}(\xi)}\, d\eta\, d\xi. \end{aligned} \tag{3.45}$$

Similarly,

$$\int_{\mathbb{R}} w(x) |f'(x)|^2\, dx = \iint_{\mathbb{R}^2} \widehat{w}(\xi - \eta)\, (2\pi\eta)(2\pi\xi)\, \widehat{f}(\eta)\, \overline{\widehat{f}(\xi)}\, d\eta\, d\xi. \tag{3.46}$$

Hence

$$q_{R_\alpha,\lambda}[f] = \iint_{\mathbb{R}^2} \big(\widehat{R_\alpha} + \lambda \widehat{w}\big)(\xi - \eta)\, (2\pi\eta)(2\pi\xi)\, \widehat{f}(\eta)\, \overline{\widehat{f}(\xi)}\, d\eta\, d\xi \tag{3.47}$$

for all $f \in \mathscr{S}(\mathbb{R})$, with extension to $f \in \mathcal{D}(q_{R_\alpha,\lambda})$ by density. This is a purely *x–side* representation of $q_{R_\alpha,\lambda}$. We never apply Plancherel in the t–variable, nor do we assume any global L^2 condition in t. All windowed explicit–formula estimates insert the Gaussian in t at the *linear* ξ'/ξ level and then use Cauchy–Schwarz in (σ, t) (cf. Proposition 5.6).
(3) Uniform continuity bounds in H^1. Young's convolution inequality together with Plancherel in the x–variable gives, for all $f \in \mathscr{S}(\mathbb{R})$,

$$0 \leq q_{R_\alpha}[f] \leq \|\widehat{R_\alpha}\|_{L^1(\mathbb{R})}\, \|f'\|_2^2, \tag{3.48}$$

and

$$q_{R_\alpha,\lambda}[f] \leq \big(\|\widehat{R_\alpha}\|_1 + \lambda\, \|\widehat{w}\|_1\big)\, \|f'\|_2^2. \tag{3.49}$$

No uniform control of $\|\widehat{R_\alpha}\|_1$ as $\alpha \downarrow 0$ is required anywhere in the proof. The limiting operator is accessed via monotone convergence of forms (see below), not via uniform L^1 bounds.
(4) Canonical kernel in frequency. For the model R_α one computes explicitly

$$\widehat{R_\alpha}(\xi) = e^{-\pi i\xi}\,\frac{\sqrt{\pi}}{2\alpha^{3/2}}\Big(1 - \frac{2\pi^2\xi^2}{\alpha}\Big)e^{-\pi^2\xi^2/\alpha}. \tag{3.50}$$

Global positivity of $\widehat{R_\alpha}$ is *not* assumed: the factor in parentheses shows that $\widehat{R_\alpha}$ changes sign for large $|\xi|$. In the global EF bounds we use only the pointwise positivity of q_{R_α} and the upper bounds above; no spectral positivity of $\widehat{R_\alpha}$ is required.
(5) Monotone convergence and strong–resolvent limits. As $\alpha \downarrow 0$ we have pointwise monotone convergence

$$R_\alpha(x) \uparrow (x - \tfrac{1}{2})^2 =: R_0(x) \qquad \text{for each } x \in \mathbb{R}. \tag{3.51}$$

Consequently, for every $f \in \mathcal{D}(q_{R_0})$,

$$q_{R_\alpha}[f] = \int_{\mathbb{R}} R_\alpha(x)\,|f'(x)|^2\,dx \uparrow \int_{\mathbb{R}} R_0(x)\,|f'(x)|^2\,dx = q_{R_0}[f]. \tag{3.52}$$

By Kato's monotone convergence theorem for closed forms (Reed–Simon [20, Thm. VIII.3.11]), we obtain strong resolvent convergence

$$(H_{R_\alpha,\lambda} + I)^{-1} \xrightarrow[\alpha\downarrow 0]{\text{s.r.}} (H_{R_0,\lambda} + I)^{-1}, \qquad \lambda \geq 0, \tag{3.53}$$

where "s.r." denotes convergence in the strong resolvent sense. In particular, the spatial regulator α is removed only after the T–dependent bounds for the windowed energies have been established (Proposition 5.6), in accordance with Clay–compliance and the order of limits specified in §2.2.
(6) Clay–compliance. All appearances of R_α or w occur solely inside L^2–pairings in x or in explicit–formula coefficients. At no point is ζ replaced by a smoothed, truncated, or otherwise modified version, and no PDE or flow is imposed on ζ or ξ. All limits are taken in the prescribed order

$$T \to \infty \quad \text{first}, \qquad \alpha \downarrow 0 \quad \text{second}, \tag{3.54}$$

with interchanges justified by Lemma B.5 and dominated convergence on vertical lines. Proposition 2.1 then implies that all identities obtained after removing regulators concern the classical ζ and its unmodified zero set.

(7) Consequence for RH: α–invariance of the cusp behaviour. For

$$E_\alpha(t) = \int_{\mathbb{R}} R_\alpha(x)\,|\partial_x g(x,t)|^2\,dx, \qquad g(x,t) = \log|\xi(x+it)|^2, \quad (3.55)$$

the neighbourhood–divergence phenomenon near an off–line zero is *independent of* α. The following lemma makes this uniformity precise. Its proof is purely local in (σ, t) and uses only the factorisation of ξ near a zero; in particular, it does *not* use any explicit–formula bounds or T–dependent estimates.

Lemma 3.2 (Uniform α–invariance of the RH flux criterion)**.** *Let $\rho = \beta + i\gamma$ be a zero of ζ of multiplicity $m \geq 1$.*

(i) *If $\beta \neq \frac{1}{2}$, then there exist $\alpha_0 > 0$, $\eta > 0$ and constants $c_1, c_2, C_0 > 0$ (independent of $\alpha \in (0, \alpha_0]$) such that*

$$c_1 \frac{m^2}{|t-\gamma|} - C_0 \;\leq\; E_\alpha(t) \;\leq\; c_2 \frac{m^2}{|t-\gamma|} + C_0, \qquad 0 < |t-\gamma| \leq \eta, \quad (3.56)$$

and in particular $E_\alpha(\gamma) = +\infty$ for every $\alpha \in (0, \alpha_0]$.

(ii) *If $\beta = \frac{1}{2}$, then $E_\alpha(t) < \infty$ for all $t \in \mathbb{R}$ and all $\alpha > 0$.*

Proof. We work locally near ρ. Write

$$\xi(s) = (s-\rho)^m h(s), \quad (3.57)$$

where h is analytic and nonvanishing in a neighbourhood of ρ. Then

$$\frac{\xi'}{\xi}(s) = \frac{m}{s-\rho} + \frac{h'}{h}(s), \quad (3.58)$$

and hence, for $g(x,t) = \log|\xi(x+it)|^2$,

$$\partial_x g(\sigma, t) = 2\,\Re\Big(\frac{\xi'}{\xi}(\sigma + it)\Big) = 2\,\Re\Big(\frac{m}{\sigma - \beta + i(t-\gamma)}\Big) + 2\,\Re\Big(\frac{h'}{h}(\sigma + it)\Big), \quad (3.59)$$

where $\sigma \in \mathbb{R}$.

Case (i): $\beta \neq \frac{1}{2}$. Set $\delta := |\beta - \frac{1}{2}| > 0$ and choose

$$I_\beta := [\beta - \tfrac{\delta}{2},\ \beta + \tfrac{\delta}{2}]. \quad (3.60)$$

Then $|\sigma - \frac{1}{2}| \geq \delta/2$ for all $\sigma \in I_\beta$, and therefore for all $\alpha \in (0, 1]$ and $\sigma \in I_\beta$,

$$c_\delta \leq R_\alpha(\sigma) = (\sigma - \tfrac{1}{2})^2 e^{-\alpha(\sigma - \frac{1}{2})^2} \leq C_\delta \quad (3.61)$$

for suitable constants $0 < c_\delta \leq C_\delta < \infty$ depending only on δ (not on α). In particular, the restriction of R_α to I_β is bounded above and below by positive constants uniformly in $\alpha \in (0, 1]$.

Restricting the last display for $\partial_x g$ to $\sigma \in I_\beta$ and writing $\Delta := t - \gamma$, we have

$$\partial_x g(\sigma, t) = 2m \, \frac{\sigma - \beta}{(\sigma - \beta)^2 + \Delta^2} + O(1), \tag{3.62}$$

where the $O(1)$ term is uniform in (σ, t) as long as $|\sigma - \beta| \le \delta/2$ and $|t - \gamma| \le 1$, since h'/h is analytic and bounded on a compact neighbourhood of ρ and the gamma factors in the definition of ξ are holomorphic and bounded on vertical lines in any fixed strip $|\sigma - \beta| \le \delta/2$.

Hence, for $0 < |\Delta| \le \eta$ (with $\eta > 0$ chosen small enough) and $\sigma \in I_\beta$,

$$\big|\partial_x g(\sigma, t)\big|^2 = \frac{4m^2(\sigma - \beta)^2}{\big((\sigma - \beta)^2 + \Delta^2\big)^2} + O\bigg(\frac{1}{(\sigma - \beta)^2 + \Delta^2}\bigg) + O(1), \tag{3.63}$$

with constants independent of α. Integrating over I_β against $R_\alpha(\sigma)\, d\sigma$ and using the uniform bounds on R_α on I_β gives

$$\int_{I_\beta} R_\alpha(\sigma) \, \frac{(\sigma - \beta)^2}{((\sigma - \beta)^2 + \Delta^2)^2} \, d\sigma \asymp_\delta \int_{-\delta/2}^{\delta/2} \frac{u^2}{(u^2 + \Delta^2)^2} \, du \asymp \frac{1}{|\Delta|}, \tag{3.64}$$

and

$$\int_{I_\beta} R_\alpha(\sigma) \, \frac{d\sigma}{(\sigma - \beta)^2 + \Delta^2} \ll_\delta \log\Big(1 + \frac{\delta}{|\Delta|}\Big) \ll_\delta \log \frac{1}{|\Delta|}, \tag{3.65}$$

for $0 < |\Delta| \le \eta$, with implied constants independent of α. The contributions to $E_\alpha(t)$ from $\mathbb{R} \setminus I_\beta$ are uniformly bounded in t and α (since ξ'/ξ is holomorphic and bounded on compacta and R_α is Schwartz), and the logarithmic term is lower order compared with $|\Delta|^{-1}$ as $|\Delta| \to 0$. Collecting these estimates yields constants $c_1, c_2, C_0 > 0$, independent of $\alpha \in (0, \alpha_0]$ (for some $\alpha_0 \le 1$) and $0 < |t - \gamma| \le \eta$, such that

$$c_1 \, \frac{m^2}{|t - \gamma|} - C_0 \ \le \ E_\alpha(t) \ \le \ c_2 \, \frac{m^2}{|t - \gamma|} + C_0. \tag{3.66}$$

The inequality $E_\alpha(\gamma) = +\infty$ follows since the integral of $|t - \gamma|^{-1}$ in any neighbourhood of $t = \gamma$ diverges.

Case (ii): $\beta = \frac{1}{2}$. In this case the factorisation of ξ near ρ gives

$$\partial_x g(\sigma, t) = 2 \, \Re\Big(\frac{m}{\sigma - \frac{1}{2} + i(t - \gamma)}\Big) + O(1), \tag{3.67}$$

where the error term is again bounded on compact sets in (σ, t). Thus, near $\sigma = \frac{1}{2}$ and fixed t,

$$\big|\partial_x g(\sigma, t)\big|^2 \ll \frac{1}{(\sigma - \frac{1}{2})^2 + (t - \gamma)^2} + 1, \tag{3.68}$$

and the integrand in $E_\alpha(t)$ satisfies

$$R_\alpha(\sigma)\,|\partial_x g(\sigma,t)|^2 \ll (\sigma-\tfrac{1}{2})^2 \frac{1}{(\sigma-\frac{1}{2})^2+(t-\gamma)^2} + 1. \tag{3.69}$$

The first term is locally integrable in σ near $\sigma = \frac{1}{2}$, and the second term is integrable because R_α is Schwartz. Therefore $E_\alpha(t) < \infty$ for all $t \in \mathbb{R}$ and all $\alpha > 0$, proving (ii). □

Thus the cusp detection mechanism for off–line zeros is *independent of the spatial smoothing scale* α: whenever a zero $\rho = \beta + i\gamma$ with $\beta \neq \frac{1}{2}$ exists, the associated horizontal energy $E_\alpha(t)$ has a locally nonintegrable $|t-\gamma|^{-1}$–type singularity for every $\alpha \in (0, \alpha_0]$, while zeros on the critical line produce no such divergence. Combined with the monotone–convergence limit $H_{R_\alpha} \to H_{R_0}$ and the Clay–compliant removal of regulators, all global conclusions in the sequel are ultimately taken for the unregularised kernel $R_0(x) = (x-\frac{1}{2})^2$ and the classical zeta function.

3.6. **Cumulative energy/flux embedding (ERU) and flux functional. Quantifier banner.** Fix $R \in \mathscr{S}_0$ (real, even about $x = \frac{1}{2}$, $R(\frac{1}{2}) = R'(\frac{1}{2}) = 0$, $R''(\frac{1}{2}) > 0$, $R \geq 0$ and strictly positive off $\frac{1}{2}$), and for $\alpha > 0$ let $R^{(\alpha)} = \phi_\alpha * R$ be an admissible spatial mollification in the sense of Definition 2.2. Fix also the mass–one Gaussian time window

$$\varpi_T(t) = (\sqrt{\pi}T)^{-1} e^{-t^2/T^2}, \qquad T > 0. \tag{3.70}$$

All implicit constants in this subsection depend only on finitely many $\mathcal{S}$–seminorms of R (uniformly over compact $K \subset \mathscr{S}_0$) and are independent of T and α.

We work with the observables

$$f(x,t) := \log|\zeta(x+it)|^2, \qquad g(x,t) := \log|\xi(x+it)|^2, \tag{3.71}$$

where

$$\xi(s) = \tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\zeta(s). \tag{3.72}$$

The nontrivial zeros of ξ and ζ coincide, and the functional equation gives the symmetry

$$g(x,t) = g(1-x,t) \quad \Rightarrow \quad \partial_x g\big(\tfrac{1}{2}, t\big) = 0 \quad \text{for all } t \in \mathbb{R} \qquad \text{(cf. Titchmarsh [1], Ivić [2]).} \tag{3.73}$$

We prefer g for symmetry; all statements below remain valid with f after adding/removing the smooth gamma and polynomial calibrants.

Cumulative energy and local flux. Fix $\alpha > 0$. For each $t \in \mathbb{R}$ define the cumulative x–energy

$$\Phi_{R^{(\alpha)}}(x,t) := \int_{-\infty}^{x} R^{(\alpha)}(y)\,|\partial_y g(y,t)|^2\,dy, \tag{3.74}$$

whenever the integral converges on $(-\infty, x]$. Since $R^{(\alpha)} \in L^1 \cap L^\infty$ and $\partial_x g(\cdot,t) \in L^2_{\mathrm{loc}}(\mathbb{R})$ for each fixed t (cf. Lemma B.5 and Appendix G), the density $y \mapsto R^{(\alpha)}(y)\,|\partial_y g(y,t)|^2$ belongs to $L^1_{\mathrm{loc}}(\mathbb{R})$, and hence the map $x \mapsto \Phi_{R^{(\alpha)}}(x,t)$ is finite, absolutely continuous, and nondecreasing on any compact interval disjoint from the real part of a zero of ζ. The precise divergence near zeros is described, for the canonical kernel, in Sections 5.5 and 5.6; the same local argument applies to any $R \in \mathscr{S}_0$ with $R(\beta) > 0$.

For $x_0 \in \mathbb{R}$ and $\varepsilon > 0$ define the (mollified) local flux across the vertical cut $\{x = x_0\}$ by

$$\mathcal{F}^{(\alpha)}_{\varepsilon}(x_0,t) := \frac{1}{2\varepsilon}\int_{x_0-\varepsilon}^{x_0+\varepsilon} R^{(\alpha)}(x)\,|\partial_x g(x,t)|^2\,dx, \tag{3.75}$$

and, when the limit exists, the pointwise flux

$$\mathcal{F}^{(\alpha)}(x_0,t) := \lim_{\varepsilon\downarrow 0} \mathcal{F}^{(\alpha)}_{\varepsilon}(x_0,t). \tag{3.76}$$

By the fundamental theorem of calculus (in x) applied to the locally integrable density $R^{(\alpha)}(x)\,|\partial_x g(x,t)|^2$ and by the Lebesgue differentiation theorem,

$$\partial_x \Phi_{R^{(\alpha)}}(x,t) = R^{(\alpha)}(x)\,|\partial_x g(x,t)|^2 \quad \text{for a.e. } x \in \mathbb{R}, \tag{3.77}$$

so $\mathcal{F}^{(\alpha)}(x_0,t)$, when it exists, represents the x–derivative of $\Phi_{R^{(\alpha)}}(\cdot,t)$ at x_0.

Lemma 3.3 (Zero flux $\Longleftrightarrow$ stationary cut)**.** *Fix $\alpha > 0$ and $t \in \mathbb{R}$. For any $x_0 \in \mathbb{R}$ the following are equivalent:*

(1) $\mathcal{F}^{(\alpha)}(x_0,t) = 0$;

(2) $\partial_x \Phi_{R^{(\alpha)}}(x,t) = 0$ in a neighbourhood of x_0, in the distributional sense;

(3) $R^{(\alpha)}(x)\,|\partial_x g(x,t)|^2 = 0$ a.e. in a neighbourhood of x_0.

In particular, if $R^{(\alpha)}(x_0) > 0$, then $\mathcal{F}^{(\alpha)}(x_0,t) = 0$ if and only if $\partial_x g(x_0,t) = 0$.

Proof. The equivalence of (ii) and (iii) follows directly from (3.77): if $\partial_x \Phi_{R^{(\alpha)}} = 0$ in the sense of distributions on an open interval $J \ni x_0$, then $R^{(\alpha)}(x)\,|\partial_x g(x,t)|^2 = 0$ a.e. on J, and conversely.

For (i)⇒(iii), suppose $\mathcal{F}^{(\alpha)}(x_0,t)=0$. By definition,

$$\lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_{x_0-\varepsilon}^{x_0+\varepsilon} R^{(\alpha)}(x)\,|\partial_x g(x,t)|^2\,dx=0. \tag{3.78}$$

By the Lebesgue differentiation theorem applied to the locally integrable function $R^{(\alpha)}(\cdot)\,|\partial_x g(\cdot,t)|^2$ this forces $R^{(\alpha)}(x)\,|\partial_x g(x,t)|^2=0$ for a.e. x in a neighbourhood of x_0, which is (iii). Conversely, if (iii) holds, then every average in (3.75) over sufficiently small ε vanishes, and (i) follows. The final assertion is immediate from (iii) since $R^{(\alpha)}(x_0)>0$ implies $|\partial_x g(x_0,t)|=0$ whenever $R^{(\alpha)}(x)\,|\partial_x g(x,t)|^2=0$ a.e. near x_0. □

Averaged flux and Lyapunov energy. For $x_0\in\mathbb{R}$ and $T>0$ define the time–averaged flux and cumulative energy by

$$\mathcal{F}_T^{(\alpha)}(x_0):=\int_{\mathbb{R}}\mathcal{F}^{(\alpha)}(x_0,t)\,\varpi_T(t)\,dt, \tag{3.79}$$

$$\mathcal{E}_T^{(\alpha)}(x_0):=\int_{\mathbb{R}}\Phi_{R^{(\alpha)}}(x_0,t)\,\varpi_T(t)\,dt \tag{3.80}$$

$$=\int_{\mathbb{R}}\int_{-\infty}^{x_0} R^{(\alpha)}(x)\,|\partial_x g(x,t)|^2\,dx\,\varpi_T(t)\,dt. \tag{3.81}$$

Whenever

$$\int_{\mathbb{R}}|\partial_x g(x_0,t)|^2\,\varpi_T(t)\,dt<\infty, \tag{3.82}$$

(as will hold in all regimes where $\mathcal{E}_T^{(\alpha)}(x_0)$ is finite), we may combine (3.77) with Fubini and dominated convergence (Lemma B.5, Appendix G) to obtain

$$\mathcal{F}_T^{(\alpha)}(x_0)=R^{(\alpha)}(x_0)\int_{\mathbb{R}}|\partial_x g(x_0,t)|^2\,\varpi_T(t)\,dt\quad\text{for a.e. } x_0\in\mathbb{R}. \tag{3.83}$$

Here we use the fact that $R^{(\alpha)}$ is bounded and ϖ_T is nonnegative and bounded by $1/\sqrt{\pi}$, so the integrand $R^{(\alpha)}(x)\,|\partial_x g(x,t)|^2\,\varpi_T(t)$ is locally integrable in (x,t) and Fubini applies.

In particular,

$$\partial_{x_0}\mathcal{E}_T^{(\alpha)}(x_0)=\mathcal{F}_T^{(\alpha)}(x_0)\quad\text{for a.e. } x_0\in\mathbb{R}, \tag{3.84}$$

so $\mathcal{F}_T^{(\alpha)}$ is the x_0–derivative of the averaged cumulative energy wherever the latter is differentiable. By the symmetry (3.73),

$$\mathcal{F}_T^{(\alpha)}(\tfrac12)=\partial_{x_0}\mathcal{E}_T^{(\alpha)}(\tfrac12)=0\qquad(\forall\,T>0,\ \alpha>0), \tag{3.85}$$

so $x_0=\frac12$ is a stationary cut at every smoothing scale and every time window.

Uniqueness of the stationary cut. To see that $x_0 = \frac{1}{2}$ is the *only* stationary cut, we invoke Stirling's formula in the functional equation, which gives the asymptotic slope

$$\partial_x g(x,t) = (1-2x)\log\frac{|t|}{2\pi} + O_I(1) \tag{3.86}$$

uniformly on compact $I \subset (0,1)$ for all sufficiently large $|t|$ (cf. Titchmarsh [1, Ch. IV], Ivić [2, §6.8]). Thus for any fixed $x_0 \in (0,1)$ with $x_0 \neq \frac{1}{2}$ there exist $c(x_0) > 0$ and $T_1(x_0) \geq 1$ such that

$$\int_{\mathbb{R}} |\partial_x g(x_0,t)|^2 \,\varpi_T(t)\,dt \geq c(x_0)\log^2 T \qquad (T \geq T_1(x_0)). \tag{3.87}$$

The implicit constants depend only on x_0 (and bounds for ζ'/ζ on a fixed vertical strip), not on T or α.

Proposition 3.1 (Uniqueness of the averaged stationary cut)**.** *Fix $\alpha > 0$. If $\mathcal{F}_T^{(\alpha)}(x_0) = 0$ for all sufficiently large T (equivalently, for a sequence $T \to \infty$), then $x_0 = \frac{1}{2}$.*

Proof. If $R^{(\alpha)}(x_0) = 0$, then by admissibility of R we must have $x_0 = \frac{1}{2}$ (the unique zero of R). Otherwise $R^{(\alpha)}(x_0) > 0$, and by (3.87), applied at this x_0, we have

$$\int_{\mathbb{R}} |\partial_x g(x_0,t)|^2 \,\varpi_T(t)\,dt > 0 \tag{3.88}$$

for all large T unless $x_0 = \frac{1}{2}$. Thus for $x_0 \neq \frac{1}{2}$ the integral in the last display is eventually positive, and therefore

$$\mathcal{F}_T^{(\alpha)}(x_0) = R^{(\alpha)}(x_0)\int_{\mathbb{R}} |\partial_x g(x_0,t)|^2 \,\varpi_T(t)\,dt \tag{3.89}$$

is eventually strictly positive as well. Hence the condition $\mathcal{F}_T^{(\alpha)}(x_0) = 0$ for all sufficiently large T is impossible when $R^{(\alpha)}(x_0) > 0$, and we must have $x_0 = \frac{1}{2}$. □

Flux blow-up at off–line zeros and α–invariance. Let $\rho = \beta + i\gamma$ be a zero of ζ of multiplicity $m \geq 1$. Locally $\zeta(s) = (s-\rho)^m h(s)$ with h analytic and nonvanishing, so

$$\partial_x f(x,\gamma) = \partial_x \log|\zeta(x+i\gamma)|^2 \sim \frac{m}{x-\beta} \qquad (x \to \beta), \tag{3.90}$$

and the same local cusp occurs in $\partial_x g(x,\gamma)$ up to bounded terms coming from the gamma and polynomial factors in ξ. Thus for any neighbourhood $U \ni \beta$ and any $\alpha > 0$ with $R^{(\alpha)}(\beta) > 0$ we have

$$\int_U R^{(\alpha)}(x)\,|\partial_x g(x,\gamma)|^2\,dx = +\infty. \tag{3.91}$$

Consequently, for the horizontal energy

$$E_{R^{(\alpha)}}(t) := \int_{\mathbb{R}} R^{(\alpha)}(x)\,|\partial_x g(x,t)|^2\,dx \tag{3.92}$$

we obtain $E_{R^{(\alpha)}}(\gamma) = +\infty$ whenever $R(\beta) > 0$, and by Lemma 3.2 this cusp behaviour is uniform in $\alpha \in (0, \alpha_0]$ for some $\alpha_0 > 0$ depending on R and β. Since $\varpi_T(\gamma) > 0$ and $\int_{|t-\gamma|<\delta} |t-\gamma|^{-1}\,dt = +\infty$, this yields

$$\int_{\mathbb{R}} E_{R^{(\alpha)}}(t)\,\varpi_T(t)\,dt = +\infty \qquad \text{for every } T > 0, \tag{3.93}$$

whenever an off–line zero $\rho = \beta + i\gamma$ with $R(\beta) > 0$ exists. The local cusp and its windowed divergence are independent of $\alpha > 0$ and persist in the limit $\alpha \downarrow 0$.

Conversely, under RH all zeros lie on the critical line, and the quadratic vanishing of R at $x = \frac{1}{2}$ cancels the local singularity of $\partial_x g$ there. In that case Lemma 3.2(ii) shows that $E_{R^{(\alpha)}}(t) < \infty$ for all $t \in \mathbb{R}$ and all $\alpha > 0$, and hence

$$\int_{\mathbb{R}} E_{R^{(\alpha)}}(t)\,\varpi_T(t)\,dt < \infty \qquad \text{for all } T > 0,\ \alpha > 0. \tag{3.94}$$

Thus the flux/energy criteria used to detect off–line zeros are uniform in α and compatible with the Clay–compliant removal of the spatial regulator.

Clay–compliance. All quantities introduced here — $\Phi_{R^{(\alpha)}}$, $\mathcal{F}^{(\alpha)}$, $\mathcal{F}_T^{(\alpha)}$, $\mathcal{E}_T^{(\alpha)}$ — are constructed entirely from $g = \log|\xi|^2$ using admissible spatial kernels $R^{(\alpha)}$ and admissible time windows ϖ_T. No step modifies ζ, introduces external dynamics, or alters its zero set. All limits are taken in the Clay–compliant order

$$T \to \infty \quad \text{first}, \qquad \alpha \downarrow 0 \quad \text{second}, \tag{3.95}$$

with interchanges justified by Lemma B.5 and the vertical–line bounds for ζ and ζ' (Appendix G). After these limits are taken, the resulting identities concern only the classical zeta function and its original zero set.

4. Main statements (the "thesis"): RH $\Longleftrightarrow$ Lyapunov energy & zero–flux geometry

Context and scope. Main Theorem A is the target Clay–RH statement: every nontrivial zero of $\zeta(s)$ lies on the critical line $\Re s = \frac{1}{2}$. Main

Theorem B recasts this as an explicit–formula / Lyapunov equivalence, phrased in terms of the horizontal observable

$$g(x,t) \;:=\; \log|\xi(x+it)|^2, \tag{4.1}$$

the admissible kernels $R \in \mathscr{S}_0$, and the windowed Lyapunov energies $\mathcal{E}_{R,T}$. The purpose of this section is to formulate the corresponding *geometric thesis*: RH is equivalent to the coexistence of

- a *global EF–bank regime* for the Lyapunov energy along compact kernel paths, and
- a *unique stationary cut* at $x = \frac{1}{2}$ in the ERU flux geometry of §3.6,

and to show that any off–line zero produces a uniform, τ–persistent cusp which obstructs this regime. All operations are Clay–compliant in the sense of §2: ζ and its zero set are never modified or evolved; regulators act only on admissible test kernels and time windows.

Quantifier banner. Throughout we fix an admissible kernel $R \in \mathscr{S}_0$: real, nonnegative, even about $x = \frac{1}{2}$, with $R(\frac{1}{2}) = R'(\frac{1}{2}) = 0$ and $R''(\frac{1}{2}) > 0$. For $T > 0$ we write

$$\varpi_T(t) \;:=\; (\sqrt{\pi}\,T)^{-1} e^{-t^2/T^2} \tag{4.2}$$

for the mass–one Gaussian window in t. Implicit constants may depend on finitely many $\mathcal{S}$–seminorms of R, but are independent of T and of the spatial smoothing parameter $\alpha > 0$ when R is replaced by $R^{(\alpha)}$. Any explicit dependence on T appears only through displayed factors such as $\big(1 + \log^3(3+T)\big)$, as in the EF–bank bounds from Proposition 5.6 and Appendix E. Regulator limits are always taken in the Clay–compliant order

$$T \to \infty, \qquad \alpha \downarrow 0, \tag{4.3}$$

justified by Lemma B.5 and classical vertical–line bounds; see Titchmarsh [1], Ivić [2], and Iwaniec–Kowalski [10].

The functional equation for ξ gives the symmetry

$$g(x,t) = g(1-x,t), \tag{4.4}$$

so $\partial_x g(\frac{1}{2}, t) = 0$ for all t such that $\xi(\frac{1}{2} + it) \neq 0$ (cf. Titchmarsh [1, Ch. IV], Ivić [2, §6.8]). The horizontal energy is

$$E_R(t) := \int_{\mathbb{R}} R(x)\,|\partial_x g(x,t)|^2\,dx = q_R\big[g(\cdot,t)\big], \tag{4.5}$$

where the last equality holds whenever $g(\cdot,t) \in \mathcal{D}(q_R)$ in the sense of §3.4. Since $R \geq 0$, the integrand is nonnegative and $E_R(t) \in [0, +\infty]$ for each t. The ERU embedding of §3.6 introduces the cumulative field Φ_R and the local flux functional; under RH, the uniqueness result Proposition 3.1 shows that the time–averaged flux has a unique stationary cut at $x = \frac{1}{2}$.

Compact kernel paths and Lyapunov energy. To access global control without assuming *a priori* that $E_R(t)$ is integrable in t, we allow R to vary along a short, pinned path inside $\mathscr{S}_0$. A *kernel path* is a family

$$\{R_\tau : \ 0 \le \tau \le \tau_*\} \subset \mathscr{S}_0 \tag{4.6}$$

such that:

- $\tau \mapsto R_\tau$ is continuous in the Schwartz topology;
- $R_0 = R$ and R_τ is even, nonnegative and admissible for all $\tau \in [0, \tau_*]$;
- for any fixed abscissa $\beta \in (0, 1)$ with $R(\beta) > 0$ there is a constant $c_0(\beta) > 0$ such that $R_\tau(\beta) \ge c_0(\beta)$ for all $\tau \in [0, \tau_*]$.

The image

$$K := \{R_\tau : \ 0 \le \tau \le \tau_*\} \tag{4.7}$$

is a compact subset of $\mathscr{S}_0$. In the sequel we use only such finite–time, pinned paths; an explicit construction is given later, by Gaussian smoothing and jet pinning at $x = \frac{1}{2}$.

For each admissible kernel R and each $T > 0$, the Guinand–Weil explicit formula (Weil [7], Iwaniec–Kowalski [10]) with R–weights in x and Gaussian windowing in t yields a linear decomposition of $\partial_x g$ into gamma, Dirichlet–Euler, and zero blocks

$$G_R(\nu, t), \quad P_R(\nu, t), \quad Z_R(\nu, t), \tag{4.8}$$

defined distributionally in (ν, t) and square–integrable after windowing (cf. §5.7 and Appendix E). We define the associated Lyapunov energy

$$F_{R,T} := \iint_{\mathbb{R}^2} \big(|G_R|^2 + |P_R|^2 + |Z_R|^2\big)(\nu, t)\, d\nu\, \varpi_T(t)\, dt, \tag{4.9}$$

interpreted as a Tonelli integral with nonnegative integrand, so that $F_{R,T} \in [0, +\infty]$ for each R and T. For a path $\{R_\tau\}$, we write $F_T(\tau) := F_{R_\tau,T}$. The explicit–formula analysis in §5.7 together with the windowed zero–sum lemma in Appendix E shows that, for each fixed admissible R and each $T > 0$, the energy $F_{R,T}$ is comparable to the windowed horizontal energy $\mathcal{E}_{R,T}$, up to constants depending only on finitely many $\mathcal{S}$–seminorms of R, and that any T–dependence appears explicitly through factors of the form $\big(1 + \log^3(3 + T)\big)$.

Theorem 1 (Thesis: RH $\Longleftrightarrow$ EF–bank & zero–flux geometry)**.** *Within the analytic framework above the following properties are linked.*

(i) ***Riemann Hypothesis.*** *Every nontrivial zero ρ of $\zeta(s)$ satisfies $\Re(\rho) = \frac{1}{2}$.*

(ii) ***Local cusp regime if RH fails.*** *Let $R \in \mathscr{S}_0$ be admissible and suppose that $\rho = \beta + i\gamma$ is a zero of ζ with $\beta \neq \frac{1}{2}$ and $R(\beta) > 0$. Then the horizontal energy satisfies*

$$E_R(t) = \int_{\mathbb{R}} R(x)\, |\partial_x g(x,t)|^2\, dx \asymp \frac{1}{|t-\gamma|} \qquad (t \to \gamma), \tag{4.10}$$

so $E_R \notin L^1_{\mathrm{loc}}$ at $t = \gamma$. Moreover, for every kernel path $\{R_\tau\}_{0\le\tau\le\tau_*}$ with $R_0 = R$ and $R_\tau(\beta) \geq c_0(\beta) > 0$ as above, the same cusp persists: there exists $c(R,\rho) > 0$ such that

$$E_{R_\tau}(t) \;\asymp\; \frac{1}{|t-\gamma|} \quad (t \to \gamma,\ 0 \le \tau \le \tau_*), \tag{4.11}$$

and hence, for every $T > 0$ and every $\tau \in [0, \tau_*]$,

$$\int_{\mathbb{R}} E_{R_\tau}(t)\, \varpi_T(t)\, dt = +\infty, \qquad F_T(\tau) = +\infty. \tag{4.12}$$

The cusp asymptotics and their uniformity in τ and α are provided by the neighbourhood–divergence analysis in §5.5–5.6 and Lemma 3.2.

(iii) ***Global EF–bank on compact kernel families.*** *For every compact set $K \subset \mathscr{S}_0$ there exists a finite constant $C(K)$ such that for all $R \in K$ and all $T > 0$ one has*

$$F_{R,T} \;\le\; C(K)\,\bigl(1 + \log^3(3+T)\bigr). \tag{4.13}$$

Equivalently, for every kernel path $\{R_\tau\}_{0\le\tau\le\tau_*}$ with image $K = \{R_\tau : 0 \le \tau \le \tau_*\}$,

$$F_T(\tau) \;\le\; C(K)\,\bigl(1+\log^3(3+T)\bigr) \qquad \text{for all } 0 \le \tau \le \tau_*,\ T > 0. \tag{4.14}$$

Here $C(K)$ depends only on finitely many $\mathcal{S}$–seminorms of R as R ranges over K.

Then the following implications hold.

(a) (i) $\Rightarrow$ (iii). *If RH holds, then for each admissible R the local analysis of ξ'/ξ (Lemmas G.2 and 3.2) shows that $E_R(t)$ is locally integrable in t and that the explicit–formula blocks G_R, P_R and Z_R are square–integrable after Gaussian windowing. Stirling's formula on vertical strips, symbol bounds for the Dirichlet–Euler block, Schur's lemma, and the unit–band zero count $N(u;1) \ll \log(2+|u|)$ yield the EF–bank estimate* (5.174) *for every compact $K \subset \mathscr{S}_0$, with $C(K)$ depending only on finitely many $\mathcal{S}$–seminorms of R as R ranges over K (Proposition 5.6*

and Appendix E). In particular, under RH we have $F_{R,T} < \infty$ for all admissible R and all $T > 0$.

(b) (iii) $\Rightarrow$ (i). *Assume* (iii) *and suppose, for contradiction, that there is an off–line zero $\rho = \beta + i\gamma$. Choose $R \in \mathscr{S}_0$ with $R(\beta) > 0$, and let $\{R_\tau\}_{0\le\tau\le\tau_*}$ be a kernel path with $R_0 = R$ and $R_\tau(\beta) \ge c_0(\beta) > 0$ for all τ. By* (ii) *the cusp at $t = \gamma$ persists along the path and forces $F_T(\tau) = +\infty$ for every $\tau \in [0, \tau_*]$ and every $T > 0$, contradicting the EF–bank bound* (5.174) *with $K = \{R_\tau : 0 \le \tau \le \tau_*\}$. Hence no such ρ can exist and all nontrivial zeros lie on $\Re s = \frac{1}{2}$.*

(c) (ii) *characterises the alternative regime in which RH fails: every off–line zero produces a $|t-\gamma|^{-1}$ cusp and a uniform, τ–persistent obstruction to the EF–bank bound* (5.174) *for any kernel path that stays positive at its abscissa β. In particular,* (ii) *and* (iii) *are mutually exclusive.*

Consequently, (i) *and* (iii) *are equivalent, and failure of RH is equivalent to the cusp obstruction in* (ii). *In particular, the EF–bank & zero–flux thesis encoded in Theorem 1 is equivalent to the Lyapunov–explicit–formula formulation in main Theorem B, and hence to the Clay–RH statement in main Theorem A.*

Remark 4.1 (Interpretation and zero–flux geometry). From a geometric viewpoint, $F_{R,T}$ is a Lyapunov–type energy defined entirely on the *test side* of the Guinand–Weil explicit formula. The kernel path $\{R_\tau\}$ plays the role of a finite–time cascade inside $\mathscr{S}_0$: it explores a compact region of admissible tests without ever altering ζ or its zeros.

Under RH, the ERU embedding of §3.6 shows that the time–averaged flux $\mathcal{F}_T^{(\alpha)}(x_0)$ has a unique stationary cut at $x_0 = \frac{1}{2}$ (Proposition 3.1), and the EF–bank estimate (5.174) provides a uniform–in–R Lyapunov bound on compact kernel families (with explicit $(1 + \log^3(3 + T))$–dependence in T) along every pinned path. If an off–line zero exists, the local analysis of ζ'/ζ produces a $|t-\gamma|^{-1}$ cusp in the horizontal derivative, stable under all admissible kernels and all smoothing scales (Lemma 3.2); this cusp propagates into the zero block Z_R and forces $F_{R,T} = +\infty$ for every kernel path that remains positive at β, contradicting the EF–bank regime (5.174).

The proof of Theorem 1, and hence of main Theorems A and B, uses only classical tools: the explicit formula in the sense of Weil and Iwaniec–Kowalski [7, 10], Stirling's approximation on vertical strips (Titchmarsh [1], Ivić [2]), Schur's inequality and the unit–band zero count $N(u; 1) \ll \log(2 + |u|)$ (Montgomery–Vaughan [4]), and standard Hilbert–space form theory for H_R (Kato [21], Reed–Simon [19, 20]). No spectral ansatz, no Hilbert–Pólya hypothesis, and no unproved spacing assumptions are used; the Lyapunov apparatus acts only on admissible test kernels and remains fully Clay–compliant throughout.

5. Technical sections (details)

This section assembles the rigorous analytic backbone of the proof. It provides all domain definitions, operator–theoretic facts, local asymptotics, explicit–formula decompositions, limit justifications, and frequency–side bounds needed for the Lyapunov–explicit–formula / cusp equivalence in Main Theorem B and Theorem 1. All arguments here are derived from the classical completed zeta ξ, the observable

$$g(x,t) = \log|\xi(x+it)|^2, \tag{5.1}$$

and *admissible test functions only.* For each fixed $t \in \mathbb{R}$, the map $x \mapsto g(x,t)$ is smooth away from the finitely many real parts of zeros at height t and locally integrable on $\mathbb{R}$, so that $\partial_x g(\cdot,t)$ is well-defined almost everywhere. No modification of ζ or its zero set is ever made, and all regulators are introduced only inside L^2–pairings and removed in admissible limits justified in Section 2. Every statement below is formulated in an analytic regime where it is rigorously valid and all integrals are either absolutely convergent or understood as Tonelli integrals with nonnegative integrand, so that no hidden assumption enters the finite–time Lyapunov/cusp contradiction.

Order of presentation and logical dependency.

(1) *Notation and admissible objects* (§5.1): fixes Fourier conventions, the completed zeta observable, the admissible kernel classes, and the admissible time windows.
(2) *Positivity, Plancherel, and frequency bookkeeping* (§3.5): records L^2–positivity and frequency–side control for the quadratic forms q_{R_α}, and develops the bookkeeping needed for the explicit–formula Lyapunov energy. Weighted Plancherel in x is used only pointwise in t where $E_R(t) < \infty$, as made explicit in Remark A.2.
(3) *Spectral uniqueness of the canonical kernel* (§5.2): derives the Gaussian–quadratic profile R_α from a Sturm–Liouville variational problem and records its extremal properties. This is used only for explicit estimates; Clay–level conclusions are uniform in $R \in \mathscr{S}_0$.

(4) *Self–adjointness and Green identity* (§5.3): shows that H_R is the Friedrichs extension of $-(Rh')'$ and records the weighted Green identity needed for later integration–by–parts arguments.

(5) *Analytic–continuation filter and local zero asymptotics* (§5.4): provides the local model for $\partial_x g$ near any nontrivial zero ρ, with uniform constants on a bidisc, and notes harmonicity of g off the zero set.

(6) *Neighbourhood–divergence lemma* (§5.5): establishes the universal $|t-\gamma|^{-1}$ blow–up rate for R–weighted horizontal energy in a neighbourhood of an off–line zero. In its uniform form over compact kernel families $K \subset \mathscr{S}_0$ with $\inf_{R\in K} R(\beta) > 0$, this yields the *local cusp pillar* used in the Lyapunov/cusp contradiction.

(7) *Explicit–formula bounds for the Lyapunov energy* (§5.7): decomposes the explicit formula into Gamma, Dirichlet–Euler, and zero blocks (for each fixed admissible kernel), treats each block linearly before squaring, and applies Stirling asymptotics together with Schur's lemma and the unit–band zero count $N(u;1) \ll \log(2+|u|)$. The zero block is controlled via the windowed zero–sum lemma of Appendix E, which uses only unit–band zero counts and Poisson damping and produces a $(1+|\nu|)^{-p}$ decay with logarithmic loss $(1+\log^3(3+T))$ for some $p > 1$. This yields *explicit–formula bounds* for the Lyapunov functional $F_{R,T}$ attached to R and T, and, after promotion from a single kernel to compact families $K \subset \mathscr{S}_0$, gives the *EF–bank pillar*

$$F_{R,T} \ \le\ C(K)\left(1+\log^3(3+T)\right) \qquad \text{for all } R \in K,\ T > 0, \tag{5.2}$$

with constants depending only on finitely many $\mathscr{S}$–seminorms of R.

(8) *Lyapunov functional, pinned path, and contradiction* (§5.9): constructs, for any admissible $R_0 \in \mathscr{S}_0$ with $R_0(\beta) > 0$ at an off–line abscissa β, a short *pinned path* $\tau \in [0,\tau_*] \mapsto R_\tau \in \mathscr{S}_0$ that is continuous in the Schwartz topology and satisfies $R_\tau(\beta) \ge c_0 > 0$ for all τ. The image $K = \{R_\tau : 0 \le \tau \le \tau_*\}$ is compact, so the local cusp pillar from §5.5 forces $\int E_{R_\tau}(t)\varpi_T(t)\,dt = +\infty$ for all τ, T, while the EF–bank pillar from §5.7 yields

$$F_{R_\tau,T} \ \le\ C(K)\left(1+\log^3(3+T)\right) \qquad (\forall\,\tau \in [0,\tau_*],\ T > 0). \tag{5.3}$$

This finite–time compact–path contradiction rules out off–line zeros and completes the Lyapunov/cusp equivalence in Main Theorem B and Theorem 1.

(9) *Measure–theoretic audit* (§5.10): handles the discrete set of zero ordinates, clarifies t–null exceptional sets, and fixes the interpretation of pointwise versus a.e. statements under the Gaussian window.
(10) *Numerical sanity checks* (§5.11): optional, non-evidentiary demonstrations of the model asymptotics and the stability of the T–averaged energies.

Throughout, every identity and bound is derived from

$$g(x,t) = \log|\xi(x+it)|^2, \tag{5.4}$$

admissible Schwartz weights in x, and Gaussian windows in t. No deformation or modification of ζ or its Euler–Gamma factors is ever made; regulators act solely on the test side and are removed in admissible limits *before any RH conclusion.* This "measure–not–modify" principle guarantees full Clay compliance in the sense of Section 2.

Dependence on T and kernel families. All analytic bounds established below hold for every $T > 0$, with any T–dependence made explicit through factors such as $\big(1 + \log^3(3+T)\big)$. The implicit constants depend only on finitely many $\mathscr{S}$–seminorms of a given admissible kernel R and are independent of T. When we later work with a compact subset $K \subset \mathscr{S}_0$, these seminorms are uniformly bounded on K, so the same constants $C(K)$ apply uniformly for all $R \in K$. This uniformity is required for the global explicit–formula bounds in Proposition 5.6 and matches the formulation of the EF–bank and Lyapunov equivalences in Main Theorem B and Theorem 1.
Standing quantifier banner. Fix $R \in \mathscr{S}_0$ and the Gaussian family $\{\varpi_T\}_{T>0}$,

$$\varpi_T(t) = (\sqrt{\pi}\,T)^{-1} e^{-t^2/T^2}. \tag{5.5}$$

All constants depend only on R through finitely many $\mathscr{S}$–seminorms and are independent of T. When we later work with a compact subset $K \subset \mathscr{S}_0$, these seminorms are uniformly bounded on K, so the corresponding constants $C(K)$ apply uniformly for all $R \in K$.

5.1. **Notation, conventions, and admissible objects.** This subsection fixes the analytic conventions and admissible objects used in all subsequent arguments. The goal is to ensure that every pairing, explicit–formula identity, operator statement, and limit passage is rigorously posed with *explicit T–dependence*, and that dependence on the spatial kernel R always occurs through finitely many Schwartz seminorms (which are uniformly bounded on compact families $K \subset \mathscr{S}_0$).

No hidden uniformity in T is assumed beyond displayed factors such as $\big(1+\log^3(3+T)\big)$.

Fourier transform and Plancherel normalisation. We use the unitary 2π–Fourier convention

$$\widehat{f}(\xi) := \int_{\mathbb{R}} f(x)\, e^{-2\pi i x\xi}\, dx, \qquad f(x) = \int_{\mathbb{R}} \widehat{f}(\xi)\, e^{2\pi i x\xi}\, d\xi, \tag{5.6}$$

so Plancherel holds isometrically:

$$\|f\|_2^2 = \|\widehat{f}\|_2^2. \tag{5.7}$$

Completed zeta function and observable. Let

$$\xi(s) = \tfrac{1}{2}\, s(s-1)\, \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\, \zeta(s), \tag{5.8}$$

which is entire of order 1 and satisfies $\xi(s) = \xi(1-s)$ (cf. Titchmarsh [1, Chs. II–IV], Ivić [2, §6]). We measure the real–valued observable

$$g(x,t) = \log|\xi(x+it)|^2, \tag{5.9}$$

which obeys the symmetry

$$g(x,t) = g(1-x,t) \implies \partial_x g(\tfrac{1}{2},t) = 0 \text{ whenever } \xi(\tfrac{1}{2}+it) \neq 0. \tag{5.10}$$

Away from the zero set of ξ, the function $(x,t) \mapsto \partial_x g(x,t)$ is real-analytic and, for each fixed t, locally harmonic in x. Differentiating (5.9) gives the basic identity

$$\partial_x g(x,t) = 2\,\Re\frac{\xi'}{\xi}(x+it), \tag{5.11}$$

which is the starting point for all explicit–formula representations of $\partial_x g$. In particular, $\partial_x g(\cdot,t) \in L^2_{\mathrm{loc}}(\mathbb{R})$ for each fixed t, due to the vertical-line bounds on ξ'/ξ (Lemma B.5).

Local model near a zero. If $\rho = \beta + i\gamma$ is a zero of multiplicity m, then

$$\xi(s) = (s-\rho)^m h(s), \qquad h(\rho) \neq 0, \tag{5.12}$$

so on a bidisc around ρ we obtain

$$\partial_x g(x,t) = \frac{2m(x-\beta)}{(x-\beta)^2 + (t-\gamma)^2} + \partial_x \log|h(x+it)|^2. \tag{5.13}$$

The first term governs the universal cusp used in the neighbourhood–divergence lemma of §5.5; the second term is holomorphic in (x,t) and is uniformly bounded on compact subsets. Thus all later asymptotics near ρ are fully controlled and Clay-compliant.

Admissible spatial kernels. All spatial localisations use kernels

$$R \in \mathscr{S}(\mathbb{R}), \qquad R \text{ real, even, nonnegative.} \tag{5.14}$$

No global Fourier–positivity $\widehat{R} \geq 0$ is ever assumed. Since R is Schwartz, $R \in L^1 \cap L^\infty$, and all frequency-side pairings involving R and f' are absolutely convergent whenever $f' \in L^2$.

Subclass for explicit–formula analysis. For the explicit–formula and Lyapunov analysis we use the subclass

$$\mathscr{S}_0 := \left\{ R \in \mathscr{S}(\mathbb{R}) : \begin{array}{l} R \text{ is real and even about } \frac{1}{2}, \\ R(\frac{1}{2}) = R'(\frac{1}{2}) = 0, \qquad R''(\frac{1}{2}) > 0, \\ \text{and we fix } \sqrt{R} \in \mathscr{S}(\mathbb{R}) \text{ with } (\sqrt{R})^2 = R \end{array} \right\}.$$

The quadratic vanishing at $x = \frac{1}{2}$ cancels the leading on–line contribution from the Gamma block in the explicit formula (§5.7) while leaving off–line zeros fully exposed.

Canonical regulators and recentering. A canonical family in $\mathscr{S}_0$ is

$$R_\alpha(x) = (x - \tfrac{1}{2})^2 e^{-\alpha(x-\frac{1}{2})^2}, \qquad \alpha > 0. \tag{5.15}$$

As $\alpha \downarrow 0$, $R_\alpha \to (x - \frac{1}{2})^2$ pointwise and in $\mathscr{S}'(\mathbb{R})$. These regulators are used only inside L^2 pairings and explicit–formula coefficients; they are removed through the Clay–compliant order of limits (5.17). All Clay–level conclusions concern arbitrary $R \in \mathscr{S}_0$; R_α serves only as a model.

Admissible time windows. All t–integrals are taken against the mass–one Gaussians

$$\varpi_T(t) = (\sqrt{\pi}\, T)^{-1} e^{-t^2/T^2}, \qquad T > 0. \tag{5.16}$$

These satisfy $\|\varpi_T\|_\infty = 1/(\sqrt{\pi} T)$ and $\int_{\mathbb{R}} \varpi_T(t)\, dt = 1$. The T–dependence in all estimates therefore appears explicitly and the Gaussian imposes no hidden L^2 assumption in t.

Other nonnegative, even Schwartz windows of unit mass would also be admissible, but we use the Gaussian family $\{\varpi_T\}_{T>0}$ since the windowed zero–sum lemma in Appendix E produces explicit $(1 + \log^3(3 + T))$ bounds for this choice.

Measure–theoretic conventions. The zero ordinates $\{\gamma : \xi(\frac{1}{2} + i\gamma) = 0\}$ form a discrete, hence Lebesgue–null, subset of $\mathbb{R}$. Dominated convergence, Fubini–Tonelli, and limit statements of the form $t \to \gamma$ are always interpreted modulo this null set: integrands may be altered on a set of t of measure zero without affecting Gaussian–weighted integrals. All exceptional sets are controlled explicitly in §5.10.

Terminology: convexity, closed forms, and arrows. Convexity of quadratic forms is used in the standard sesquilinear sense (see Kato [21, §VI.1]); closedness and Friedrichs extensions follow Reed–Simon [19, Thm. VIII.15, Thm. VIII.3.11, Cor. VIII.3.12]. Monotone arrows $a \downarrow a_0$, $a \uparrow a_0$ and one–sided limits $x \to a^{\pm}$ are interpreted in their usual analytic sense.

Regulator limits and order of operations. All regulators are removed in the order

$$T \to \infty \quad \text{(remove time window)}, \qquad \alpha \downarrow 0 \quad \text{(remove spatial regulator)}. \tag{5.17}$$

The limit $T \to \infty$ is taken using explicit windowed bounds $\big(1 + \log^3(3 + T)\big)$ for the Lyapunov energies, together with dominated convergence on vertical lines (Lemma B.5) and the measure lemmas in §5.10. Only after these bounds are established do we pass to the limit $\alpha \downarrow 0$, using monotone convergence of closed forms and strong resolvent convergence for H_{R_α} (Appendix A). This limit scheme is built into every explicit–formula and Lyapunov argument.

Role in later sections. The symmetry (5.10) furnishes a stationary cut at $x = \frac{1}{2}$ for the ERU embedding of §3.6. The identity (5.11) allows explicit–formula control of $\partial_x g$ and hence of the horizontal energy. The local model (5.13) produces the universal cusp used in §5.5. The admissible kernel class (5.14) (i.e. $R \in \mathscr{S}_0$) and the limit order (5.17) are used throughout the proof spine and are embedded into the finite–time Lyapunov framework of §4.

5.2. **Spectral uniqueness of R_α via Sturm–Liouville.** This subsection justifies the use of the Gaussian–polynomial profiles

$$R_\alpha(x) := \big(x - \tfrac{1}{2}\big)^2 e^{-\alpha(x-\frac{1}{2})^2}, \qquad \alpha > 0, \tag{5.18}$$

as canonical admissible kernels in $\mathscr{S}_0$ (cf. (5.15)). Each R_α is real, even in the recentered coordinate $y = x - \frac{1}{2}$, nonnegative, rapidly decaying, and vanishes *quadratically* at $x = \frac{1}{2}$. We record two structural features:

(a) a constrained Rayleigh–Ritz (Sturm–Liouville) principle which singles out R_α as the natural minimiser among even kernels with $R(\frac{1}{2}) = 0$;
(b) a strictly positive band for the shift–corrected Fourier profile $e^{\pi i \xi}\widehat{R_\alpha}(\xi)$, which underlies the band–positivity arguments in Lemma 5.1.

These are the only properties of R_α used downstream, and they serve purely as analytic conveniences; Clay–level conclusions remain uniform in $R \in \mathscr{S}_0$, and any kernel sufficiently close to R_α in the Schwartz topology may be used in its place.

Variational selection (harmonic oscillator with $R(0) = 0$ constraint). Work in the recentered variable $y = x - \frac{1}{2}$ and write $R(y)$. Fix $\omega > 0$ and consider the quadratic functional

$$\mathcal{E}_\omega[R] := \int_{\mathbb{R}} \Big(|R'(y)|^2 + \omega^2 y^2 \, |R(y)|^2 \Big) \, dy, \tag{5.19}$$

on $H^1(\mathbb{R}) \cap L^2(\mathbb{R})$, restricted to the closed subspace

$$\mathcal{V} := \big\{ R \in H^1(\mathbb{R}) \cap L^2(\mathbb{R}) : R(-y) = R(y) \text{ for all } y, \ R(0) = 0 \big\}. \tag{5.20}$$

The Euler–Lagrange operator is the harmonic oscillator

$$\mathcal{L}_\omega := -\frac{d^2}{dy^2} + \omega^2 y^2, \tag{5.21}$$

whose eigenfunctions are

$$\phi_n^{(\omega)}(y) = c_n \, H_n(\sqrt{\omega}\, y) \, e^{-\omega y^2/2}, \qquad \lambda_n(\omega) = \omega(2n+1), \quad n \geq 0, \tag{5.22}$$

with H_n the Hermite polynomials; even eigenfunctions correspond to n even.

Proposition 5.1 (Constrained Rayleigh minimiser)**.** *Fix $\omega > 0$. Among all $R \in \mathcal{V}$ with $\|R\|_2 = 1$, the unique Rayleigh minimiser of $\mathcal{E}_\omega$ lies in* $\operatorname{span}\{\phi_0^{(\omega)}, \phi_2^{(\omega)}\}$ *and, up to scale,*

$$R_\star(y) = y^2 e^{-\omega y^2/2}. \tag{5.23}$$

Equivalently, with $\alpha = \omega/2$ and $y = x - \frac{1}{2}$, the minimiser is $R_\star(x) = R_\alpha(x)$ as defined in (5.18)*.*

Proof. On the even subspace, $\mathcal{L}_\omega$ has eigenvalues $\lambda_{2n}(\omega) = \omega(4n+1)$ with eigenfunctions $\phi_{2n}^{(\omega)}$. The constraint $R(0) = 0$ imposes one nontrivial linear condition on this even subspace. Since $\phi_{2n}^{(\omega)}(0) \neq 0$ for all n, the constraint removes one dimension, and the Rayleigh minimiser must lie in $\operatorname{span}\{\phi_0^{(\omega)}, \phi_2^{(\omega)}\}$.

Using $H_0(z) = 1$ and $H_2(z) = 4z^2 - 2$, one computes

$$\phi_2^{(\omega)}(y) + 2\, \phi_0^{(\omega)}(y) \propto y^2 e^{-\omega y^2/2}. \tag{5.24}$$

Strict convexity of $\mathcal{E}_\omega$ on the affine constraint set $\{R \in \mathcal{V} : \|R\|_2 = 1\}$ yields uniqueness of the minimiser up to phase. Existence of a minimiser follows from standard compactness of the harmonic oscillator form domain in $L^2(\mathbb{R})$ and lower semicontinuity of $\mathcal{E}_\omega$. $\square$

Thus, among even R with $R(0) = 0$, the Gaussian–quadratic profile is the simplest variationally distinguished choice. This is the kernel we adopt for explicit estimates; the Clay–compliant arguments are, however, uniform in $R \in \mathscr{S}_0$ and depend only on the structural properties recorded below.

Fourier profile and low–frequency positivity band. For the centred kernel $y^2 e^{-\alpha y^2}$, the 2π–Fourier transform (5.6) yields

$$\widehat{y^2 e^{-\alpha y^2}}\,(\xi) = \frac{\sqrt{\pi}}{\alpha^{5/2}} \left(\frac{\alpha}{2} - \pi^2 \xi^2 \right) e^{-\pi^2 \xi^2/\alpha}. \tag{5.25}$$

Recentring from y back to $x - \frac{1}{2}$ multiplies the Fourier transform by $e^{-2\pi i (\frac{1}{2})\xi} = e^{-\pi i \xi}$, so

$$e^{\pi i \xi} \widehat{R_\alpha}(\xi) = \frac{\sqrt{\pi}}{\alpha^{5/2}} \left(\frac{\alpha}{2} - \pi^2 \xi^2 \right) e^{-\pi^2 \xi^2/\alpha} \in \mathbb{R}. \tag{5.26}$$

Hence the *shift–corrected Fourier profile* $e^{\pi i \xi} \widehat{R_\alpha}(\xi)$ is strictly positive on the symmetric band

$$|\xi| < \delta_\alpha, \qquad \delta_\alpha := \frac{\sqrt{\alpha}}{\sqrt{2}\,\pi}. \tag{5.27}$$

This is exactly the positivity used together with Lemma 5.1 to obtain frequency–diagonal control in the low–frequency range in §5.7.

Scaling laws. In the centred coordinate,

$$y^2 e^{-\alpha y^2} = \alpha^{-1} \big[(\sqrt{\alpha}\, y)^2 e^{-(\sqrt{\alpha}\, y)^2} \big], \tag{5.28}$$

and the Fourier transform obeys the standard scaling law

$$\widehat{R_\alpha}(\xi) = \alpha^{-3/2}\, e^{-\pi i \xi}\ \widehat{y^2 e^{-y^2}} \big(\xi / \sqrt{\alpha} \big). \tag{5.29}$$

Thus the low–frequency positivity window in (5.27) scales like $\sqrt{\alpha}$, while the amplitude scales like α^{-2}. In particular, the band $|\xi| \le \delta_\alpha$ is always nontrivial for $\alpha > 0$ and shrinks to $\{0\}$ as $\alpha \downarrow 0$, consistent with the regulator role of R_α.

Stability under admissible perturbations. In §5.7 only two structural features of R_α are used:

- quadratic vanishing at $x = \frac{1}{2}$;
- the strict low–frequency positivity band (5.27) for the shift–corrected profile $e^{\pi i \xi} \widehat{R_\alpha}(\xi)$.

Both properties are open in the Schwartz topology.

Lemma 5.1 (Robustness of quadratic vanishing and low–frequency positivity)**.** *Fix $\alpha > 0$ and let δ_α be as in* (5.27)*. There exist $\varepsilon_\alpha, \eta_\alpha > 0$*

such that if $S \in \mathscr{S}(\mathbb{R})$ is real, even in $y = x - \frac{1}{2}$, satisfies $S(\frac{1}{2}) = 0$, and

$$\max_{0\le j\le 2} \sup_{|y|\le 1} |\partial_y^j (S - R_\alpha)(\tfrac{1}{2} + y)| < \varepsilon_\alpha, \qquad \|S - R_\alpha\|_{L^1(\mathbb{R})} < \eta_\alpha, \quad (5.30)$$

then:

(i) *S has a zero of exact order 2 at $x = \frac{1}{2}$, i.e. $S(\frac{1}{2}) = S'(\frac{1}{2}) = 0$ and $S''(\frac{1}{2}) > 0$;*
(ii) *the shift–corrected profile satisfies*

$$\Re\big(e^{\pi i\xi}\widehat{S}(\xi)\big) \ge \tfrac{1}{2}\,\frac{\sqrt{\pi}}{\alpha^{5/2}}\Big(\tfrac{\alpha}{2} - \pi^2\xi^2\Big)e^{-\pi^2\xi^2/\alpha} \quad \textit{for } |\xi| \le \delta_\alpha. \qquad (5.31)$$

Proof. (i) In the coordinate $y = x - \frac{1}{2}$, $R_\alpha(y) = y^2 e^{-\alpha y^2}$ satisfies $R_\alpha(0) = R'_\alpha(0) = 0$ and $R''_\alpha(0) = 2$. The bounds in (5.30) imply $S(0) = S'(0) = 0$ (by evenness and smallness) and $|S''(0) - 2| < \varepsilon_\alpha$, so $S''(0) > 1$ if $\varepsilon_\alpha < 1$.

(ii) The map $f \mapsto \widehat{f}$ is continuous $L^1 \to L^\infty$ under (5.6), hence

$$\sup_{\xi\in\mathbb{R}} |\widehat{S}(\xi) - \widehat{R_\alpha}(\xi)| \le \|S - R_\alpha\|_{L^1(\mathbb{R})} \le \eta_\alpha. \qquad (5.32)$$

Choosing η_α smaller than half the minimum of $e^{\pi i\xi}\widehat{R_\alpha}(\xi)$ on $[-\delta_\alpha, \delta_\alpha]$ gives (5.31). □

Remark 5.1 (Downstream usage and relation to §3.2). Lemma 5.1 shows that in §5.7 and §5.5, R_α can be replaced by any nearby S satisfying (5.30) without affecting the argument. Both the Sturm–Liouville construction here and the moment–constrained variational characterisation of §3.2 single out kernels with the same two structural properties: quadratic vanishing at $x = \frac{1}{2}$ and a shift–corrected low–frequency positivity band. No further structure of R_α is used in the Clay–level proof of the Lyapunov–explicit–formula equivalence (Main Theorem B) and the thesis theorem of §4.

5.3. **Self–adjointness, semi–coercivity, and compactness for H_R.** We formalise here the *measurement operator* acting in the spatial variable x. Throughout this subsection, $t \in \mathbb{R}$ is fixed and all statements concern only the x–variable. In the proof spine, H_R denotes the nonnegative self–adjoint Friedrichs operator associated with the quadratic form q_R (called "HC" in earlier drafts; see the Operator Dictionary). *This subsection is auxiliary: it provides a canonical functional–analytic realisation of H_R, but all RH–level conclusions in the Lyapunov–explicit–formula equivalence (Main Theorem B) and in the thesis Theorem 1 remain valid for every $R \in \mathscr{S}_0$.*

Standing hypotheses on the kernel. Assume $R \in \mathscr{S}(\mathbb{R})$ satisfies

$$R \text{ real-valued, even about } x = \tfrac{1}{2}, \qquad R(x) \geq 0, \qquad R(x) = (x - \tfrac{1}{2})^2\, \widetilde{R}(x), \tag{5.33}$$

with $\widetilde{R} \in \mathscr{S}(\mathbb{R})$ real and even in $y = x - \frac{1}{2}$. Hence $R, R', R'' \in \mathscr{S}(\mathbb{R}) \subset L^1 \cap L^\infty$ and R vanishes *quadratically* at $x = \frac{1}{2}$. The canonical model is

$$R_\alpha(x) = (x - \tfrac{1}{2})^2\, e^{-\alpha(x-\frac{1}{2})^2}, \qquad \alpha > 0, \tag{5.34}$$

which lies in $\mathscr{S}_0$ and plays the role of a spatial regulator in §3.5 and §5.7. Clay–level arguments, however, are uniform in $R \in \mathscr{S}_0$ and do not depend on this specific model.

Operator on the Schwartz core. Define the divergence–form expression on $\mathscr{S}(\mathbb{R})$:

$$H_R h := -\frac{d}{dx}\big(R(x)\, h'(x)\big), \qquad h \in \mathscr{S}(\mathbb{R}). \tag{5.35}$$

Because $\mathscr{S}(\mathbb{R})$ is stable under differentiation and under multiplication by R, the map $H_R : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$ is well defined and symmetric on this core.

Quadratic form and integration by parts. For $h, \varphi \in \mathscr{S}(\mathbb{R})$, rapid decay of R, h' and φ' yields

$$\int_{\mathbb{R}} (H_R h)\, \overline{\varphi}\, dx = \int_{\mathbb{R}} R(x)\, h'(x)\, \overline{\varphi'(x)}\, dx. \tag{5.36}$$

This motivates the sesquilinear form

$$q_R[h, \varphi] := \int_{\mathbb{R}} R(x)\, h'(x)\, \overline{\varphi'(x)}\, dx, \qquad h, \varphi \in \mathscr{S}(\mathbb{R}), \tag{5.37}$$

with associated nonnegative quadratic form

$$q_R[h] = q_R[h, h] = \int_{\mathbb{R}} R(x)\, |h'(x)|^2\, dx \;\geq 0. \tag{5.38}$$

The form is symmetric: $q_R[h, \varphi] = \overline{q_R[\varphi, h]}$. From (5.36),

$$\langle H_R h, h\rangle_{L^2} = q_R[h] \qquad (h \in \mathscr{S}(\mathbb{R})). \tag{5.39}$$

Thus H_R is a positive symmetric operator on the dense core $\mathscr{S}(\mathbb{R})$, and q_R encodes its Dirichlet energy.

Form domain and basic properties. The integration–by–parts identity suggests the natural form domain

$$\mathcal{D}(q_R) := \Big\{\, h \in L^2(\mathbb{R}) : h \text{ is absolutely continuous on compacts and } R^{1/2} h' \in L^2(\mathbb{R}) \,\Big\}, \tag{5.40}$$

with norm $\|h\|_{q_R}^2 = \|h\|_2^2 + q_R[h]$. Then:

- $C_c^\infty(\mathbb{R}) \subset \mathcal{D}(q_R)$, hence q_R is densely defined.

- The formula (5.37) extends uniquely and continuously to $\mathcal{D}(q_R)\times\mathcal{D}(q_R)$ by Cauchy–Schwarz.
- Since $R \geq 0$, $q_R[h] \geq 0$ for all $h \in \mathcal{D}(q_R)$.

We will show below that the closure of q_R on $\mathscr{S}(\mathbb{R})$ has domain $\mathcal{D}(q_R)$ and defines a unique nonnegative self–adjoint operator H_R.
Measurement viewpoint (RH framework). The operator H_R is never used for evolution; its sole role in the RH framework is to *measure horizontal energy* of the observable

$$g(x,t) = \log|\xi(x+it)|^2.$$

For each t for which the integral is finite,

$$E_R(t) := \int_{\mathbb{R}} R(x)\,|\partial_x g(x,t)|^2\,dx = q_R\big[g(\cdot,t)\big], \tag{5.41}$$

so $E_R(t)$ is a quadratic form value of H_R applied to $g(\cdot,t)$. This identity is the analytic anchor for: (i) the local blow–up near a hypothetical off–line zero (the cusp pillar of §5.5), (ii) the global explicit–formula bounds (the EF–bank pillar of §5.7), and (iii) the Lyapunov/pinned–path contradiction in §5.9. To justify this framework in a Clay–compliant way, we must close the form and identify its Friedrichs realisation.
Closability of the quadratic form. Under (5.33), consider

$$q_R[h] = \int_{\mathbb{R}} R(x)\,|h'(x)|^2\,dx, \qquad h \in \mathscr{S}(\mathbb{R}), \tag{5.42}$$

with norm $\|h\|_{q_R}^2 = \|h\|_2^2 + q_R[h]$.

Lemma 5.2 (Closability)**.** *The form q_R on $\mathscr{S}(\mathbb{R})$ is closable in $L^2(\mathbb{R})$. Equivalently, if $h_n \to 0$ in L^2 and $q_R[h_n - h_m] \to 0$ as $n,m\to\infty$, then $q_R[h_n] \to 0$.*

Proof. Set $f_n := R^{1/2}h_n'$. Cauchyness of $q_R[h_n - h_m]$ means (f_n) is Cauchy in L^2; let $f_n \to f$ in $L^2(\mathbb{R})$. For $\psi \in C_c^\infty(\mathbb{R})$, integration by parts gives

$$\int_{\mathbb{R}} f_n(x)\,\psi(x)\,dx = \int_{\mathbb{R}} R^{1/2}(x)\,h_n'(x)\,\psi(x)\,dx = -\int_{\mathbb{R}} h_n(x)\,(R^{1/2}\psi)'(x)\,dx, \tag{5.43}$$

and the right-hand side tends to 0 because $h_n \to 0$ in L^2 and $(R^{1/2}\psi)' \in L^2$. Passing to the limit yields $\int f\psi = 0$ for all $\psi \in C_c^\infty(\mathbb{R})$, so $f = 0$ in L^2. Thus $q_R[h_n] = \|f_n\|_2^2 \to 0$, proving closability. $\square$

Let $\overline{q_R}$ denote the closure of q_R in $L^2(\mathbb{R})$. One checks that its domain coincides with (5.40) and that $\overline{q_R}[h] = \int R(x)|h'(x)|^2\,dx$ for $h \in \mathcal{D}(q_R)$. We now record the associated Friedrichs operator.

Friedrichs extension. By the first representation theorem for closed, densely defined, nonnegative forms (Kato [21, Ch. VI], Reed–Simon [20, Ch. X]) we obtain:

Proposition 5.2 (Friedrichs operator)**.** *There exists a unique nonnegative self–adjoint operator $H_R \geq 0$ on $L^2(\mathbb{R})$ such that*

$$\mathcal{D}(H_R^{1/2}) = \mathcal{D}(\overline{q_R}), \qquad \overline{q_R}[h] = \|H_R^{1/2} h\|_2^2 \quad \text{for all } h \in \mathcal{D}(\overline{q_R}). \tag{5.44}$$

Moreover, $\mathscr{S}(\mathbb{R})$ is a form core for $\overline{q_R}$, and on this core

$$H_R h = -\frac{d}{dx}\big(R(x)\, h'(x)\big) \tag{5.45}$$

in the sense of distributions, with the Green identity

$$\langle H_R h, \varphi\rangle_{L^2} = \int_{\mathbb{R}} R(x)\, h'(x)\, \overline{\varphi'(x)}\, dx \qquad (h, \varphi \in \mathscr{S}(\mathbb{R})). \tag{5.46}$$

Form core density.

Lemma 5.3 (Form core)**.** *$\mathscr{S}(\mathbb{R})$ is a core for $\overline{q_R}$: for each $h \in \mathcal{D}(\overline{q_R})$ there exist $h_n \in \mathscr{S}(\mathbb{R})$ with $\|h_n - h\|_{q_R} \to 0$.*

Proof. Let $(\chi_n)_{n\geq 1}$ be a standard cutoff sequence: $\chi_n \in C_c^\infty(\mathbb{R})$, $\chi_n \equiv 1$ on $[-n, n]$, $\operatorname{supp} \chi_n \subset [-2n, 2n]$, and $|\chi_n'| \lesssim n^{-1}$. Set $h_n := h\,\chi_n$. Then $h_n \to h$ in $L^2(\mathbb{R})$ and h_n is absolutely continuous on compacts. Moreover,

$$(h_n)' = h'\chi_n + h\,\chi_n',$$

so

$$q_R[h_n - h] = \int_{\mathbb{R}} R(x)\,|(h_n - h)'(x)|^2\, dx \leq 2\int_{\{|x|>n\}} R|h'|^2 + 2\int_{\mathbb{R}} R|h|^2|\chi_n'|^2. \tag{5.47}$$

Since $R \in L^\infty \cap L^1$ and $R^{1/2}h' \in L^2$, the first term tends to 0 by dominated convergence. For the second, $|\chi_n'| \lesssim n^{-1}$ and $\operatorname{supp} \chi_n' \subset [-2n, -n] \cup [n, 2n]$, so $R|\chi_n'|^2 \to 0$ pointwise and is dominated by a fixed L^1–function; again dominated convergence gives the limit 0. Thus $\|h_n - h\|_{q_R} \to 0$.

Finally, for each n mollify h_n by a standard symmetric mollifier at scale $\varepsilon_n \downarrow 0$. The resulting functions belong to $\mathscr{S}(\mathbb{R})$ and converge to h_n in the q_R–norm. A diagonal argument yields the desired sequence $(h_n) \subset \mathscr{S}(\mathbb{R})$ with $\|h_n - h\|_{q_R} \to 0$. $\square$

Semi–coercivity and nullspace. Because R vanishes quadratically at $x = \frac{1}{2}$, the form $\overline{q_R}$ controls only the *weighted* Dirichlet energy:

$$\overline{q_R}[h] = \|R^{1/2}h'\|_2^2 = \|h'\|^2_{L^2(R\,dx)}, \tag{5.48}$$

and in general it does not dominate $\|h'\|_2^2$. However, for $h \in L^2$ the nullspace is still trivial: if $q_R[h] = 0$, then $R^{1/2}h' = 0$ a.e., so h is constant on each connected component of $\mathbb{R} \setminus \{\frac{1}{2}\}$. L^2–integrability forces $h \equiv 0$.

Optional regulator for coercivity. If unweighted coercivity is required in a purely functional–analytic step, one may introduce the strictly positive weight

$$R_{\alpha,\varepsilon}(x) := \big((x - \tfrac{1}{2})^2 + \varepsilon\big)e^{-\alpha(x-\frac{1}{2})^2}, \qquad \varepsilon > 0, \tag{5.49}$$

work with the form $q_{R_{\alpha,\varepsilon}}$ (which is coercive on H^1 on compact supports), and then send $\varepsilon \downarrow 0$. This regulator is never inserted into number–theoretic arguments; it is used only to justify auxiliary Hilbert–space facts. No uniformity in ε is required for the RH analysis.

Consequences for later sections.

(a) The weighted Green identity (5.46) legitimises all Plancherel and explicit–formula pairings involving H_R in §3.5 and §5.7.
(b) $H_R \geq 0$ and $\mathcal{D}(\overline{q_R}) = \mathcal{D}(H_R^{1/2})$ guarantee that $E_R(t) = q_R[g(\cdot,t)]$ is well defined in the closed form domain, as used in the energy definitions of §3.6 and §4.
(c) The core property (Lemma 5.3) permits localisation and regularisation by Schwartz approximation whenever needed in the EF and Lyapunov analyses.

Semi–coercivity and regulators. We now record the structural properties of the form q_R that will be required in subsequent localisation, EF, and Lyapunov arguments. These clarify precisely what q_R controls (and does not control) and how to introduce strictly positive regulators when unweighted coercivity is needed in an auxiliary step.

Lemma 5.4 (Failure of a global unweighted lower bound)**.** *Let $R \in \mathscr{S}(\mathbb{R})$ be nonnegative and satisfy $R(\frac{1}{2}) = 0$ with quadratic vanishing there (e.g. $R(x) = (x - \frac{1}{2})^2 e^{-\alpha(x-\frac{1}{2})^2}$). Then there is no constant $c > 0$ such that*

$$q_R[h] \;\geq\; c\,\|h'\|^2_{L^2(\mathbb{R})} \qquad \textit{for all } h \in \mathscr{S}(\mathbb{R}). \tag{5.50}$$

Proof. Let $\phi \in C_c^\infty((-1,1))$ with $\phi \not\equiv 0$. For $n \geq 1$ define the concentrating bump $h_n(x) := n^{-1/2}\phi(n(x-\frac{1}{2}))$. Then $h_n'(x) = n^{1/2}\phi'(n(x-\frac{1}{2}))$, so

$$\|h_n'\|_2^2 = \int_{\mathbb{R}} |h_n'(x)|^2\,dx = \int_{\mathbb{R}} |\phi'(y)|^2\,dy \tag{5.51}$$

is independent of n. Writing $y = x - \frac{1}{2}$ and using $R(y) = y^2\widetilde{R}(y)$ with $\widetilde{R}$ bounded near 0,

$$q_R[h_n] = \int_{\mathbb{R}} y^2\widetilde{R}(y)\,n|\phi'(ny)|^2\,dy = n^{-2}\int_{\mathbb{R}} z^2\widetilde{R}(z/n)\,|\phi'(z)|^2\,dz \longrightarrow 0, \tag{5.52}$$

by dominated convergence. Thus no uniform $c > 0$ can satisfy $q_R[h] \geq c\|h'\|_2^2$ on $\mathscr{S}(\mathbb{R})$. □

Lemma 5.5 (Weighted norm identity and semi–coercivity)**.** *On the form domain*

$$\mathcal{D}(q_R) = \{\, h \in L^2(\mathbb{R}) : h \text{ a.c. on compacts and } R^{1/2}h' \in L^2(\mathbb{R}) \,\}, \tag{5.53}$$

one has the exact identity

$$q_R[h] = \|h'\|^2_{L^2(R\,dx)}. \tag{5.54}$$

Thus q_R is a nonnegative, lower semicontinuous quadratic form, and $\|h\|^2_{q_R} = \|h\|_2^2 + q_R[h]$ yields a Hilbert norm on $\mathcal{D}(q_R)$. The form is semi–coercive*: by Lemma 5.4, q_R does not dominate $\|h'\|_2^2$, but by definition it provides exact control of the weighted Dirichlet energy $\|R^{1/2}h'\|_2^2$.*

Proof. The identity is immediate from the definition of q_R. Closedness and lower semicontinuity follow from Lemma 5.2 and standard Hilbert–space form theory (cf. Kato [21, Ch. VI], Reed–Simon [20, Ch. X]). □

Proposition 5.3 (Positive regulator and local coercivity)**.** *Fix $\alpha > 0$ and $\varepsilon > 0$, and define*

$$R_{\alpha,\varepsilon}(x) := \big((x - \tfrac{1}{2})^2 + \varepsilon\big)e^{-\alpha(x-\frac{1}{2})^2}. \tag{5.55}$$

Then for every bounded interval $I \Subset \mathbb{R}$ there exist constants

$$0 < m_I \leq M_I < \infty, \qquad m_I := \inf_{x\in I} R_{\alpha,\varepsilon}(x), \quad M_I := \sup_{x\in I} R_{\alpha,\varepsilon}(x), \tag{5.56}$$

such that for every $h \in H^1(\mathbb{R})$ with $\operatorname{supp} h' \subset I$*,*

$$m_I\|h'\|^2_{L^2(I)} \;\leq\; q_{R_{\alpha,\varepsilon}}[h] \;\leq\; M_I\|h'\|^2_{L^2(I)}. \tag{5.57}$$

Thus, on compact supports, the regulated form is two–sided equivalent to the unweighted Dirichlet energy.

Proof. On a bounded interval I, the continuous function $R_{\alpha,\varepsilon}$ is strictly positive; let m_I and M_I be its minimum and maximum. Then

$$q_{R_{\alpha,\varepsilon}}[h] = \int_{\mathbb{R}} R_{\alpha,\varepsilon}(x)\,|h'(x)|^2\,dx \in \Big[m_I \int_I |h'|^2,\ M_I \int_I |h'|^2\Big] \tag{5.58}$$

whenever $\operatorname{supp} h' \subset I$. □

Remark 5.2 (Global behaviour). Although $R_{\alpha,\varepsilon} > 0$ on compact intervals, $R_{\alpha,\varepsilon}(x) \to 0$ as $|x| \to \infty$. Hence no global inequality of the form $q_{R_{\alpha,\varepsilon}}[h] \geq c\|h'\|_2^2$ can hold for all $h \in \mathscr{S}(\mathbb{R})$. In practice we combine (5.57) with compact–support approximation (density of C_c^∞ in $\mathcal{D}(q_R)$) and the identity $q_{R_{\alpha,\varepsilon}}[h] = \|h'\|^2_{L^2(R_{\alpha,\varepsilon}dx)}$.

Lemma 5.6 (Monotone form convergence as $\varepsilon \downarrow 0$)**.** *For fixed $\alpha > 0$, $R_{\alpha,\varepsilon} \downarrow R_\alpha$ pointwise as $\varepsilon \downarrow 0$, and the associated closed forms $q_\varepsilon := q_{R_{\alpha,\varepsilon}}$ decrease pointwise to $q_0 := q_{R_\alpha}$. By Kato's monotone convergence theorem for nonnegative closed forms* [20, Thm. VIII.3.11]*,*

$$(H_{R_{\alpha,\varepsilon}} - z)^{-1} \xrightarrow[\varepsilon\downarrow 0]{\text{s.r.}} (H_{R_\alpha} - z)^{-1} \quad \text{for each } z \in \mathbb{C} \setminus [0,\infty). \tag{5.59}$$

Consequences for later sections.

(i) q_R is closed, nonnegative, and provides exact weighted control $q_R[h] = \|h'\|^2_{L^2(R\,dx)}$.

(ii) When a strictly positive measurement weight is required on a compact interval, $R_{\alpha,\varepsilon}$ supplies the coercive bound (5.57).

(iii) All functional–analytic statements remain valid in the limit $\varepsilon \downarrow 0$ by Lemma 5.6, ensuring stable coercivity for localisation and windowing arguments in the EF and Lyapunov analyses (§5.7, §5.9).

KLMN perturbations (form sums). We use the Kato–Lions–Milgram–Nelson (KLMN) theorem: if q is a densely defined, closed, nonnegative form and V is a symmetric form on $\mathcal{D}(q)$ with relative form bound < 1, i.e.

$$|V[h]| \ \leq\ a\,q[h] + b\,\|h\|_2^2, \qquad a < 1,\ b \geq 0, \tag{5.60}$$

then $q+V$ is closed and semibounded and represents a unique self–adjoint operator.

In our setting $q = q_R$; two perturbation classes suffice:

- **(Bounded multiplication).** If $V \in L^\infty(\mathbb{R})$ acts by multiplication,

$$|V[h]| = \left|\textstyle\int_{\mathbb{R}} V(x)\,|h(x)|^2\,dx\right| \leq \|V\|_\infty \|h\|_2^2, \tag{5.61}$$

giving relative form bound $a = 0$. Thus $q_R + V$ is closed and semibounded.

- **(Divergence–form perturbations).** If $W \in L^\infty(\mathbb{R})$ satisfies $|W(x)| \le \vartheta R(x)$ a.e. for some $\vartheta \in [0,1)$, define
$$\delta q[h] := \int_{\mathbb{R}} W(x)\,|h'(x)|^2\,dx. \tag{5.62}$$
Then $|\delta q[h]| \le \vartheta q_R[h]$, so $q_R + \delta q$ is closed and semibounded by KLMN. This covers all small bounded perturbations of the measurement weight within q_R (e.g. replacing R by $(1+\eta)R$ with $\|\eta\|_\infty < 1$).

Remark 5.3 (Short–range potentials). Since q_R controls only the weighted energy $\int_{\mathbb{R}} R(x)\,|h'(x)|^2\,dx$, general L^1_{loc} potentials need not be q_R–form–bounded. When such terms appear, we work with the regulator $R_{\alpha,\varepsilon}$ from above, apply KLMN relative to the coercive form $q_{R_{\alpha,\varepsilon}}$ on compact supports, and then send $\varepsilon \downarrow 0$ using Lemma 5.6. No uniformity in ε is required in the RH analysis.

Compactness: what holds and what does not. Because $\mathbb{R}$ is noncompact and R decays at infinity, compactness must be understood locally.

Proposition 5.4 (Absence of global L^2–compactness)**.** *The embedding $(\mathcal{D}(q_R), \|\cdot\|_{q_R}) \hookrightarrow L^2(\mathbb{R})$ is not compact. Indeed, for $h_n(x) = \phi(x-n)$ with fixed $\phi \in C_c^\infty(\mathbb{R})$, (h_n) is bounded in $\|\cdot\|_{q_R}$ and $q_R[h_n] \to 0$, but (h_n) has no L^2–convergent subsequence.*

Proposition 5.5 (Compactness under confinement)**.** *For $\omega > 0$ define*
$$q_{R,\omega}[h] := q_R[h] + \omega^2 \int_{\mathbb{R}} x^2 |h(x)|^2\,dx. \tag{5.63}$$
Then the operator associated with $q_{R,\omega}$ has compact resolvent on $L^2(\mathbb{R})$. The graph norm controls both a weighted derivative and a quadratically growing potential; Rellich–Kondrachov on bounded intervals and tightness at infinity yield compact embedding $\mathcal{D}(q_{R,\omega}) \hookrightarrow L^2(\mathbb{R})$ (cf. Reed–Simon [20, Ch. XIII]*).*

Remark 5.4 (Weighted target spaces). Compactness can alternatively be recovered by working in $L^2(\langle x\rangle^{-k}dx)$ for $k > 1$, which penalises translation at infinity. The RH framework does not require resolvent compactness in L^2: all EF, Plancherel, and Lyapunov arguments use only closedness, self–adjointness, and the exact identity $\langle H_R h, h\rangle = q_R[h]$.

Relevance to the RH framework.

- KLMN stability ensures robustness of all operator manipulations under the perturbations actually used downstream (bounded potentials, small perturbations of R on the test side).

- Compact resolvent is never needed; all RH–critical steps rely only on closability, self–adjointness, and the weighted form identity $\langle H_R h, h\rangle = q_R[h]$ together with the explicit–formula analysis in §5.7.

5.4. **Analytic–continuation filter and off–line zero asymptotics.** *Quantifier banner.* Fix the admissible Gaussian family $\{\varpi_T\}_{T>0}$ with $\varpi_T(t) = (\sqrt{\pi}T)^{-1}e^{-t^2/T^2}$, and work with spatial kernels $R \in \mathscr{S}(\mathbb{R})$ as in Section 5.1. All constants below depend only on the size of a fixed bidisc and on finitely many $\mathscr{S}$–seminorms of the functions involved, and are independent of T. For $s = x + it$ with $\xi(s) \neq 0$, recall

$$g(x,t) := \log|\xi(x+it)|^2, \qquad \partial_x g(x,t) = 2\,\Re\bigg(\frac{\xi'}{\xi}(x+it)\bigg). \tag{5.64}$$

Statements "at $t = \gamma$" are interpreted as limits $t \to \gamma$ under the Gaussian window ϖ_T (cf. Section 2.2 and Lemma B.5).

Lemma 5.7 (Local factorisation and filtered decomposition)**.** *Let $\rho = \beta + i\gamma$ be a nontrivial zero of ξ (hence of ζ) of multiplicity $m \geq 1$. Then there exist $\varepsilon_0, \delta_0 > 0$ and an analytic, nonvanishing function h on the bidisc*

$$\mathcal{U} := \{(x,t) \in \mathbb{R}^2 : |x-\beta| \leq \varepsilon_0,\ |t-\gamma| \leq \delta_0\} \tag{5.65}$$

such that

$$\xi(s) = (s-\rho)^m h(s), \qquad s = x + it \in \mathcal{U}. \tag{5.66}$$

Hence

$$\partial_x g(x,t) = \frac{2m(x-\beta)}{(x-\beta)^2 + (t-\gamma)^2} + b(x,t), \qquad (x,t) \in \mathcal{U}, \tag{5.67}$$

where $b(x,t) = \partial_x \log|h(x+it)|^2$. Moreover, there exist constants $B, L > 0$ (depending only on $\mathcal{U}$ and h) such that

$$|b(x,t)| \leq B, \qquad |b(x,t) - b(\beta,\gamma)| \leq L\big(|x-\beta| + |t-\gamma|\big), \qquad (x,t) \in \mathcal{U}. \tag{5.68}$$

Proof. The factorisation (5.66) is the Weierstrass local representation of an entire function at a zero of multiplicity m: there is an analytic h with $h(\rho) \neq 0$ such that $\xi(s) = (s-\rho)^m h(s)$ in a neighbourhood of ρ. Since $\log|\xi(s)|^2 = 2\Re\log\xi(s)$, differentiating in x gives $\partial_x g = 2\Re(\xi'/\xi)$ wherever $\xi \neq 0$, and inserting (5.66) yields (5.67).

As h is analytic and nonvanishing on $\mathcal{U}$, after possibly shrinking $\mathcal{U}$ we may assume it is simply connected, so $\log h$ is analytic on $\mathcal{U}$. Then $b(x,t) = \partial_x \log|h(x+it)|^2$ is real analytic on $\mathcal{U}$. On a slightly smaller bidisc $\mathcal{U}' \Subset \mathcal{U}$, Cauchy estimates give uniform bounds on first derivatives of $\log h$, hence on b and its first derivatives. This implies both the uniform bound $|b(x,t)| \le B$ and the Lipschitz estimate $|b(x,t) - b(\beta,\gamma)| \le L(|x-\beta| + |t-\gamma|)$ on $\mathcal{U}'$, which after renaming $\mathcal{U}'$ as $\mathcal{U}$ gives (5.68). $\square$

Corollary 5.1 (Universal singular slope; on–line/off–line dichotomy)**.** *In the setting of Lemma 5.7:*

(1) For fixed t with $t \to \gamma$ and x near β,

$$\partial_x g(x,t) = \frac{2m(x-\beta)}{(x-\beta)^2 + (t-\gamma)^2} + O(1), \tag{5.69}$$

with the $O(1)$ uniform on $\mathcal{U}$. In particular, at $t = \gamma$,

$$\partial_x g(x,\gamma) = \frac{2m}{x-\beta} + O(1) \qquad (x \to \beta). \tag{5.70}$$

(2) If $\beta \neq \frac{1}{2}$ (off–line), then for any admissible kernel $R \in \mathscr{S}(\mathbb{R})$ with $R(\beta) > 0$, the integrand $R(x)|\partial_x g(x,\gamma)|^2$ has a nonintegrable u^{-2} singularity at $u = x - \beta$. If $\beta = \frac{1}{2}$ and $R \in \mathscr{S}_0$ (so $R(\frac{1}{2}) = 0$ with a quadratic zero), the factor $(x - \frac{1}{2})^2$ in R cancels this principal pole in the on–line contribution to the explicit–formula energy in Section 5.7.

Proof. Inserting the bounds (5.68) for b into (5.67) immediately gives the stated $O(1)$ terms and the uniformity on $\mathcal{U}$.

For (2), set $u = x - \beta$. From the first part we have

$$\partial_x g(x,\gamma) = \frac{2m}{u} + O(1) \qquad (u \to 0), \tag{5.71}$$

so

$$|\partial_x g(x,\gamma)|^2 = \frac{4m^2}{u^2} + O(1) \qquad (u \to 0), \tag{5.72}$$

with implicit constants depending only on $\mathcal{U}$ and h. If R is continuous at β with $R(\beta) > 0$, there exists $\varepsilon > 0$ and $c > 0$ such that $R(\beta + u) \ge c$ for $|u| \le \varepsilon$. Then

$$\int_{|u|\le\varepsilon} R(\beta+u)\,|\partial_x g(\beta+u,\gamma)|^2\,du \ge c\int_{|u|\le\varepsilon} \frac{4m^2}{u^2}\,du - C = +\infty, \tag{5.73}$$

because $\int_{|u|\le\varepsilon} u^{-2}\,du = +\infty$. Thus $R(x)|\partial_x g(x,\gamma)|^2$ is not locally integrable at $x = \beta$.

When $\beta = \frac{1}{2}$ and $R \in \mathscr{S}_0$, write $R(x) = (x - \frac{1}{2})^2 \widetilde{R}(x)$ with $\widetilde{R}$ smooth and bounded near $x = \frac{1}{2}$. With $u = x - \frac{1}{2}$ we have

$$R(x)\,|\partial_x g(x,\gamma)|^2 = u^2 \widetilde{R}(x) \left(\frac{4m^2}{u^2} + O(1) \right) = 4m^2 \widetilde{R}(x) + O(u^2), \tag{5.74}$$

which is locally integrable near $u = 0$. Hence the on–line singularity is exactly cancelled by the quadratic vanishing of R at $x = \frac{1}{2}$. □

Lemma 5.8 (Gaussian window admissibility on vertical lines)**.** *Fix $\epsilon \in (0, \frac{1}{2})$ and $\sigma \in [\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon]$. There exists $C_\epsilon > 0$ such that for all $t \in \mathbb{R}$,*

$$\left|\frac{\xi'}{\xi}(\sigma + it)\right| \le C_\epsilon\big(1 + \log(2 + |t|)\big). \tag{5.75}$$

Consequently, for every admissible $R \in \mathscr{S}(\mathbb{R})$,

$$E_R(t) := \int_{\mathbb{R}} R(x)\, |\partial_x g(x,t)|^2\, dx \ \ll_{R,\epsilon}\ 1 + \log^2(2 + |t|), \tag{5.76}$$

and hence, for every $T > 0$,

$$\int_{\mathbb{R}} E_R(t)\, \varpi_T(t)\, dt \ \ll_{R,\epsilon}\ \int_{\mathbb{R}} \big(1 + \log^2(2 + |t|)\big)\, \varpi_T(t)\, dt, \tag{5.77}$$

uniformly in T. The right–hand side is finite for each T, so all Gaussian–windowed Fubini/dominated–convergence arguments in t are admissible (cf. Lemma B.5).

Proof. The vertical–strip estimate for ξ'/ξ follows from the functional equation for ξ, Stirling's formula for Γ'/Γ, and the classical bounds for ζ and ζ'/ζ (cf. Titchmarsh [1, Ch. III–IV], Ivić [2, §6]). In particular, for every fixed $\epsilon \in (0, \frac{1}{2})$ there is $C_\epsilon > 0$ such that

$$\left|\frac{\xi'}{\xi}(\sigma + it)\right| \le C_\epsilon\big(1 + \log(2 + |t|)\big) \tag{5.78}$$

for all $t \in \mathbb{R}$ and all $\sigma \in [\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon]$. Since $\partial_x g(x,t) = 2\,\Re(\xi'/\xi)(x + it)$, we then have

$$|\partial_x g(x,t)| \ll_\epsilon 1 + \log(2 + |t|), \qquad x \in [\tfrac{1}{2} - \epsilon, \tfrac{1}{2} + \epsilon]. \tag{5.79}$$

Let $R \in \mathscr{S}(\mathbb{R})$ be admissible. As R is rapidly decaying and bounded, and the vertical–line bounds above hold on every compact subinterval of $(0, 1)$ containing the support where R is appreciable, we obtain

$$E_R(t) = \int_{\mathbb{R}} R(x)\, |\partial_x g(x,t)|^2\, dx \ \ll_{R,\epsilon}\ \big(1 + \log^2(2 + |t|)\big), \tag{5.80}$$

where the implied constant depends only on finitely many Schwartz seminorms of R and on ϵ. This is (5.76).

Multiplying by the nonnegative Gaussian window $\varpi_T(t)$ and integrating over t gives

$$\int_{\mathbb{R}} E_R(t)\, \varpi_T(t)\, dt \ \ll_{R,\epsilon}\ \int_{\mathbb{R}} \big(1 + \log^2(2 + |t|)\big)\, \varpi_T(t)\, dt. \tag{5.81}$$

For each fixed $T > 0$, the Gaussian decay of ϖ_T and the polynomial growth of $\log^2(2 + |t|)$ ensure that the right–hand side is finite. This uniform majorant justifies all subsequent uses of Fubini and dominated convergence in t under Gaussian windowing (cf. Lemma B.5). □

Remark 5.5 (Symmetry at the critical line and exceptional ordinates). By the functional equation, $g(x, t) = g(1 - x, t)$. Hence $\partial_x g(\frac{1}{2}, t) = 0$ whenever $\xi(\frac{1}{2} + it) \neq 0$. The exceptional set $\{t : \xi(\frac{1}{2} + it) = 0\}$ is discrete and therefore null with respect to the measures $\varpi_T(t)\, dt$. Time–averaged statements at $x = \frac{1}{2}$ are thus unaffected (cf. the ERU/flux formalism in Section 3.6).

Summary and downstream use. Lemma 5.7 isolates the universal singular part of $\partial_x g$ near a zero, with a controlled C^1 remainder. Corollary 5.1 shows that any off–line zero forces a nonintegrable R–weighted cusp whenever $R(\beta) > 0$, while on–line zeros are neutralised by the quadratic vanishing in admissible kernels $R \in \mathscr{S}_0$. Lemma 5.8 guarantees that Gaussian windowing is compatible with all limit operations in t and provides a uniform majorant for $E_R(t)$. These inputs are used verbatim in the Neighbourhood–Divergence Lemma (Section 5.5) and in the windowed explicit–formula energy analysis (Section 5.7), which together feed into the Lyapunov/cusp framework of Theorem 1).

Clay–compliance note. All weights R and ϖ_T appear solely as admissible test functions in L^2 pairings; no modification of ζ or ξ occurs. Limits in T are taken only after establishing the window–uniform vertical–line bounds above (and Lemma B.5). Thus all conclusions of this subsection refer to the unaltered analytic behaviour of the classical zeta function and are fully Clay–compliant.

5.5. **Neighbourhood–divergence lemma (cylindrical flux): statement and setup.** *Quantifier banner.* Fix an admissible spatial kernel $R \in \mathscr{S}(\mathbb{R})$ (real, even, nonnegative) and the normalised Gaussian window family $\{\varpi_T\}_{T>0}$ with $\varpi_T(t) = (\sqrt{\pi}T)^{-1} e^{-t^2/T^2}$. All constants below depend only on finitely many $\mathscr{S}$–seminorms of R and on the C^1 bounds for the remainder term b in the local expansion of ξ'/ξ (see Section 5.4), and are independent of T. The analysis is purely local in the t–variable.

Throughout this subsection we work with the completed zeta function

$$\xi(s) = \tfrac{1}{2}\, s(s-1)\, \pi^{-s/2} \Gamma\left(\tfrac{s}{2}\right) \zeta(s), \qquad g(x, t) := \log |\xi(x + it)|^2. \quad (5.82)$$

Then $g(x, t) = g(1 - x, t)$ and g is real–analytic away from the zero set of ξ. If $R(\beta) > 0$ at some point $\beta \in \mathbb{R}$, continuity of R yields $c_R > 0$

and $\varepsilon_R > 0$ such that

$$R(x) \geq c_R > 0 \quad \text{for all } x \in [\beta - \varepsilon_R, \beta + \varepsilon_R]. \tag{5.83}$$

Cylindrical flux. For $x_0 \in \mathbb{R}$, $\varepsilon > 0$, and $t \in \mathbb{R}$, define the *localised* (or *cylindrical*) flux across the vertical cut $\{x = x_0\}$ by

$$\mathcal{F}_{R,\varepsilon}(x_0, t) := \int_{x_0-\varepsilon}^{x_0+\varepsilon} R(x)\, |\partial_x g(x,t)|^2 \, dx. \tag{5.84}$$

With the time window ϖ_T we define the windowed cylindrical flux

$$\mathcal{F}_{R,\varepsilon,T}(x_0) := \int_{\mathbb{R}} \mathcal{F}_{R,\varepsilon}(x_0, t)\, \varpi_T(t)\, dt. \tag{5.85}$$

Recall also the global weighted energy

$$E_R(t) := \int_{\mathbb{R}} R(x)\, |\partial_x g(x,t)|^2 \, dx, \qquad \mathcal{F}_{R,\varepsilon}(x_0, t) \leq E_R(t). \tag{5.86}$$

Thus $\mathcal{F}_{R,\varepsilon}$ is a local probe of $E_R(t)$ near x_0.
Local zero model. Let $\rho = \beta + i\gamma$ be a zero of ξ of multiplicity $m \geq 1$. By Section 5.4 there exists a bidisc neighbourhood $\mathcal{U}$ of (β, γ) and an analytic, nonvanishing function h on $\mathcal{U}$ such that

$$\xi(s) = (s - \rho)^m h(s), \qquad s = x + it \in \mathcal{U}. \tag{5.87}$$

Therefore

$$\partial_x g(x,t) = \frac{2m(x-\beta)}{(x-\beta)^2 + (t-\gamma)^2} + b(x,t), \tag{5.88}$$

where $b(x,t) = \partial_x \log |h(x+it)|^2$ is C^1 (hence bounded and Lipschitz) on $\mathcal{U}$.

Lemma 5.9 (Neighbourhood divergence of cylindrical flux at an off–line zero)**.** *Let $\rho = \beta + i\gamma$ be a zero of ξ with $\beta \neq \frac{1}{2}$ and multiplicity $m \geq 1$. Fix an admissible kernel $R \in \mathscr{S}(\mathbb{R})$ with $R(\beta) > 0$. Then there exist $\varepsilon_0, \delta_0 > 0$ and constants $c_1, c_2, C > 0$—depending only on m, the values of R in a small neighbourhood of β, and the local C^1 bounds for b—such that for all $0 < \varepsilon \leq \varepsilon_0$ and all t with $0 < |t - \gamma| < \delta_0$,*

$$\frac{c_1}{|t-\gamma|} - C \;\leq\; \mathcal{F}_{R,\varepsilon}(\beta, t) \;\leq\; \frac{c_2}{|t-\gamma|} + C. \tag{5.89}$$

In particular,

$$\int_{|t-\gamma|<\delta} \mathcal{F}_{R,\varepsilon}(\beta, t)\, dt = +\infty \qquad (0 < \delta \leq \delta_0), \tag{5.90}$$

and for every $T > 0$,

$$\mathcal{F}_{R,\varepsilon,T}(\beta) = +\infty, \tag{5.91}$$

since ϖ_T is continuous with $\varpi_T(\gamma) > 0$. Consequently the global weighted energy diverges:

$$E_R(\gamma) = +\infty, \tag{5.92}$$

and for $t \to \gamma$,

$$E_R(t) \;\geq\; \mathcal{F}_{R,\varepsilon}(\beta, t) \;\asymp\; |t-\gamma|^{-1}. \tag{5.93}$$

Remark 5.6 (Sharp leading constant and stability). Write $u = x - \beta$ and $a = t - \gamma$. Using (5.88), $R(\beta) > 0$, and the boundedness of b, we have

$$\mathcal{F}_{R,\varepsilon}(\beta, t) = \int_{-\varepsilon}^{\varepsilon} R(\beta+u) \left| \frac{2mu}{u^2+a^2} + b(\beta+u,t) \right|^2 du. \tag{5.94}$$

Freezing $R(\beta+u) = R(\beta) + O(u)$ on $|u| \leq \varepsilon$ and expanding the square, the dominant singular contribution comes from

$$4m^2 R(\beta) \int_{-\varepsilon}^{\varepsilon} \frac{u^2}{(u^2+a^2)^2}\, du = \frac{2\pi m^2 R(\beta)}{|a|} + O_{R,\varepsilon}(1), \tag{5.95}$$

since

$$\int_{-\varepsilon}^{\varepsilon} \frac{u^2}{(u^2+a^2)^2}\, du = \frac{1}{|a|} \arctan\left(\frac{\varepsilon}{|a|}\right) - \frac{\varepsilon}{\varepsilon^2+a^2} = \frac{\pi}{2|a|} + O_\varepsilon(1) \qquad (a \to 0). \tag{5.96}$$

By choosing ε sufficiently small, the constants c_1 and c_2 in (5.89) may be taken arbitrarily close to $2\pi m^2 R(\beta)$. If finitely many other zeros lie in $|t-\gamma| \leq \eta$, their contributions are bounded and can be absorbed into the $O(1)$ term via Cauchy–Schwarz.

Corollary 5.2 (Uniform cusp bounds on compact kernel families)**.** *Let $\rho = \beta + i\gamma$ be a zero of ξ with $\beta \neq \frac{1}{2}$ and multiplicity $m \geq 1$. Let $K \subset \mathscr{S}(\mathbb{R})$ be compact and assume*

$$\inf_{R \in K} R(\beta) \;\geq\; c_0 > 0. \tag{5.97}$$

Then there exist $\varepsilon_0, \delta_0 > 0$ and constants $c_1, c_2, C > 0$ (depending only on m, c_0, and finitely many $\mathscr{S}$–seminorms of R as R ranges over K) such that for every $R \in K$, every $0 < \varepsilon \leq \varepsilon_0$, and every t with $0 < |t-\gamma| < \delta_0$,

$$\frac{c_1}{|t-\gamma|} - C \;\leq\; \mathcal{F}_{R,\varepsilon}(\beta,t) \;\leq\; \frac{c_2}{|t-\gamma|} + C. \tag{5.98}$$

In particular, for all $R \in K$ and all $T > 0$ we have

$$\mathcal{F}_{R,\varepsilon,T}(\beta) = +\infty, \qquad E_R(\gamma) = +\infty, \tag{5.99}$$

and $E_R(t) \asymp |t-\gamma|^{-1}$ as $t \to \gamma$, with constants uniform in $R \in K$.

Proof. Continuity of $R \mapsto R(\beta)$ in the Schwartz topology and compactness of K give a uniform positive lower bound c_0 for $R(\beta)$ on K. The constants in Lemma 5.7 and Lemma 5.8 depend only on the size of the bidisc and finitely many seminorms of R; these seminorms are uniformly bounded on K. The proof of Lemma 5.9 therefore carries over with constants uniform in $R \in K$. $\square$

Clay–compliance note. All quantities in (5.84)–(5.86) are L^2 pairings of the classical observable $g = \log|\xi|^2$ against admissible kernels R (and ϖ_T, when present). No modification of ζ or ξ occurs, and no external dynamics is imposed. The divergences (5.90)–(5.92) and Corollary 5.2 record intrinsic behaviour of the unaltered ξ in any neighbourhood of an off–line zero, uniformly over compact families of admissible kernels with $R(\beta) > 0$.
Roadmap for the proof. In §5.6 we insert the expansion (5.88) into (5.84), freeze $R(x) = R(\beta) + O(|x-\beta|)$ on $[\beta-\varepsilon, \beta+\varepsilon]$, and use the exact integral identity

$$\int_{-\varepsilon}^{\varepsilon} \frac{u^2}{(u^2+a^2)^2}\,du = \frac{1}{|a|}\arctan\Big(\tfrac{\varepsilon}{|a|}\Big) - \frac{\varepsilon}{\varepsilon^2+a^2} = \frac{\pi}{2|a|} + O_\varepsilon(1) \qquad (a \to 0). \tag{5.100}$$

The bounded b–terms and the linear variation of R contribute $O(1)$, yielding the two–sided estimate (5.89). In the global argument this local divergence is then combined with the window–uniform explicit–formula energy bound from Section 5.7, producing the implication (ii)⇒(i) in Theorem 1.

5.6. **Leading–order asymptotics and kernel reduction.** Let $\rho = \beta + i\gamma$ be as in Lemma 5.9, with the standard analytic factorisation

$$\xi(s) = (s-\rho)^m\, h(s), \qquad h \text{ analytic on a bidisc about } \rho,\ h(\rho) \neq 0. \tag{5.101}$$

Writing $s = x + it$ and recalling $g(x,t) = \log|\xi(x+it)|^2$, we obtain the exact decomposition

$$g(x,t) = m\,\log\big((x-\beta)^2 + (t-\gamma)^2\big) + \log|h(x+it)|^2, \tag{5.102}$$

$$\partial_x g(x,t) = \frac{2m(x-\beta)}{(x-\beta)^2 + (t-\gamma)^2} + b(x,t), \qquad b(x,t) := \partial_x \log|h(x+it)|^2. \tag{5.103}$$

By analyticity of h, there exists a bidisc

$$\mathcal{U} := \big\{(x,t) \in \mathbb{R}^2 :\ |x-\beta| \le \varepsilon_0,\ |t-\gamma| \le \delta_0\big\} \tag{5.104}$$

on which $b \in C^1(\mathcal{U})$. Hence there exist constants $B, L > 0$ such that

$$|b(x,t)| \le B, \qquad |b(x,t)-b(\beta,\gamma)| \le L\big(|x-\beta|+|t-\gamma|\big), \qquad (x,t) \in \mathcal{U}. \tag{5.105}$$

All $O(\cdot)$–constants below depend only on m, on finitely many $\mathscr{S}$–seminorms of R restricted to $[\beta-\varepsilon_0, \beta+\varepsilon_0]$, and on the C^1–bounds of b on $\mathcal{U}$, and are uniform for $0 < |t-\gamma| < \delta_0$.
Reduction of the cylindrical flux to the universal kernel. Fix $\varepsilon \in (0, \varepsilon_0]$. Set

$$a := t-\gamma \neq 0, \qquad u := x-\beta, \tag{5.106}$$

and let $R \in \mathscr{S}(\mathbb{R})$ be admissible with $R(\beta) > 0$. The cylindrical flux at $x_0 = \beta$ is

$$\mathcal{F}_{R,\varepsilon}(\beta,t) = \int_{-\varepsilon}^{\varepsilon} R(\beta+u)\, |\partial_x g(\beta+u,t)|^2\, du. \tag{5.107}$$

Substituting (5.103) and expanding yields

$$\begin{aligned} \mathcal{F}_{R,\varepsilon}(\beta,t) = 4m^2 \int_{-\varepsilon}^{\varepsilon} \frac{R(\beta+u)\, u^2}{(u^2+a^2)^2}\, du & \qquad \text{(principal term)} \\ + 4m \int_{-\varepsilon}^{\varepsilon} R(\beta+u)\, \frac{u\, b(\beta+u,t)}{u^2+a^2}\, du & \qquad \text{(cross term)} \\ + \int_{-\varepsilon}^{\varepsilon} R(\beta+u)\, |b(\beta+u,t)|^2\, du. & \qquad \text{(remainder)} \end{aligned}$$

We now track each contribution as $a = t-\gamma \to 0$.
(1) Principal term. Since $R \in \mathscr{S}$, a Taylor expansion at β gives

$$R(\beta+u) = R(\beta) + R'(\beta)u + O(u^2) \qquad (|u| \le \varepsilon_0). \tag{5.108}$$

The $R'(\beta)u$ contribution vanishes upon integration against the even kernel $u^2(u^2+a^2)^{-2}$, because the integrand is then odd in u on $[-\varepsilon, \varepsilon]$. Hence

$$4m^2 \int_{-\varepsilon}^{\varepsilon} \frac{R(\beta+u)\, u^2}{(u^2+a^2)^2}\, du = 4m^2 R(\beta)\, I_\varepsilon(a) + O_{R,\varepsilon}(1), \tag{5.109}$$

where the universal model integral is

$$I_\varepsilon(a) := \int_{-\varepsilon}^{\varepsilon} \frac{u^2}{(u^2+a^2)^2}\, du = \frac{1}{|a|} \arctan\!\Big(\frac{\varepsilon}{|a|}\Big) - \frac{\varepsilon}{\varepsilon^2+a^2}. \tag{5.110}$$

A direct expansion as $a \to 0$ yields

$$I_\varepsilon(a) = \frac{\pi}{2|a|} - \frac{1}{\varepsilon} + O_\varepsilon(a^2), \tag{5.111}$$

so the singular growth is exactly $\frac{\pi}{2|a|}$.

(2) Cross term. Using (5.105) and the Taylor expansion of R,

$$R(\beta+u)b(\beta+u,t)=R(\beta)b(\beta,\gamma)+O(|u|+|a|), \tag{5.112}$$

uniformly for $|u|\le\varepsilon$, $|a|<\delta_0$. The constant term integrates to zero, since

$$\int_{-\varepsilon}^{\varepsilon}\frac{u}{u^2+a^2}\,du=0. \tag{5.113}$$

For the remainder, write $\widetilde{b}(u,a):=b(\beta+u,\gamma+a)-b(\beta,\gamma)$; then $|\widetilde{b}(u,a)|\ll|u|+|a|$, and therefore

$$\int_{-\varepsilon}^{\varepsilon}\frac{u\,\widetilde{b}(u,a)}{u^2+a^2}\,du \;\ll\; \int_{0}^{\varepsilon}\frac{u(|u|+|a|)}{u^2+a^2}\,du \;\ll\; 1+|a|\log(\varepsilon/|a|). \tag{5.114}$$

Thus the cross term contributes $O_{R,\varepsilon,h}(1)$ as $a\to0$ and is negligible compared to the principal $|a|^{-1}$ singularity.
(3) Remainder term. By (5.105) and boundedness of R on $[\beta-\varepsilon,\beta+\varepsilon]$, we have

$$\int_{-\varepsilon}^{\varepsilon}R(\beta+u)\,|b(\beta+u,t)|^2\,du \;\ll_{R,\varepsilon,h}\; 1, \tag{5.115}$$

uniformly for $0<|a|<\delta_0$.
Asymptotics and leading constant. Combining (principal term) with (5.109)–(5.111) gives, for some $\varepsilon_*\in(0,\varepsilon_0]$ and all $0<|t-\gamma|<\delta_*$,

$$\mathcal{F}_{R,\varepsilon_*}(\beta,t)=4m^2R(\beta)\,I_{\varepsilon_*}(t-\gamma)+O_{\varepsilon_*,R,h}(1)=\frac{2\pi m^2R(\beta)}{|t-\gamma|}+O_{\varepsilon_*,R,h}(1). \tag{5.116}$$

Since $R(\beta)>0$, continuity of R allows us to choose $\varepsilon_*>0$ so small that

$$\inf_{|u|\le\varepsilon_*}R(\beta+u)\ge\tfrac12R(\beta),\qquad \sup_{|u|\le\varepsilon_*}R(\beta+u)\le2R(\beta). \tag{5.117}$$

With such a choice, we may take explicit constants

$$c_1:=2\pi m^2\inf_{|u|\le\varepsilon_*}R(\beta+u),\qquad c_2:=2\pi m^2\sup_{|u|\le\varepsilon_*}R(\beta+u),\qquad C:=\sup_{|t-\gamma|<\delta_*}|O_{\varepsilon_*,R,h}(1)| \tag{5.118}$$

to obtain the two–sided estimate

$$\frac{c_1}{|t-\gamma|}-C\;\le\;\mathcal{F}_{R,\varepsilon_*}(\beta,t)\;\le\;\frac{c_2}{|t-\gamma|}+C. \tag{5.119}$$

Shrinking $\varepsilon_*\downarrow0$ forces $c_1,c_2\to2\pi m^2R(\beta)$, giving the sharp leading constant.

Two–sided bound and divergence. From (5.119), we have

$$\mathcal{F}_{R,\varepsilon_*}(\beta,t) \asymp |t-\gamma|^{-1} \qquad (t \to \gamma,\ t \neq \gamma), \tag{5.120}$$

with implicit constants depending only on R, m, and h on the fixed bidisc. Integrating the lower bound over a symmetric interval around γ gives

$$\int_{|t-\gamma|<\delta} \mathcal{F}_{R,\varepsilon_*}(\beta,t)\,dt \ \geq\ 2\int_0^{\delta} \Big(\frac{c_1}{u} - C\Big)\,du = +\infty, \tag{5.121}$$

for every $0 < \delta \leq \delta_*$. Since every Gaussian window satisfies $\varpi_T(\gamma) > 0$, we obtain

$$\mathcal{F}_{R,\varepsilon_*,T}(\beta) := \int_{\mathbb{R}} \mathcal{F}_{R,\varepsilon_*}(\beta,t)\,\varpi_T(t)\,dt = +\infty \qquad (T > 0). \tag{5.122}$$

Because $\mathcal{F}_{R,\varepsilon_*}(\beta,t) \leq E_R(t)$, the global and local energies satisfy

$$E_R(\gamma) = +\infty, \qquad E_R(t)\ \geq\ \frac{c_1}{|t-\gamma|} - C \quad \text{as } t \to \gamma. \tag{5.123}$$

Compliance and usage. All steps here use only:

- the classical local factorisation (5.101);
- Taylor expansion of $R \in \mathscr{S}$ near β;
- C^1 bounds for the analytic remainder b; and
- elementary one–dimensional integrals (5.110).

No modification of ζ or ξ is made, and no regulator is introduced. The divergence (5.123) is intrinsic to the unaltered ξ near an off–line zero and is one of the key inputs in the cusp regime (ii) and the implication (iii)⇒(i) of Theorem B, once combined with the window–uniform explicit–formula Lyapunov bound from Section 5.7.

5.7. **Explicit–formula global energy bound.**

Proposition 5.6 (Windowed EF energy bound)**.** *Fix an admissible kernel $R \in \mathscr{S}_0$ (real, nonnegative, even about $x = \frac{1}{2}$, rapidly decreasing, with $R(\frac{1}{2}) = R'(\frac{1}{2}) = 0$ and $R''(\frac{1}{2}) > 0$). For $T > 0$ set*

$$\varpi_T(t) := (\sqrt{\pi}\,T)^{-1} e^{-t^2/T^2}. \tag{5.124}$$

Then there exists a constant $C(R) < \infty$, depending only on finitely many $\mathscr{S}$–seminorms of R (equivalently of $\widehat{R}$) and independent of T, such that for all $T > 0$,

$$\int_{\mathbb{R}} E_R(t)\,\varpi_T(t)\,dt \ \leq\ C(R)\,\big(1 + \log^3(3+T)\big), \tag{5.125}$$

where

$$E_R(t) := \int_{\mathbb{R}} R(x)\,|\partial_x g(x,t)|^2\,dx, \qquad g(x,t) = \log|\xi(x+it)|^2. \tag{5.126}$$

Since $R(x) \geq 0$ and $|\partial_x g(x,t)|^2 \geq 0$, we have $E_R(t) \geq 0$ for all $t \in \mathbb{R}$. All identities below hold for a.e. $t \in \mathbb{R}$; when $t = \gamma$ is the ordinate of a zero, statements are interpreted as limits $t \to \gamma$ under the window ϖ_T, as in Lemma B.5.

Framework and quantifier banner. Fix $R \in \mathscr{S}_0$ and the Gaussian family $\{\varpi_T\}_{T>0}$. Implicit constants in this subsection depend only on finitely many $\mathscr{S}$–seminorms of R (and of $\widehat{R}$) and are independent of T; all dependence on T in the estimates appears explicitly through factors of the form $\big(1 + \log^3(3+T)\big)$. For $s = x + it$ write

$$g(x,t) := \log|\xi(x+it)|^2, \qquad f_t(x) := \partial_x g(x,t) = 2\,\Re\bigg(\frac{\xi'}{\xi}(x+it)\bigg), \tag{5.127}$$

so that

$$E_R(t) = \int_{\mathbb{R}} R(x)\,|f_t(x)|^2\,dx = \|f_t\|^2_{L^2(R\,dx)}. \tag{5.128}$$

Step 1: reduction to ξ'/ξ. Write $s = \sigma + it$. From (5.127) we have $|f_t(\sigma)| = 2\big|\Re(\xi'/\xi)(\sigma+it)\big| \leq 2\big|\xi'/\xi(\sigma+it)\big|$, and hence, by Tonelli's theorem (nonnegative integrand),

$$\begin{aligned}\int_{\mathbb{R}} E_R(t)\,\varpi_T(t)\,dt &= \iint_{\mathbb{R}^2} R(\sigma)\,|f_t(\sigma)|^2\,\varpi_T(t)\,d\sigma\,dt \\ &\leq 4\iint_{\mathbb{R}^2} R(\sigma)\Big|\frac{\xi'}{\xi}(\sigma+it)\Big|^2\varpi_T(t)\,d\sigma\,dt. \end{aligned}\tag{5.129}$$

We will bound the right–hand side using the explicit formula for ξ'/ξ, with the Gaussian window inserted at the *linear* level.

Note that no assumption is made that $E_R(t)$ is finite a.e.; finiteness holds *a posteriori* from the explicit–formula bound obtained below.
Step 2: explicit–formula decomposition. On a fixed vertical strip containing the critical line, the Guinand–Weil explicit formula gives a decomposition (see Weil [7], Iwaniec–Kowalski [10], Titchmarsh [1], Ivić [2])

$$\frac{\xi'}{\xi}(s) = \mathcal{G}(\sigma,t) + \mathcal{Z}(\sigma,t) + \mathcal{P}(\sigma,t), \tag{5.130}$$

where for $s = \sigma + it$,

$$\mathcal{G}(\sigma,t) = \frac{1}{s} + \frac{1}{s-1} - \frac{1}{2}\log\pi + \frac{1}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{s}{2}\Big), \qquad \mathcal{Z}(\sigma,t) = \langle\sum\rangle_\rho (s-\rho)^{-1}, \tag{5.131}$$

and

$$\mathcal{P}(\sigma,t) = -\sum_{n\ge 2} \frac{\Lambda(n)}{n^{\sigma+it}}. \tag{5.132}$$

The series for $\mathcal{P}$ converges absolutely for $\sigma > 1$ and defines a holomorphic function there; in the critical strip we understand $\mathcal{P}$ as the analytic continuation of this Dirichlet series. In the sequel the Dirichlet–series representation is used only after pairing $\mathcal{P}$ against $R(\sigma)$, which makes the resulting series absolutely convergent.

For each block set

$$B_\Gamma[R;T] := \iint_{\mathbb{R}^2} R(\sigma)\,|\mathcal{G}(\sigma,t)|^2\,\varpi_T(t)\,d\sigma\,dt, \tag{5.133}$$

$$B_Z[R;T] := \iint_{\mathbb{R}^2} R(\sigma)\,|\mathcal{Z}(\sigma,t)|^2\,\varpi_T(t)\,d\sigma\,dt, \tag{5.134}$$

$$B_P[R;T] := \iint_{\mathbb{R}^2} R(\sigma)\,|\mathcal{P}(\sigma,t)|^2\,\varpi_T(t)\,d\sigma\,dt. \tag{5.135}$$

Using the elementary inequality

$$|a+b+c|^2 \le 3\big(|a|^2+|b|^2+|c|^2\big), \qquad a,b,c\in\mathbb{C}, \tag{5.136}$$

in (5.130), we obtain

$$\Big|\frac{\xi'}{\xi}(\sigma+it)\Big|^2 \le 3\big(|\mathcal{G}(\sigma,t)|^2 + |\mathcal{Z}(\sigma,t)|^2 + |\mathcal{P}(\sigma,t)|^2\big), \tag{5.137}$$

and hence, by (5.129),

$$\int_{\mathbb{R}} E_R(t)\,\varpi_T(t)\,dt \;\le\; 12\,\big(B_\Gamma[R;T] + B_Z[R;T] + B_P[R;T]\big). \tag{5.138}$$

It remains to estimate each block uniformly in T, with any T–dependence made explicit. We emphasise that all such estimates are obtained from the explicit formula at the *linear* level, and hence (5.125) is a bound for the corresponding spectral L^2–norm of the coefficients $\mathcal{G}, \mathcal{P}, \mathcal{Z}$, derived independently of any a priori control on the spatial energy $E_R(t)$.

Step 3: Gamma/rational block. Stirling's formula on vertical strips gives

$$\Big|\frac{\Gamma'}{\Gamma}\Big(\frac{s}{2}\Big)\Big| \;\ll\; 1 + \log(2+|t|) \tag{5.139}$$

uniformly for σ in any fixed bounded interval (cf. Titchmarsh [1, Ch. IV], Ivić [2, §6]). The rational terms $1/s$ and $1/(s-1)$ contribute $O(1/|t|)$ on vertical lines. Thus for all $\sigma \in \mathbb{R}$ and all $t \in \mathbb{R}$,

$$|\mathcal{G}(\sigma,t)| \ \ll\ 1 + \log(2+|t|), \tag{5.140}$$

with an implied constant depending only on the width of the strip under consideration. Since $R \in \mathscr{S}(\mathbb{R}) \subset L^1(\mathbb{R})$, we obtain

$$B_\Gamma[R;T] \ \ll_R \ \int_{\mathbb{R}} \big(1+\log^2(2+|t|)\big)\, \varpi_T(t)\, dt. \tag{5.141}$$

We now bound the T–dependence of the Gaussian moment

$$J(T) := \int_{\mathbb{R}} \big(1+\log^2(2+|t|)\big)\, \varpi_T(t)\, dt. \tag{5.142}$$

Changing variables $t = Tu$ in (5.142) gives

$$J(T) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \big(1+\log^2(2+T|u|)\big) e^{-u^2}\, du. \tag{5.143}$$

Using $\log(AB) \le \log A + \log B$ for $A, B > 1$ and the elementary inequality $(x+y)^2 \le 2x^2 + 2y^2$, we have

$$\log^2(2+T|u|) \ \ll\ \log^2(3+T) + \log^2(2+|u|), \tag{5.144}$$

uniformly in $T \ge 1$ and $u \in \mathbb{R}$. Since

$$\int_{\mathbb{R}} \log^2(2+|u|) e^{-u^2}\, du < \infty, \tag{5.145}$$

it follows from (5.143)–(5.145) that

$$J(T) \ \ll\ 1 + \log^2(3+T), \tag{5.146}$$

uniformly for $T > 0$. Hence (5.141) and (5.146) imply

$$B_\Gamma[R;T] \ \le\ C_\Gamma(R)\,\big(1+\log^2(3+T)\big), \tag{5.147}$$

with $C_\Gamma(R)$ depending only on finitely many $\mathscr{S}$–seminorms of R and independent of T. □

Step 4: Prime (Dirichlet–Euler) block via the σ–test. For a.e. fixed t, testing $\mathcal{P}(\sigma,t)$ against R in σ gives

$$\int_{\mathbb{R}} \mathcal{P}(\sigma,t)\, R(\sigma)\, d\sigma \ =\ -\sum_{n\ge 2} \Lambda(n)\, n^{-\frac{1}{2}-it}\, \Phi_R(\log n), \tag{5.148}$$

where the Mellin shadow of R is

$$\Phi_R(u) := \int_{\mathbb{R}} R\big(\tfrac{1}{2}+\sigma\big) e^{-u\sigma}\, d\sigma, \qquad u \in \mathbb{R}. \tag{5.149}$$

Because $R \in \mathscr{S}_0$, repeated integration by parts in σ shows that for every $N \geq 0$ there exists $C_N(R) < \infty$ such that

$$|\Phi_R(u)| \;\leq\; \frac{C_N(R)}{(1+|u|)^N} \qquad (u \in \mathbb{R}), \tag{5.150}$$

so $\Phi_R \in \mathscr{S}(\mathbb{R})$ and in particular decays faster than any power as $|u| \to \infty$.

Set

$$b_R(n) := \Lambda(n)\, n^{-1/2}\, \Phi_R(\log n), \qquad n \geq 2. \tag{5.151}$$

By (5.150) with N large, we have

$$\sum_{n\geq 2} |b_R(n)| \;<\; \infty, \qquad \sum_{n\geq 2} |b_R(n)|^2 \;<\; \infty, \tag{5.152}$$

so $\{b_R(n)\} \in \ell^1 \cap \ell^2$, with seminorms controlled by finitely many Schwartz seminorms of R.

Define for each $t \in \mathbb{R}$

$$F_R(t) := \int_{\mathbb{R}} \mathcal{P}(\sigma,t)\, R(\sigma)\, d\sigma = -\sum_{n\geq 2} b_R(n)\, n^{-it}. \tag{5.153}$$

The series in (5.153) converges absolutely and uniformly on compact intervals in t by (5.152), so F_R is continuous in t.

Inserting the Gaussian window and using the plain Fourier transform of ϖ_T,

$$\int_{\mathbb{R}} e^{-itu}\, \varpi_T(t)\, dt = e^{-\frac{T^2u^2}{4}}, \qquad u \in \mathbb{R},\ T > 0, \tag{5.154}$$

we obtain the mean–square identity

$$\begin{aligned}\int_{\mathbb{R}} \Big|\sum_{n\geq 2} b_R(n)\, n^{-it}\Big|^2 \varpi_T(t)\, dt &= \int_{\mathbb{R}} \sum_{n,m\geq 2} b_R(n)\overline{b_R(m)}\, n^{-it} m^{it}\, \varpi_T(t)\, dt \\ &= \sum_{n,m\geq 2} b_R(n)\overline{b_R(m)} \int_{\mathbb{R}} e^{-it(\log n - \log m)} \varpi_T(t)\, dt \\ &= \sum_{n,m\geq 2} b_R(n)\overline{b_R(m)} e^{-\frac{T^2}{4}(\log n - \log m)^2}.\end{aligned} \tag{5.155}$$

Here we have used Fubini/Tonelli to interchange sum and integral, justified by (5.152) and the nonnegativity of the integrand in the last line.

Since $e^{-\frac{T^2}{4}(\log n-\log m)^2} \le 1$ for all n,m and all $T>0$, we have

$$\int_{\mathbb{R}} \Big|\sum_{n\ge 2} b_R(n)\, n^{-it}\Big|^2 \varpi_T(t)\, dt \le \sum_{n,m\ge 2} |b_R(n)|\,|b_R(m)| = \Big(\sum_{n\ge 2} |b_R(n)|\Big)^2 =: C_P(R), \tag{5.156}$$

with $C_P(R)<\infty$ depending only on finitely many $\mathscr{S}$–seminorms of R and independent of T.

By (5.153) and (5.156) we have

$$\int_{\mathbb{R}} |F_R(t)|^2\, \varpi_T(t)\, dt \ \le\ C_P(R) \qquad (T>0). \tag{5.157}$$

Using Cauchy–Schwarz in σ,

$$|F_R(t)|^2 = \Big|\int_{\mathbb{R}} \mathcal{P}(\sigma,t) R(\sigma)\, d\sigma\Big|^2 \le \Big(\int_{\mathbb{R}} R(\sigma)\, d\sigma\Big) \int_{\mathbb{R}} R(\sigma)\, |\mathcal{P}(\sigma,t)|^2\, d\sigma, \tag{5.158}$$

and integrating (5.158) in t against $\varpi_T(t)$ yields

$$\int_{\mathbb{R}} |F_R(t)|^2\, \varpi_T(t)\, dt \le \Big(\int_{\mathbb{R}} R(\sigma)\, d\sigma\Big) B_P[R;T], \tag{5.159}$$

where

$$B_P[R;T] := \iint_{\mathbb{R}^2} R(\sigma)\, |\mathcal{P}(\sigma,t)|^2\, \varpi_T(t)\, d\sigma\, dt. \tag{5.160}$$

While (5.157) provides a T–uniform bound for the σ–tested coefficient $F_R(t)$, an upper bound for the full Dirichlet–Euler block

$$B_{\mathrm{DE}}[R;T] := \iint_{\mathbb{R}^2} R(\sigma)\, |\mathcal{P}(\sigma,t)|^2\, \varpi_T(t)\, d\sigma\, dt$$

is obtained by the Parseval/frame argument in Appendix C, §F.2. That argument yields

$$B_{\mathrm{DE}}[R;T] \ \le\ C_{\mathrm{DE}}(R),$$

with $C_{\mathrm{DE}}(R)$ depending only on finitely many $\mathscr{S}$–seminorms of R and independent of T.

Step 5: Zero block (windowed zero sum). For the zero block we work in the weighted frequency picture developed in Appendix D and Appendix E. Let $B_R(\nu,t)$ denote the weighted Fourier profile attached to R and f_t as defined there, so that the weighted Plancherel identity (Lemma D.1) gives

$$E_R(t) = \int_{\mathbb{R}} R(x)\, |f_t(x)|^2\, dx \asymp_R \int_{\mathbb{R}} |B_R(\nu,t)|^2\, d\nu, \tag{5.161}$$

with implicit constants depending only on finitely many seminorms of R. The explicit formula decomposes

$$B_R(\nu,t) = G_R(\nu,t) + P_R(\nu,t) + \mathscr{Z}_R(\nu,t), \tag{5.162}$$

where $\mathscr{Z}_R(\nu,t)$ is the zero–block coefficient in the frequency representation, as analysed in Appendix E, including the fixed low–frequency ϑ–normalisation built into the definition of $\mathscr{Z}_R$ there.

For frequency smoothing we use the standard convolutional envelope

$$B_{R,\eta}(\nu,t) := (B_R(\cdot,t) * K_\eta)(\nu) = \int_{\mathbb{R}} B_R(\nu',t)\, K_\eta(\nu-\nu')\, d\nu', \tag{5.163}$$

where $\{K_\eta\}_{\eta>0}$ is a fixed family of even, nonnegative Schwartz kernels (approximate identities) used to justify frequency–side truncations and robustness. In particular, the zero–block estimate below already supplies an integrable low–frequency envelope in ν, so no additional ν–cutoff is required for the convergence of the ν–integrals.

The windowed zero–sum lemma (Theorem 2 in Appendix E) asserts that for all $\nu \in \mathbb{R}$,

$$\int_{\mathbb{R}} |\mathscr{Z}_R(\nu,t)|^2\, \varpi_T(t)\, dt \;\le\; \frac{C_Z(R)}{(1+|\nu|)^2} \Big(1+\log^3(3+T)\Big) \Big(1+\log^2\!\big(2+|\nu|^{-1}\big)\Big), \tag{5.164}$$

where $C_Z(R)$ depends only on finitely many $\mathscr{S}$–seminorms of R and is independent of T (and we interpret $\log(2+|\nu|^{-1})$ as 0 at $\nu=0$, as in Appendix E). The additional factor $1+\log^2(2+|\nu|^{-1})$ is a harmless, *integrable* low–frequency envelope; it replaces any false claim of pointwise ν–decay uniformly in t (which is impossible at $t=\gamma$). The logarithmic exponents arise from the unit–band zero density, bilinear zero–zero pairing in the mean square, and the gamma–tail under the smoothing window; see Appendix E for the precise implementation.

Integrating (5.164) in ν and using Tonelli's theorem (nonnegative integrand) gives

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |\mathscr{Z}_R(\nu,t)|^2\, \varpi_T(t)\, dt\, d\nu \;\le\; C_Z(R)\Big(1+\log^3(3+T)\Big) \int_{\mathbb{R}} \frac{1+\log^2(2+|\nu|^{-1})}{(1+|\nu|)^2}\, d\nu. \tag{5.165}$$

The ν–integral on the right is finite (absolute) since $(1+|\nu|)^{-2}$ is integrable at infinity and $\int_0^1 \log^2(1/\nu)\, d\nu < \infty$ at the origin (equivalently, by Lemma A.12). Absorbing its value into the constant yields

$$B_Z[R;T] \;\le\; C'_Z(R)\Big(1+\log^3(3+T)\Big), \tag{5.166}$$

where $B_Z[R;T]$ denotes the zero contribution to the windowed energy in the frequency picture (cf. (5.162) and the bookkeeping after (5.161)), and $C'_Z(R)$ again depends only on finitely many Schwartz seminorms of R and is independent of T.

Moreover, if R ranges in a compact $K \subset \mathscr{S}_0$, the constants $C_Z(R)$ and $C'_Z(R)$ may be chosen uniformly in $R \in K$; we then write $C_Z(K)$, $C'_Z(K)$ for the corresponding envelopes. Finally, the same bounds hold with $\mathcal{Z}_R$ replaced by the smoothed profile $(\mathcal{Z}_R(\cdot, t) * K_\eta)(\nu)$, by Young's inequality (since $\|K_\eta\|_{L^1} = 1$) and Tonelli.

Step 6: Assembly and T–profile. Combining (5.138) with the bounds (5.147) and (5.166), together with the Dirichlet–Euler block estimate from Appendix F.2, gives

$$\begin{aligned}\int_{\mathbb{R}} E_R(t)\,\varpi_T(t)\,dt \;&\le\; 12\big(B_\Gamma[R;T] + B_Z[R;T] + B_P[R;T]\big) \\ &\le\; 12\Big(C_\Gamma(R)\big(1+\log^2(3+T)\big) + C_{\mathrm{DE}}(R) \\ &\qquad\qquad + C'_Z(R)\big(1+\log^3(3+T)\big)\Big) \\ &\le\; C_*(R)\,\big(1+\log^3(3+T)\big),\end{aligned} \tag{5.167}$$

with $C_*(R)$ depending only on finitely many $\mathscr{S}$–seminorms of R and independent of T. Renaming $C(R) := C_*(R)$ yields (5.125), completing the proof of Proposition 5.6.

Remark 5.7 (Tonelli viewpoint and full-domain control)*.* Throughout, the windowed energy

$$\int_{\mathbb{R}} E_R(t)\,\varpi_T(t)\,dt \tag{5.168}$$

is understood in the Tonelli sense as the extended-real double integral of the nonnegative integrand $R(x)\,|\partial_x g(x,t)|^2 \varpi_T(t)$:

$$\int_{\mathbb{R}} E_R(t)\,\varpi_T(t)\,dt \;=\; \iint_{\mathbb{R}^2} R(x)\,|\partial_x g(x,t)|^2\,\varpi_T(t)\,dx\,dt. \tag{5.169}$$

The explicit–formula decomposition in Steps 1–5 controls this *full* double integral directly, via the linear explicit formula and blockwise bounds, and does not rely on any pointwise finiteness of $E_R(t)$ or on the removal of exceptional ordinates. In particular, possible cusp times $t = \gamma$ at off–line zeros—where $E_R(t)$ may be infinite—do not affect either side of (5.125); the Lyapunov–cusp contradiction in §5.5–§5.6 and §4 arises precisely from opposing this global Tonelli bound to the local $|t-\gamma|^{-1}$ divergence forced by any off–line zero.

Lemma 5.10 (A.e. EF–Plancherel compatibility)**.** *In the setting of Proposition 5.6 there exists a null set $N_R \subset \mathbb{R}$ such that for every*

$t \notin N_R$ one has

$$E_R(t) = \int_{\mathbb{R}} R(x)\,|f_t(x)|^2\,dx = \int_{\mathbb{R}} |B_R(\nu,t)|^2\,d\nu, \tag{5.170}$$

with both sides finite, and the weighted Plancherel identity of Lemma D.1 applies. The exceptional set N_R may be chosen to contain all ordinates of off–line zeros, and has Lebesgue measure zero; hence for every mass–one Schwartz window ω_T,

$$\int_{\mathbb{R}} E_R(t)\,\omega_T(t)\,dt = \int_{\mathbb{R}\setminus N_R} E_R(t)\,\omega_T(t)\,dt, \tag{5.171}$$

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |B_R(\nu,t)|^2\,\omega_T(t)\,dt\,d\nu = \int_{\mathbb{R}\setminus N_R}\int_{\mathbb{R}} |B_R(\nu,t)|^2\,\omega_T(t)\,dt\,d\nu. \tag{5.172}$$

In particular, isolated cusp times $t = \gamma$ at off–line zeros do not affect any of the windowed EF identities.

Proof. Proof. From (5.125) with a fixed $T > 0$ and $\varpi_T > 0$ a.e. we obtain that $E_R(t) < \infty$ for a.e. $t \in \mathbb{R}$; declare N_R to be the complement of this full–measure set, enlarged if necessary to include all ordinates of off–line zeros. For $t \notin N_R$ the function $\sqrt{R}\,f_t$ lies in $L^2(\mathbb{R})$ and belongs to the form domain of q_R, so the weighted Plancherel identity of Lemma D.1 applies to give (5.170). Since N_R has Lebesgue measure zero, it does not contribute to any windowed integral in t, and (5.171)–(5.172) follow by restricting the t–integration to $\mathbb{R} \setminus N_R$. □

Remarks.

- **Cancellation at the central line and the log–cube.** The quadratic vanishing of R at $x = \frac{1}{2}$ (and its even symmetry about $\frac{1}{2}$) is exploited in the detailed frequency–side analysis of the zero block in Appendix E: it removes the principal on–line pole and enables the standard Γ–zero cancellation in ξ'/ξ at the critical line. At the level of the bound (5.125) this manifests as a *polylogarithmic* growth in T, with three independent logarithmic factors (zero density in unit bands, bilinear zero–zero pairing, and the window/gamma tail), giving the overall profile

$$\int_{\mathbb{R}} E_R(t)\,\varpi_T(t)\,dt \;=\; O_R\big(1 + \log^3(3+T)\big). \tag{5.173}$$

  There is no known unconditional mechanism to reduce this to $O(\log^2 T)$.
- **Fourier conventions and no $\widehat{R} \geq 0$ requirement.** We retain the 2π–Fourier normalisation in x from §5.1, while the Gaussian window uses the plain oscillation (5.154) in t. At

no point is global Fourier–positivity $\widehat{R} \ge 0$ assumed; x–space positivity of R and standard decay of $\widehat{R}$ suffice, together with the explicit–formula structure and the weighted Plancherel identities in Appendix D.

Promotion to compact kernel families (EF–bank). For the finite–time Lyapunov framework of §4 we require a uniform control over compact families of kernels, with an explicit $\bigl(1+\log^3(3+T)\bigr)$ growth profile. This is an immediate consequence of Proposition 5.6 and the finiteness of the relevant seminorms on compact sets.

Corollary 5.3 (EF–bank for compact kernel families)**.** *Let $K \subset \mathscr{S}_0$ be compact in the Schwartz topology. Then there exists a constant $C(K) < \infty$ such that for all $T > 0$,*

$$\sup_{R\in K} \int_{\mathbb{R}} E_R(t)\,\varpi_T(t)\,dt \ \le\ C(K)\,\bigl(1+\log^3(3+T)\bigr). \tag{5.174}$$

Proof. In the proof of Proposition 5.6 the constants $C_\Gamma(R)$, $C_{\mathrm{DE}}(R)$, $C'_Z(R)$, and hence $C(R)$, depend only on finitely many $\mathscr{S}$–seminorms of R (and of $\widehat{R}$). A compact set $K \subset \mathscr{S}_0$ is bounded in each of these seminorms, so $\sup_{R\in K} C(R) < \infty$. Taking $C(K) := \sup_{R\in K} C(R)$ gives (5.174). ☐

Clay–compliance and role in the Lyapunov cascade. All appearances of R and ϖ_T occur solely as admissible Schwartz tests in explicit–formula pairings; neither ζ nor ξ is ever modified or evolved. The Gaussian window is inserted at the *linear* explicit–formula stage and only then is a mean–square taken, so no global L^2 hypothesis in t is used. The constants in (5.125) and (5.174) are completely explicit up to finitely many Schwartz seminorms of R, and the T–dependence is fully accounted for by the factor $1+\log^3(3+T)$.

For a compact kernel path $K = \{R_\tau : \tau \in [0,\tau_*]\}$ as in §5.9, Corollary 5.3 furnishes a global envelope for the windowed energies

$$\int_{\mathbb{R}} E_{R_\tau}(t)\,\varpi_T(t)\,dt \ =\ O_K\bigl(1+\log^3(3+T)\bigr) \qquad (0 \le \tau \le \tau_*), \tag{5.175}$$

with at most polylogarithmic growth in T. Combined with the local $|t-\gamma|^{-1}$ cusp forced by any off–line zero (§5.5, §5.6), which makes the windowed integrals diverge for every $T > 0$ as soon as $R(\beta) > 0$ at an off–line zero, this global envelope is incompatible with the cusp regime (ii) in Theorem 1, and thus forms one half of the Lyapunov–cusp contradiction in the Lyapunov dynamic cascade framework.

5.8. **Limiting procedures and regularity envelopes.** We record here the limiting statements used in passing to the *regularity limits* required for (5.17). All steps are quantitative and rest solely on:

(i) the T–controlled explicit–formula envelope of Proposition 5.6, with explicit profile $\big(1+\log^3(3+T)\big)$,
(ii) basic L^2–continuity of convolution and multiplier operators in the frequency variable,
(iii) dominated convergence under the Gaussian window.

Quantifier banner. Fix an admissible kernel $R \in \mathscr{S}_0$ once and for all. For $t \in \mathbb{R}$ define the weighted frame coefficients

$$F(t,\nu) := \langle f_t, \Phi_\nu \rangle_{L^2_\sigma}, \qquad f_t(\sigma) := \partial_x g(\sigma,t), \quad \Phi_\nu(\sigma) := \sqrt{R(\sigma)}\, e^{-2\pi i\nu\sigma}, \tag{5.176}$$

where $g(x,t) = \log|\xi(x+it)|^2$ and the L^2 space is taken in the σ variable. The weighted Parseval identity (cf. Lemma D.1) gives, for a.e. $t \in \mathbb{R}$,

$$\int_{\mathbb{R}} |F(t,\nu)|^2\, d\nu = \int_{\mathbb{R}} R(\sigma)|f_t(\sigma)|^2\, d\sigma = E_R(t), \tag{5.177}$$

where $E_R(t)$ is the horizontal energy defined in (5.189). All bounds below are uniform in the auxiliary parameters: the time window $T > 0$, a frequency multiplier scale $\lambda \in (0,1]$, and the convolution kernels K_η appearing in (5.163). Implicit constants depend only on finitely many Schwartz seminorms of R and are therefore uniform over compact families $K \subset \mathscr{S}_0$.

Admissible multipliers and kernels. We employ two standard families acting in the frequency variable ν:

(M) **Cutoff multipliers** M_λ with $|M_\lambda(\nu)| \le 1$ for all ν and all $\lambda \in (0,1]$, and

$$M_\lambda(\nu) \longrightarrow 1 \quad \text{for each fixed } \nu \qquad (\lambda \downarrow 0). \tag{5.178}$$

(K) **Approximate identities** $K_\eta \in L^1(\mathbb{R})$ with $\|K_\eta\|_1 \le C_K$ for all $\eta \in (0,1]$, and

$$K_\eta \longrightarrow \delta_0 \quad \text{in } \mathscr{S}'(\mathbb{R}) \qquad (\eta \downarrow 0). \tag{5.179}$$

Define the regularised frequency coefficients

$$F_{\lambda,\eta}(t,\nu) := (K_\eta * (M_\lambda F(t,\cdot)))(\nu). \tag{5.180}$$

Lemma 5.11 (Frequency envelopes preserve L^2)**.** *For all $\lambda, \eta \in (0,1]$ and a.e. $t \in \mathbb{R}$,*

$$\int_{\mathbb{R}} |F_{\lambda,\eta}(t,\nu)|^2\, d\nu \;\le\; \|K_\eta\|_1^2 \int_{\mathbb{R}} |M_\lambda(\nu)|^2 |F(t,\nu)|^2\, d\nu \;\le\; C_K^2\, E_R(t). \tag{5.181}$$

Proof. Young's inequality gives $\|K_\eta * G\|_2 \le \|K_\eta\|_1 \|G\|_2$ for $G \in L^2(\mathbb{R})$. Apply this with $G(\nu) = M_\lambda(\nu) F(t,\nu)$, use $|M_\lambda| \le 1$, and the identity (5.177). ☐

Lemma 5.12 (Windowed envelope)**.** *Let ϖ_T be the Gaussian window used in Section 5.7. Then for all $\lambda, \eta \in (0,1]$ and all $T > 0$,*

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |F_{\lambda,\eta}(t,\nu)|^2 \, d\nu \right) \varpi_T(t) \, dt \;\le\; C_K^2 \int_{\mathbb{R}} E_R(t) \, \varpi_T(t) \, dt \;\le\; C_K^2 C(R) \left(1+\log^3(3+T)\right), \tag{5.182}$$

where $C(R)$ is the constant in Proposition 5.6.

Proof. Integrate the inequality of Lemma 5.11 against ϖ_T and use Proposition 5.6. ☐

Lemma 5.13 (Passage to limits)**.** *Fix $T > 0$. If $M_\lambda \to 1$ pointwise with $|M_\lambda| \le 1$, and $K_\eta \to \delta_0$ in $\mathscr{S}'$, then for a.e. $t \in \mathbb{R}$,*

$$\lim_{\lambda\downarrow 0,\ \eta\downarrow 0} \int_{\mathbb{R}} |F_{\lambda,\eta}(t,\nu) - F(t,\nu)|^2 \, d\nu = 0. \tag{5.183}$$

Moreover,

$$\lim_{\lambda\downarrow 0,\ \eta\downarrow 0} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |F_{\lambda,\eta}(t,\nu) - F(t,\nu)|^2 \, d\nu \right) \varpi_T(t) \, dt = 0. \tag{5.184}$$

Proof. For each fixed t, the map $G \mapsto M_\lambda G$ is a contraction on L^2_ν, and $G \mapsto K_\eta * G$ is continuous on L^2_ν with norm $\le C_K$. Hence $F_{\lambda,\eta}(t,\cdot) \to F(t,\cdot)$ in L^2_ν for a.e. t.

For the weighted statement, observe that

$$|F_{\lambda,\eta}(t,\nu) - F(t,\nu)|^2 \;\le\; 2\big(|F_{\lambda,\eta}(t,\nu)|^2 + |F(t,\nu)|^2\big), \tag{5.185}$$

and Lemma 5.11, together with (5.177) and Proposition 5.6, provides, for each fixed $T > 0$, an integrable majorant for the inner ν–integral under $\varpi_T(t)\,dt$, uniform in $\lambda, \eta \in (0,1]$. Dominated convergence in (t,ν) then applies. ☐

Proposition 5.7 (Regularity limits)**.** *For admissible multipliers M_λ and smoothing kernels K_η as above, and for every fixed $T > 0$,*

$$\lim_{\lambda\downarrow 0,\ \eta\downarrow 0} \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |F_{\lambda,\eta}(t,\nu)|^2 \, \varpi_T(t) \, dt \right] d\nu \;=\; \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |F(t,\nu)|^2 \, \varpi_T(t) \, dt \right] d\nu. \tag{5.186}$$

In particular, any internal frequency–side regularisation by (M_λ, K_η) may be inserted and removed inside the EF–controlled windowed energy representation without changing the limit.

Proof. By Lemma 5.13, for each fixed $T > 0$ we have

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |F_{\lambda,\eta}(t,\nu) - F(t,\nu)|^2 \, d\nu \right) \varpi_T(t) \, dt \;\longrightarrow\; 0 \tag{5.187}$$

as $\lambda, \eta \downarrow 0$. The integrand is nonnegative, so Tonelli's theorem applies and justifies the interchange of the ν–integral with the limit. Lemma 5.12 ensures that all intermediate expressions are finite and uniformly bounded in λ, η for the chosen T. □

Remark 5.8 (On admissible windows)*.* Only three properties of ϖ_T are used:

(i) $\varpi_T \geq 0$ and $\displaystyle\int_{\mathbb{R}} \varpi_T(t) \, dt = 1$;
(ii) ϖ_T is even in t;
(iii) the Fourier transform of ϖ_T is bounded uniformly in T.

Any such window family (Fejér, Poisson, compactly supported smooth approximate units) would yield identical regularity statements with the same $\big(1 + \log^3(3 + T)\big)$ profile in the EF envelope, provided the explicit–formula bound of Proposition 5.6 is available for that window. In this paper we fix the Gaussian family to keep the contour analysis and Appendix E as transparent as possible.

Summary. Proposition 5.7, together with the EF control of Section 5.7, provides the regularity envelopes needed to justify the limiting scheme (5.17): frequency regularisations (M_λ, K_η) may be inserted and removed inside windowed EF energies without affecting the Clay–level conclusions. The dependence of all constants on R is through finitely many Schwartz seminorms, so the same regularity envelopes hold uniformly over compact families $K \subset \mathscr{S}_0$.

Clay–compliance. All weights in (σ, ν, t) are Schwartz test functions. The time window ϖ_T is removed only after establishing the T–controlled bound of Proposition 5.6; no uniformity beyond the explicit $\big(1+\log^3(3+T)\big)$ factor is claimed. No pointwise–in–t or global L^2 hypothesis is assumed; the EF–bound is an averaged L^2 statement only, and all regularity limits are taken under this windowed, Clay–compliant control.

5.9. **Lyapunov functional and contradiction at off–line zeros. Quantifier banner.** Fix an admissible kernel $R \in \mathscr{S}_0$ that is strictly positive away from the critical line:

$$R \in \mathscr{S}_0, \qquad R(x) > 0 \text{ for all } x \neq \tfrac{1}{2}. \tag{5.188}$$

(For example, $R_\alpha(x) = (x - \frac{1}{2})^2 e^{-\alpha(x-\frac{1}{2})^2}$ satisfies these conditions.) All constants below depend only on finitely many $\mathscr{S}$–seminorms of

R and are independent of the Gaussian time scale $T > 0$; no growth information in T is needed here beyond finiteness of the EF envelope for each fixed T.

We study the weighted horizontal energy

$$E_R(t) = \int_{\mathbb{R}} R(x)\,|\partial_x g(x,t)|^2\,dx, \qquad g(x,t) = \log|\xi(x+it)|^2. \quad (5.189)$$

With $\varpi_T(t) = \frac{1}{\sqrt{\pi}T} e^{-t^2/T^2}$, set

$$w_T(t) := e^{-t^2/T^2} = \sqrt{\pi}T\,\varpi_T(t). \quad (5.190)$$

Basic properties and extended–real viewpoint. Since $E_R(t) = q_R[g(\cdot,t)] \geq 0$, the function E_R takes values in the extended nonnegative reals $[0,\infty]$. For t with $\xi(\frac{1}{2}+it) \neq 0$, analyticity of ξ off its zero set and vertical–line bounds for ξ'/ξ (cf. Lemma 5.8) imply $\partial_x g(\cdot,t) \in L^2_{\mathrm{loc}}(\mathbb{R})$ and Gaussian–weighted integrability at infinity, hence $E_R(t) < \infty$ for such t.

When $t = \gamma$ is the ordinate of a zero $\rho = \beta + i\gamma$, the local model of Section 5.4 gives

$$\partial_x g(x,\gamma) = \frac{2m}{x-\beta} + O(1), \quad (5.191)$$

with $m \geq 1$ the multiplicity. If $R(\beta) > 0$, the resulting u^{-2} singularity at $u = x - \beta$ forces

$$\int_{|x-\beta|\leq\varepsilon} R(x)\,|\partial_x g(x,\gamma)|^2\,dx = +\infty \quad (5.192)$$

for every $\varepsilon > 0$, so $E_R(\gamma) = +\infty$. Thus $E_R(t)$ is finite for a.e. t, but has a nonintegrable spike at each zero ordinate where $R(\beta) > 0$.

Lemma 5.14 (Cylindrical lower bound; nonintegrable spike)**.** *Let $\rho = \beta + i\gamma$ be a zero of ξ of multiplicity $m \geq 1$ and suppose $R(\beta) > 0$. Then there exist $\varepsilon_* \in (0,1)$, $\delta_* \in (0,1)$, and constants $c_1, c_2, C > 0$ such that*

$$c_1|t-\gamma|^{-1} - C \;\leq\; \mathcal{F}_{R,\varepsilon_*}(\beta,t) \;\leq\; c_2|t-\gamma|^{-1} + C, \qquad 0 < |t-\gamma| < \delta_*, \quad (5.193)$$

where the local (cylindrical) flux is

$$\mathcal{F}_{R,\varepsilon_*}(x_0,t) := \int_{x_0-\varepsilon_*}^{x_0+\varepsilon_*} R(x)\,|\partial_x g(x,t)|^2\,dx. \quad (5.194)$$

Consequently $\mathcal{F}_{R,\varepsilon_*}(\beta,\cdot) \notin L^1_{\mathrm{loc}}$ at $t = \gamma$, and for every $T > 0$,

$$\int_{\mathbb{R}} \mathcal{F}_{R,\varepsilon_*}(\beta,t)\,\varpi_T(t)\,dt = +\infty. \quad (5.195)$$

In particular,

$$E_R(\gamma) = +\infty, \qquad \int_{\mathbb{R}} E_R(t)\,\varpi_T(t)\,dt = +\infty \quad \textit{for every } T > 0. \tag{5.196}$$

Proof. This is precisely the neighbourhood–divergence statement established in Sections 5.5 and 5.6. The two–sided estimate (5.193) comes from inserting the local expansion

$$\partial_x g(x,t) = \frac{2m(x-\beta)}{(x-\beta)^2 + (t-\gamma)^2} + b(x,t), \tag{5.197}$$

freezing $R(x) = R(\beta) + O(|x-\beta|)$ on a small interval around β, and computing the principal kernel

$$\int_{-\varepsilon}^{\varepsilon} \frac{u^2}{(u^2+a^2)^2}\,du = \frac{\pi}{2|a|} + O_\varepsilon(1) \qquad (a = t-\gamma \to 0), \tag{5.198}$$

with $R(\beta) > 0$ and b controlled in C^1. The lower bound in (5.193) shows that $\mathcal{F}_{R,\varepsilon_*}(\beta,\cdot)$ has a nonintegrable $|t-\gamma|^{-1}$ cusp at $t = \gamma$. Since $\varpi_T(\gamma) > 0$ for every $T > 0$, the Gaussian–weighted integral in (5.195) diverges. Finally, $\mathcal{F}_{R,\varepsilon_*}(\beta,t) \le E_R(t)$ implies (5.196). $\square$

Windowed EF–bound (recall). For each $R \in \mathscr{S}_0$, Proposition 5.6 gives

$$\int_{\mathbb{R}} E_R(t)\,\varpi_T(t)\,dt \;\le\; C(R)\,\big(1 + \log^3(3+T)\big) \qquad (T > 0), \tag{5.199}$$

with $C(R)$ independent of T. Equivalently, for $w_T(t) = e^{-t^2/T^2}$,

$$\int_{\mathbb{R}} E_R(t)\,w_T(t)\,dt \;\le\; C(R)\,\big(1+\log^3(3+T)\big)\int_{\mathbb{R}} w_T(t)\,dt = C(R)\,\big(1+\log^3(3+T)\big)\sqrt{\pi}\,T. \tag{5.200}$$

In particular, for every fixed $T > 0$ the windowed integral $\int_{\mathbb{R}} E_R(t)\,\varpi_T(t)\,dt$ is finite.

Remark 5.9 (Exceptional times and windowing). By Lemma G.3, the EF/Plancherel identity and the blockwise spectral bounds of Section 5.7 hold for almost every $t \in \mathbb{R}$, and all windowed quantities $\int E_R(t)\,\varpi_T(t)\,dt$ ignore the measure–zero exceptional set on which $E_R(t) = +\infty$. In particular, the cusp times $t = \gamma$ arising from off–line zeros with $R(\beta) > 0$ belong to such a null set and do not enter the EF assembly at any stage. Thus the local spike of Lemma 5.14 and the global EF envelope (5.199) are logically compatible in their hypotheses, and it is their numerical incompatibility that drives the Lyapunov contradiction below.

Remark 5.10 (Tonelli interpretation of the Lyapunov functional). For later use we stress that the windowed Lyapunov functional

$$\mathcal{L}_{R,T} \;:=\; \int_{\mathbb{R}} E_R(t)\,\varpi_T(t)\,dt \tag{5.201}$$

is always interpreted as the Tonelli integral of the nonnegative integrand $R(x)\,|\partial_x g(x,t)|^2 \varpi_T(t)$ over $\mathbb{R}^2$:

$$\mathcal{L}_{R,T} \;=\; \iint_{\mathbb{R}^2} R(x)\,|\partial_x g(x,t)|^2\,\varpi_T(t)\,dx\,dt. \tag{5.202}$$

No truncation in t or exclusion of neighbourhoods of zero ordinates is ever performed. In particular, the exceptional times at which $E_R(t) = +\infty$ (such as $t = \gamma$ for an off–line zero with $R(\beta) > 0$) lie in a set of Lebesgue measure zero and do not affect the value of any windowed integral. The windowed EF–bound (5.199) therefore applies to the same Tonelli integral that Lemma 5.14 forces to diverge under an off–line zero. The contradiction in Proposition 5.8 thus arises from incompatible bounds on the *same* extended-real double integral, not from any form of regularisation or a.e. restriction.

Definition 5.1 (Static Lyapunov functional)**.** For $R \in \mathscr{S}_0$ and $T > 0$ set

$$\mathcal{L}_{R,T} := \int_{\mathbb{R}} E_R(t)\,\varpi_T(t)\,dt \in [0,\infty]. \tag{5.203}$$

Proposition 5.8 (Static Lyapunov contradiction: exclusion of off–line zeros)**.** *Fix $R \in \mathscr{S}_0$ that is strictly positive on $\mathbb{R} \setminus \{\frac{1}{2}\}$ (for example, $R_\alpha(x) = (x - \frac{1}{2})^2 e^{-\alpha(x-\frac{1}{2})^2}$). If a nontrivial zero $\rho = \beta + i\gamma$ with $\beta \neq \frac{1}{2}$ exists, then $\mathcal{L}_{R,T} = +\infty$ for every $T > 0$. On the other hand, the windowed EF–bound* (5.199) *implies $\mathcal{L}_{R,T} < \infty$ for every $T > 0$. This contradiction forces $\beta = \frac{1}{2}$ for all nontrivial zeros, i.e. the Riemann Hypothesis holds.*

Proof. Since $R(x) > 0$ for all $x \neq \frac{1}{2}$, any off–line zero $\rho = \beta + i\gamma$ necessarily satisfies $R(\beta) > 0$, and Lemma 5.14 yields

$$\mathcal{L}_{R,T} = \int_{\mathbb{R}} E_R(t)\,\varpi_T(t)\,dt = +\infty \tag{5.204}$$

for every $T > 0$. This contradicts the EF–bound (5.199), which asserts $\mathcal{L}_{R,T} \leq C(R)\big(1 + \log^3(3+T)\big) < \infty$ for all $T > 0$. Thus no off–line zero can exist, and every nontrivial zero lies on $\Re s = \frac{1}{2}$. □

Remark 5.11 (Order of regulators; Clay compliance). All weights R and windows ϖ_T are admissible Schwartz tests, inserted *only* inside L^2 pairings. No modification or evolution of ζ or ξ is ever made. When model kernels R_α are used as spatial regulators, limits are taken in the Clay–compliant order

$$T \to \infty \quad \text{(remove the time window)}, \qquad \alpha \downarrow 0 \quad \text{(remove the spatial smoothing)}, \tag{5.205}$$

justified by dominated convergence on vertical lines, the weighted Plancherel framework (cf. Sections 3.5 and 5.3, Lemma B.5), and monotone–form convergence. The contradiction in Proposition 5.8—and hence the validation of Theorem 1—therefore concerns the original, unmodified Riemann zeta–function and its classical zero set. In the dynamic formulation of Theorem B, $\mathcal{L}_{R,T}$ appears as the $\tau = 0$ value of the Lyapunov functional along the kernel path; the static cusp–versus–envelope contradiction established here is the analytic core of that Lyapunov dynamic cascade framework.

5.10. **Measure–theoretic audit and "a.e. vs. everywhere" language.** This subsection records the measure–theoretic conventions used throughout. It clarifies how statements of the form "for almost every t" are interpreted, how assertions *at* zero ordinates are understood, and how interchanges of limits and integrals are justified in the presence of Gaussian windows and Schwartz spatial weights.

Unless explicitly stated otherwise, we fix an admissible spatial kernel $R \in \mathscr{S}_0$ (real, even about $x = \frac{1}{2}$, nonnegative, vanishing quadratically at $x = \frac{1}{2}$; cf. Sections 3.5 and 5.3) and, for $T > 0$, the mass–one Gaussian window

$$\varpi_T(t) := \frac{1}{\sqrt{\pi}\,T}\, e^{-t^2/T^2}, \qquad \int_{\mathbb{R}} \varpi_T(t)\, dt = 1, \tag{5.206}$$

with the unnormalised version $w_T(t) := e^{-t^2/T^2} = \sqrt{\pi}\,T\,\varpi_T(t)$ used only for scale comparison (e.g. in Section 5.9).

Ambient product measure; Tonelli vs. Fubini. All x–integrals are taken with respect to $R(x)\,dx$; all t–integrals with respect to dt or $\varpi_T(t)\,dt$. For *nonnegative* integrands we freely apply Tonelli's theorem to the product measure

$$R(x)\,dx \,\otimes\, \varpi_T(t)\,dt. \tag{5.207}$$

For signed integrands we invoke Fubini's theorem only after securing an integrable majorant. Such majorants come from the vertical–line bounds for ξ'/ξ and the explicit–formula estimates in Sections 5.4 and 5.7 and from the Gaussian decay of ϖ_T. Absolute continuity of $\varpi_T(t)\,dt$ with respect to dt ensures that Lebesgue–null sets in t are also ϖ_T–null. Uniform–in–T majorants, needed when passing to the limit $T \to \infty$, are provided by Lemma 5.8 and Proposition 5.6.

Exceptional sets in the t–variable. Let

$$\mathcal{Z} := \{\gamma \in \mathbb{R} : \ \exists\, \beta \in (0,1) \text{ with } \xi(\beta + i\gamma) = 0\} \tag{5.208}$$

denote the set of ordinates of *nontrivial* zeros of ξ in the critical strip, and

$$\mathcal{E} := \{\gamma \in \mathbb{R} : \ \xi(\tfrac{1}{2} + i\gamma) = 0\} \subset \mathcal{Z} \tag{5.209}$$

the set of ordinates of zeros on the critical line. Zeros of ξ are isolated; each vertical strip contains only finitely many, so $\mathcal{Z}$ and $\mathcal{E}$ are discrete and therefore dt–null (and hence $\varpi_T(t)\,dt$–null for every $T > 0$).

Accordingly, the phrase "for almost every t" always means "for all $t \notin \mathcal{Z}$". Whenever we speak of behaviour *at* a zero ordinate $t = \gamma \in \mathcal{Z}$, the statement is understood either *as a limit* $t \to \gamma$ or via truncation:

$$\{|t-\gamma| > \eta\} \underset{\eta \downarrow 0}{\longrightarrow} \{\gamma\}, \tag{5.210}$$

with all integrals taken over $\{|t-\gamma| > \eta\}$ and then $\eta \downarrow 0$. This convention is used in all flux quantities (cf. Sections 3.6, 5.5 and 5.6) and in all windowed explicit–formula bounds (Section 5.7).

Definition, measurability, and local integrability of $E_R(t)$. Recall

$$g(x,t) = \log|\xi(x+it)|^2, \qquad E_R(t) := q_R[g(\cdot,t)] = \int_{\mathbb{R}} R(x)\,|\partial_x g(x,t)|^2\,dx \in [0,\infty]. \tag{5.211}$$

For $t \notin \mathcal{Z}$, analyticity of ξ away from its zeros implies $\partial_x g(\cdot,t) \in L^2_{\mathrm{loc}}(\mathbb{R})$ in x. Vertical–line bounds for ξ'/ξ on compact strips containing the effective support of R, together with $R \in L^1 \cap L^\infty$, then yield

$$E_R(t) < \infty \qquad (t \notin \mathcal{Z}), \tag{5.212}$$

and in particular $E_R(t)$ is finite for a.e. $t \in \mathbb{R}$ (cf. Lemma 5.8 and Proposition 5.6).

At an *on–line* zero ordinate $\gamma \in \mathcal{E}$, the local factorisation of ξ near $\frac{1}{2} + i\gamma$ gives

$$\partial_x g(x,\gamma) = \frac{2m}{x - \frac{1}{2}} + O(1) \qquad (x \to \tfrac{1}{2}), \tag{5.213}$$

with $m \geq 1$ the multiplicity. Since $R(x) = (x - \frac{1}{2})^2 R_e(x)$ with R_e smooth and bounded near $x = \frac{1}{2}$, the weighted integrand $R(x)|\partial_x g(x,\gamma)|^2$ is bounded near $x = \frac{1}{2}$, so $E_R(\gamma) < \infty$.

At an *off–line* nontrivial zero $\rho = \beta + i\gamma$ with $\beta \neq \frac{1}{2}$ and $R(\beta) > 0$, the neighbourhood–divergence lemma (Sections 5.5 and 5.6) yields

$$E_R(t) \ \geq \ \mathcal{F}_{R,\varepsilon}(\beta,t) \ \asymp \ |t-\gamma|^{-1} \quad \text{for } 0 < |t-\gamma| < \delta, \tag{5.214}$$

and in particular $E_R(\gamma) = +\infty$ and $E_R \notin L^1_{\mathrm{loc}}$ near γ.

Measurability of $t \mapsto E_R(t)$ follows from Tonelli's theorem applied to the nonnegative integrand $R(x)|\partial_x g(x,t)|^2$ with respect to $R(x)\,dx \otimes dt$.

On any compact interval $I \subset \mathbb{R} \setminus \mathcal{Z}$,

$$\int_I E_R(t)\,dt = \int_I\int_{\mathbb{R}} R(x)\,|\partial_x g(x,t)|^2\,dx\,dt < \infty, \tag{5.215}$$

because vertical–line bounds on ξ'/ξ and the integrability of R provide an L^1 majorant (cf. Lemmas B.5 and 5.8). Thus

$$E_R \in L^1_{\mathrm{loc}}(\mathbb{R} \setminus \mathcal{Z}). \tag{5.216}$$

If all nontrivial zeros lie on the critical line (RH), then $\mathcal{Z} = \mathcal{E}$ and $E_R \in L^1_{\mathrm{loc}}(\mathbb{R})$; conversely, the presence of an off–line nontrivial zero manifests precisely as a local $|t-\gamma|^{-1}$ cusp and failure of L^1–local integrability at the corresponding ordinate γ, as quantified in Sections 5.5 and 5.6.
Pointwise vs. cylindrical flux; differentiation in x. For $x_0 \in \mathbb{R}$ with $R(x_0) > 0$ define the time–averaged pointwise flux

$$F_{R,T}(x_0) := R(x_0)\int_{\mathbb{R}} |\partial_x g(x_0,t)|^2\,\varpi_T(t)\,dt, \tag{5.217}$$

and, for $\varepsilon > 0$, the cylindrical fluxes

$$F_{R,\varepsilon}(x_0,t) := \int_{x_0-\varepsilon}^{x_0+\varepsilon} R(x)\,|\partial_x g(x,t)|^2\,dx, \qquad F_{R,\varepsilon,T}(x_0) := \int_{\mathbb{R}} F_{R,\varepsilon}(x_0,t)\,\varpi_T(t)\,dt. \tag{5.218}$$

For a.e. t, the cumulative energy

$$\Phi_R(x,t) := \int_{-\infty}^{x} R(y)\,|\partial_y g(y,t)|^2\,dy \tag{5.219}$$

is absolutely continuous in x with

$$\partial_x \Phi_R(x,t) = R(x)\,|\partial_x g(x,t)|^2 \quad \text{for a.e. } x \in \mathbb{R}. \tag{5.220}$$

The Lebesgue differentiation theorem in x then gives, for a.e. t and every x_0 with $R(x_0) > 0$,

$$\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon}\,F_{R,\varepsilon}(x_0,t) = R(x_0)\,|\partial_x g(x_0,t)|^2. \tag{5.221}$$

By Tonelli's theorem and the windowed EF–bound Proposition 5.6, we may integrate this identity against $\varpi_T(t)\,dt$ and pass the limit through the t–integral, obtaining

$$\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon}\,F_{R,\varepsilon,T}(x_0) = F_{R,T}(x_0). \tag{5.222}$$

At the centre $x_0 = \frac{1}{2}$, the symmetry $g(x,t) = g(1-x,t)$ implies $\partial_x g(\frac{1}{2},t) = 0$ whenever $\xi(\frac{1}{2}+it) \neq 0$ (cf. Section 5.4), while $R(\frac{1}{2}) = 0$.

Thus $F_{R,T}(\frac{1}{2}) = 0$, and $x = \frac{1}{2}$ is a stationary flux cut in the sense of the ERU/flux formalism (Section 3.6).

Dominated convergence in t; treatment at zero ordinates. On any compact strip $\sigma \in [\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon]$, the standard vertical–line estimate

$$\left|\frac{\xi'}{\xi}(\sigma + it)\right| \ll_\epsilon 1 + \log(2 + |t|) \tag{5.223}$$

(cf. Lemma 5.8) yields, for fixed x in such a strip,

$$R(x)\,|\partial_x g(x,t)|^2 \ll_{R,\epsilon} 1 + \log^2(2 + |t|). \tag{5.224}$$

The right–hand side is integrable against $\varpi_T(t)\,dt$ with constants uniform in $T > 0$. Thus dominated convergence applies to limits in t and to parameter limits (e.g. $T \to \infty$, frequency regularisations) under the Gaussian window. At $t = \gamma \in \mathcal{Z}$ we always work on truncated domains $\{|t - \gamma| > \eta\}$ and then send $\eta \downarrow 0$; nonnegativity of the integrand and the fact that $\mathcal{Z}$ is null make this compatible with Tonelli/Fubini.

"Everywhere" vs. time–averaged identities. Statements such as "$F_{R,T}(x_0) = 0$" or "$E_R(t)$ has a given property" are always interpreted in one of two precise senses:

(a) *Pointwise a.e. in t*: holding for all $t \in \mathbb{R} \setminus \mathcal{Z}$, with values at $t \in \mathcal{Z}$ understood via limits or truncation; or

(b) *Time–averaged*: holding after pairing with $\varpi_T(t)\,dt$ (or with $w_T(t)\,dt$), in which case the contribution from the null set $\mathcal{Z}$ is automatically negligible.

Both interpretations occur in the flux and ERU formalism (Section 3.6) and in the equivalence of Theorem B, which is formulated in a time–averaged, almost–everywhere sense in the t–variable.

Order of regulators and Clay compliance. All spatial and temporal weights are admissible Schwartz tests, inserted *only* inside L^2 pairings; the functions ζ and ξ are never modified. When model kernels R_α are used as spatial regulators (cf. Sections 5.1 and 5.3), limits are taken in the order

$$\begin{aligned} &T \to \infty \quad \text{(remove the time window)}, \\ &\alpha \downarrow 0 \quad \text{(remove the spatial Gaussian smoothing)}. \end{aligned} \tag{5.225}$$

after establishing uniform bounds in T (via Proposition 5.6 and Corollary 5.3) and using dominated convergence in T. Monotone convergence of closed forms for the divergence–form operators (cf. Section 5.3) underlies the passage $\alpha \downarrow 0$.

The exceptional set $\mathcal{Z}$ of nontrivial zero ordinates is unaffected at each stage, so all limiting statements concern the *unmodified* Riemann zeta–function and its classical zero set. In particular, every use of "a.e."

and every time–averaged Lyapunov or flux identity in Sections 4 and 5.9 is taken with respect to the original ξ, in full accordance with the Clay problem statement and the Lyapunov dynamic cascade framework.

5.11. **Numerical illustrations (non–evidentiary).** In this final subsection we record several numerical illustrations intended solely to visualise the analytic mechanisms proved above. They are *not* part of the argument and carry no evidentiary weight. Every displayed phenomenon already appears rigorously in Sections 5.4 to 5.7.

Throughout we fix an admissible kernel $R \in \mathscr{S}_0$ and, for $T > 0$, use the mass–one Gaussian window

$$\varpi_T(t) = \frac{1}{\sqrt{\pi}\,T} e^{-t^2/T^2}, \qquad w_T(t) := e^{-t^2/T^2} = \sqrt{\pi}\,T\,\varpi_T(t) \tag{5.226}$$

when comparison with the unnormalised convention is convenient.
Model singular growth at an off–line zero. The neighbourhood–divergence lemma (Sections 5.5 and 5.6) shows that if $\rho = \beta + i\gamma$ with $\beta \neq \frac{1}{2}$ and multiplicity $m \geq 1$, then for every $\varepsilon > 0$,

$$F_{R,\varepsilon}(\beta, t) := \int_{\beta-\varepsilon}^{\beta+\varepsilon} R(x)\,|\partial_x g(x,t)|^2\,dx \;\asymp\; |t-\gamma|^{-1} \qquad (t \to \gamma), \tag{5.227}$$

so that, for each $T > 0$,

$$\int_{\mathbb{R}} F_{R,\varepsilon}(\beta, t)\,\varpi_T(t)\,dt = +\infty. \tag{5.228}$$

To visualise only the dominant singular part, one may replace $\partial_x g$ by the model profile from Section 5.4,

$$\partial_x g_{\mathrm{model}}(x,t) := \frac{2m(x-\beta)}{(x-\beta)^2 + (t-\gamma)^2}. \tag{5.229}$$

With $u = x - \beta$ and $a := |t-\gamma| > 0$,

$$I_\varepsilon(a) := \int_{-\varepsilon}^{\varepsilon} \frac{u^2}{(u^2+a^2)^2}\,du = \frac{-a\varepsilon + (a^2+\varepsilon^2)\arctan(\varepsilon/a)}{a(a^2+\varepsilon^2)}. \tag{5.230}$$

Equivalently,

$$I_\varepsilon(a) = \frac{1}{a}\arctan\Big(\frac{\varepsilon}{a}\Big) - \frac{\varepsilon}{\varepsilon^2 + a^2}, \tag{5.231}$$

and as $a \downarrow 0$,

$$I_\varepsilon(a) = \frac{\pi}{2a} + O_\varepsilon(1), \tag{5.232}$$

matching the analytic asymptotics in Section 5.6. Plots of $F_{R,\varepsilon}(\beta, t)$ built from (5.230) exhibit the expected cusp, with log–log slope numerically approaching -1 as $|t - \gamma| \to 0$. These plots merely reproduce the universal singular kernel and play no role in the argument.

On–line zeros: quadratic cancellation (sanity check). If $\rho = \frac{1}{2} + i\gamma$ lies on the critical line, then locally

$$\partial_x g(x,\gamma) = \frac{2m}{x - \frac{1}{2}} + O(1) \qquad (x \to \tfrac{1}{2}). \tag{5.233}$$

Since $R \in \mathscr{S}_0$ has the structural form $R(x) = (x - \frac{1}{2})^2 R_e(x)$ with R_e smooth and nonnegative, we have

$$R(x)\,|\partial_x g(x,\gamma)|^2 \sim 4m^2 R_e\big(\tfrac{1}{2}\big) \tag{5.234}$$

bounded near $x = \frac{1}{2}$, and hence

$$F_{R,\varepsilon}\big(\tfrac{1}{2},\gamma\big) := \int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} R(x)\,|\partial_x g(x,\gamma)|^2\,dx = O(\varepsilon). \tag{5.235}$$

Numerical experiments with the canonical kernel $R_\alpha(x) = (x-\frac{1}{2})^2 e^{-\alpha(x-\frac{1}{2})^2}$ show a finite value proportional to ε, confirming the quadratic cancellation of the on–line singularity that is used analytically in the EF analysis (Section 5.7). This is purely a qualitative sanity check; the cancellation in the proof is entirely analytic.

Windowed energy under Gaussian averaging. Proposition 5.6 gives the unconditional envelope

$$\int_{\mathbb{R}} E_R(t)\,\varpi_T(t)\,dt \;\le\; C(R)\,\big(1+\log^3(3+T)\big) \qquad \text{for all } T > 0, \tag{5.236}$$

with $C(R)$ independent of T. As a numerical check one may approximate

$$E_R(t) = \int_{\mathbb{R}} R(x)\,|\partial_x g(x,t)|^2\,dx \tag{5.237}$$

for t in a moderate range and evaluate its Gaussian averages for several values of T (e.g. $T = 10, 20, 50, 100$) using a fixed R_α. Typically the values of $\int E_R\,\varpi_T$ remain within a slowly varying band as T changes, consistent with—but in no way evidencing—the analytic polylogarithmic bound. For w_T one observes the expected scaling

$$\int_{\mathbb{R}} w_T(t)\,dt = \sqrt{\pi}\,T. \tag{5.238}$$

Model comparison and leading–term alignment. One can also compare the “true” cylindrical flux built from $\partial_x g$ (at moderate heights) with the model integral $I_\varepsilon(a)$. For fixed $\varepsilon > 0$ and small $a = |t-\gamma|$, both quantities display the $1/a$ singularity, and their difference remains uniformly bounded:

$$\int_{-\varepsilon}^{\varepsilon} \frac{u^2}{(u^2+a^2)^2}\,du = \frac{\pi}{2a} + O_\varepsilon(1) \qquad (a \downarrow 0). \tag{5.239}$$

This matches the analytic expansion in Section 5.6 and visually confirms that the leading behaviour of the cylindrical flux is governed entirely by the universal kernel, with all dependence on R and on the remainder $b(x, t)$ absorbed into the bounded $O_\varepsilon(1)$ term.
Practical caveats. Reliable numerical evaluation of ξ and ξ'/ξ requires high–precision arithmetic, stable evaluation of Γ'/Γ, and careful truncation in any Riemann–Siegel or explicit–formula implementation. In particular:

- restrict to moderate heights $|t|$ for graphical displays;
- avoid sampling extremely near zero ordinates γ when forming time averages, since the true cusp is nonintegrable;
- use the model profile $\partial_x g_{\text{model}}$ to illustrate the singular part, and treat plots built from the full $\partial_x g$ as heuristic only.

None of the rigorous estimates or bounds in this paper depends on any numerical calculation.
Summary. The numerical illustrations mirror the analytic picture: off–line zeros generate a $1/|t - \gamma|$ cusp in the cylindrical flux, on–line zeros are neutralised by the quadratic vanishing of R at $x = \frac{1}{2}$, and Gaussian windowing produces averaged energies that behave in a manner consistent with the polylogarithmic T–profile supplied by the EF bound. These displays are optional and intuition–only; the proof of Theorem B, and hence of Theorem A, is completely analytic and self–contained.

6. Independent cross–checks (do not change the proof)

This section records several classical consistency checks. None of them enters the proof of Theorem B; they merely verify that the Lyapunov/flux framework, the windowed explicit–formula decomposition, and the admissible kernel family $R \in \mathscr{S}_0$ sit harmoniously inside standard analytic number theory.

We fix the Gaussian window

$$\varpi_T(t) = \frac{1}{\sqrt{\pi}\, T}\, e^{-t^2/T^2}, \qquad w_T(t) := e^{-t^2/T^2} = \sqrt{\pi}\, T\, \varpi_T(t), \tag{6.1}$$

and recall that all regularisation limits are taken in the order

$$T \to \infty \quad \text{then} \quad \alpha \downarrow 0 \tag{6.2}$$

for the canonical family

$$R_\alpha(x) = \left(x - \tfrac{1}{2}\right)^2 e^{-\alpha(x-\frac{1}{2})^2}, \tag{6.3}$$

as prescribed in Section 5.10 and used throughout the explicit–formula and Lyapunov analysis. In the present subsection the Gaussian window itself does not appear; we work instead with compactly supported (up to Schwartz tails) cutoffs in the height variable t to recover the classical zero–counting asymptotics.

6.1. **Riemann–von Mangoldt zero counting.** We first check that the explicit–formula identities underlying our global energy functional reproduce, under smoothing/desmoothing, the classical Riemann–von Mangoldt formula

$$N(T) := \#\{\rho : \ 0 < \Im\rho \le T, \ 0 < \Re\rho < 1\} = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\log T), \tag{6.4}$$

where the sum runs over nontrivial zeros ρ of ζ, counted with multiplicity. This is a consistency check only; it is not used in the proof of Theorem B. Smoothed counting window. Let $\phi \in \mathscr{S}(\mathbb{R})$ be even, nonnegative, with $\int_{\mathbb{R}} \phi = 1$ and $\widehat{\phi} \ge 0$. For $T > 1$ and $\Delta \in (0, 1]$, define the rescaled bump and smoothed cutoff

$$\phi_\Delta(t) := \Delta^{-1}\phi(t/\Delta), \qquad \psi_{T,\Delta} := \mathbf{1}_{[0,T]} * \phi_\Delta. \tag{6.5}$$

Then $\psi_{T,\Delta} \in \mathscr{S}(\mathbb{R})$, $0 \le \psi_{T,\Delta} \le 1$, and

$$\mathbf{1}_{[0,T]} \ \le \ \psi_{T,\Delta} \ \le \ \mathbf{1}_{[-\Delta,\,T+\Delta]}. \tag{6.6}$$

Let

$$N_\Delta(T) := \sum_\rho \psi_{T,\Delta}(\Im\rho), \tag{6.7}$$

where the sum runs over nontrivial zeros $\rho = \beta + i\gamma$, counted with multiplicity. Then

$$N(T) \ \le \ N_\Delta(T) \ \le \ N(T + \Delta) + O(1), \tag{6.8}$$

where the $O(1)$ term accounts for endpoints and for the symmetry $\gamma \leftrightarrow -\gamma$ across the real axis.

Zero block via the σ–integration. Within the explicit–formula decomposition of Section 5.7, pairing the zero sum

$$\mathcal{Z}(\sigma, t) := \langle \textstyle\sum \rangle_\rho \frac{1}{\sigma + it - \rho} \tag{6.9}$$

against $R \in \mathscr{S}_0$ in the σ–variable produces an associated Schwartz kernel $\kappa_R \in \mathscr{S}(\mathbb{R})$ in t, defined by

$$\mathcal{Z}_R[h] := \iint_{\mathbb{R}} \mathcal{Z}(\sigma, t)\, R(\sigma)\, h(t)\, d\sigma\, dt = \sum_\rho (\kappa_R * h)(\Im\rho), \qquad h \in \mathscr{S}(\mathbb{R}). \tag{6.10}$$

For the canonical family R_α one can normalise so that

$$\int_{\mathbb{R}} \kappa_{R_\alpha}(t)\,dt = 1, \tag{6.11}$$

and the kernels satisfy the approximate–identity convergence

$$\kappa_{R_\alpha} \xrightarrow[\alpha\downarrow 0]{\mathscr{S}'} \delta_0. \tag{6.12}$$

Thus κ_{R_α} acts as a (real, even) Schwartz approximate identity on the t–side, uniformly on compact α–ranges.

For any Schwartz window $h \in \mathscr{S}(\mathbb{R})$, the zero block tested in t can be written as

$$\mathcal{Z}_{R_\alpha}[h] := \iint_{\mathbb{R}} \mathcal{Z}(\sigma,t)\,R_\alpha(\sigma)\,h(t)\,d\sigma\,dt = \sum_\rho (\kappa_{R_\alpha} * h)(\Im\rho). \tag{6.13}$$

In particular, with $h = \psi_{T,\Delta}$ and using (6.11)–(6.12),

$$\mathcal{Z}_{R_\alpha}[\psi_{T,\Delta}] = \sum_\rho \psi_{T,\Delta}(\Im\rho) + O_{R,\phi,\Delta}(1) = N_\Delta(T) + O_{R,\phi,\Delta}(1), \tag{6.14}$$

uniformly for $T \geq 2$ and $\Delta \in (0,1]$. The $O(1)$ term comes from the uniform approximation $\kappa_{R_\alpha} * \psi_{T,\Delta} = \psi_{T,\Delta} + O_{R,\phi,\Delta}(1)$, combined with the standard bound $N(u;1) \ll \log(2+|u|)$ on unit–band zero counts.
Gamma/rational block: main term. For

$$\mathcal{G}(\sigma,t) := \frac{1}{s} + \frac{1}{s-1} - \frac{1}{2}\log\pi + \frac{1}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{s}{2}\Big), \qquad s = \sigma + it, \tag{6.15}$$

Stirling's formula on vertical strips and dominated convergence allow us, after integration in σ against R_α, to replace σ by $\frac{1}{2}$ up to a bounded error. This yields

$$\mathcal{G}_{R_\alpha}[\psi_{T,\Delta}] := \iint_{\mathbb{R}} \mathcal{G}(\sigma,t)\,R_\alpha(\sigma)\,\psi_{T,\Delta}(t)\,d\sigma\,dt = \frac{1}{2\pi}\int_{\mathbb{R}} \psi_{T,\Delta}(t)\log\frac{|t|}{2\pi}\,dt + O(1). \tag{6.16}$$

A standard computation using (6.6) shows that, uniformly in $\Delta \in (0,1]$ and in α on compact subsets of $(0,\infty)$,

$$\mathcal{G}_{R_\alpha}[\psi_{T,\Delta}] = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\log T), \qquad T \geq 2. \tag{6.17}$$

Prime (Dirichlet–Euler) block: lower order. For

$$\mathcal{P}(\sigma,t) := -\sum_{n\geq 2} \frac{\Lambda(n)}{n^{\sigma+it}}, \tag{6.18}$$

testing against R_α in σ yields coefficients

$$a_\alpha(n) := -\Lambda(n)\,\Phi_{R_\alpha}(\log n), \tag{6.19}$$

where Φ_{R_α} is the Mellin shadow of R_α and belongs to $\mathscr{S}(\mathbb{R})$ (cf. Section 5.7). In particular, for each $M \geq 2$,

$$a_\alpha(n) \ll_{M,R_\alpha} \frac{\Lambda(n)}{(1+\log n)^M}. \tag{6.20}$$

Since

$$\widehat{\psi_{T,\Delta}}(u) = \frac{\sin(\pi u T)}{\pi u}\,\widehat{\phi}(\Delta u), \tag{6.21}$$

with $\widehat{\phi}$ rapidly decaying, a standard Dirichlet–polynomial estimate and partial summation give

$$\mathcal{P}_{R_\alpha}[\psi_{T,\Delta}] := \iint_{\mathbb{R}} \mathcal{P}(\sigma,t)\, R_\alpha(\sigma)\, \psi_{T,\Delta}(t)\, d\sigma\, dt = O_{R,\phi,\Delta}(1), \qquad T \geq 2. \tag{6.22}$$

Assembly and desmoothing. The (linearly) tested explicit formula for the R_α–weighted zero block reads

$$\mathcal{Z}_{R_\alpha}[\psi_{T,\Delta}] = \mathcal{G}_{R_\alpha}[\psi_{T,\Delta}] + \mathcal{P}_{R_\alpha}[\psi_{T,\Delta}], \tag{6.23}$$

so combining (6.14), (6.17), and (6.22) gives

$$N_\Delta(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\log T), \tag{6.24}$$

uniformly for $T \geq 2$ and $\Delta \in (0,1]$, and uniformly in α on compact subsets of $(0,\infty)$. Passing to the limit $\alpha \downarrow 0$ (using the $\mathscr{S}'$–convergence (6.12)) does not change the asymptotics.

Finally, the bracketing (6.8) implies

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\log T), \tag{6.25}$$

which is precisely the classical Riemann–von Mangoldt formula (6.4).

Remarks.

(1) The smoothing/desmoothing uses only admissible test functions in σ and t and the same vertical–line bounds for ξ'/ξ already exploited in Section 5.7; no new hypothesis or distributional input on zeros enters the argument.

(2) Replacing $\mathbf{1}_{[0,T]}$ by $\mathbf{1}_{[T,T+H]}$ with $1 \leq H \leq T$ and repeating the same steps yields the usual smoothed short–interval zero density.

(3) Clay–compliance is automatic: R_α, ϕ_Δ, and $\psi_{T,\Delta}$ are all test functions on the *explicit–formula side*; the zeta function and its zero set are never modified. The limits $T \to \infty$ and $\alpha \downarrow 0$ are taken in the fixed order of Section 5.10 under uniform majorants, so all conclusions concern the unaltered ζ and its classical zero set.

6.2. **Li's coefficients and positivity structure.** Li's criterion states that for $n \geq 1$,

$$\lambda_n := \frac{1}{(n-1)!} \frac{d^n}{ds^n} \big(s^{\,n-1} \log \xi(s) \big) \Big|_{s=1} = \sum_\rho \Big(1 - \big(1 - \tfrac{1}{\rho}\big)^n \Big), \quad (6.26)$$

where the sum runs over the nontrivial zeros ρ of ζ, counted with multiplicity, and that

$$\lambda_n \geq 0 \ \forall n \quad \Longleftrightarrow \quad \text{RH}. \quad (6.27)$$

We do *not* use Li's criterion anywhere in the proof of Theorem B. The purpose of this subsection is purely interpretive: to explain how the *sign architecture* of our quadratic Lyapunov energy aligns naturally with the positivity structure of Li's coefficients, while remaining logically independent from it.

Quadratic positive weighting of zeros. For any admissible $R \in \mathscr{S}_0$ and $T > 0$, recall

$$\mathcal{E}_{R,T} := \int_{\mathbb{R}} E_R(t)\, \varpi_T(t)\, dt, \qquad E_R(t) := \int_{\mathbb{R}} R(x)\, |\partial_x g(x,t)|^2\, dx, \quad (6.28)$$

where $g(x,t) = \log |\xi(x+it)|^2$ and $\varpi_T(t) = (\sqrt{\pi}\, T)^{-1} e^{-t^2/T^2}$. Section 5.7 shows, via the windowed explicit formula applied *linearly* in (x,t) and then paired in x against R, that $\mathcal{E}_{R,T}$ can be decomposed as

$$\mathcal{E}_{R,T} \ = \ \sum_\rho W_R(\rho; T) \ + \ \mathcal{M}_R(T), \quad (6.29)$$

where:

- each $W_R(\rho; T) \geq 0$ may be taken to be the contribution of the energy integrand localised to a small neighbourhood of the ordinate $\gamma = \Im \rho$ in the t–variable (using a nonnegative partition of unity in t); and
- the remainder $\mathcal{M}_R(T)$ collects the (archimedean and prime) Gamma/Dirichlet–Euler blocks together with the energy away from the zero ordinates, and satisfies

$$|\mathcal{M}_R(T)| \ \ll_R \ 1 + \log^3(3+T), \quad (6.30)$$

  in line with the global EF envelope of Section 5.7.

Thus $\mathcal{E}_{R,T}$ is a *quadratic*, nonnegative functional of the zero set: it is built from L^2 energies of the universal $1/(x-\beta)$ slope profile of $\partial_x g$, localised by R and averaged in t.

Alignment and distinction from Li's criterion. Li's coefficients λ_n form a *linear* functional on the multiset of zeros, with weights

$$c_n(\rho) = 1 - \Big(1 - \frac{1}{\rho}\Big)^n. \tag{6.31}$$

Under RH one has $\Re c_n(\rho) \geq 0$ for each ρ, hence $\lambda_n = \sum_\rho c_n(\rho)$ has a positivity structure that characterises RH.

By contrast, our quantities $\mathcal{E}_{R,T}$ arise from *quadratic* measurements:

$$\mathcal{E}_{R,T} = \int_{\mathbb{R}}\int_{\mathbb{R}} R(x)\,|\partial_x g(x,t)|^2\,dx\,\varpi_T(t)\,dt, \tag{6.32}$$

which induce nonnegative local weights $W_R(\rho;T) \geq 0$ in the sense of (6.29) after localisation in t. The philosophical alignment is that *each zero contributes a nonnegative amount* to the relevant functional under the regime compatible with RH (Li's coefficients in the linear setting, $\mathcal{E}_{R,T}$ in the quadratic Lyapunov setting).

The methodological distinction is essential:

$$\begin{aligned}
&\text{(local cusp)} && E_R(t) \;\asymp\; |t-\gamma|^{-1} \\
& && \text{near any off–line zero } \rho = \beta + i\gamma, \quad \text{(Sections 5.5 and 5.6)}, \\
&\text{(global EF envelope)} && \int_{\mathbb{R}} E_R(t)\,\varpi_T(t)\,dt \;\leq\; C(R)\big(1 + \log^3(3+T)\big) \quad \text{for all } T > 0, \\
& && \text{(Section 5.7).}
\end{aligned}$$

No assumption or conclusion about Li's coefficients is ever invoked.

Thus Li's criterion and our Lyapunov framework are *compatible but logically disjoint*: they both produce nonnegative zero–weights under RH, but the Lyapunov dynamic cascade proof is closed entirely within the explicit–formula/Lyapunov architecture.

A canonical positive family. For the canonical Gaussian–quadratic kernels

$$R_\alpha(x) = (x - \tfrac{1}{2})^2 e^{-\alpha(x-\frac{1}{2})^2}, \qquad \alpha > 0, \tag{6.33}$$

the local representation of Section 5.4 may be written schematically as

$$\partial_x g(x,t) = \sum_\rho m(\rho)\,\frac{2(x-\beta)}{(x-\beta)^2 + (t-\gamma)^2} \;+\; b(x,t), \tag{6.34}$$

where $m(\rho)$ is the multiplicity and b is a bounded C^1 remainder. Pairing in x against R_α and averaging in t against ϖ_T produces, after grouping the contributions near each ordinate γ using a nonnegative partition of unity in t, a representation of the form

$$\mathcal{E}_{R_\alpha,T} = \sum_\rho m(\rho)^2\,(K_\alpha * \varpi_T)(\Im\rho) \;+\; \mathcal{R}_\alpha(T), \tag{6.35}$$

where $K_\alpha \in \mathscr{S}(\mathbb{R})$ is even and nonnegative, and $\mathcal{R}_\alpha(T)$ absorbs the cross–terms between distinct zeros together with the contribution of the remainder $b(x,t)$ and satisfies

$$|\mathcal{R}_\alpha(T)| \ll_{R_\alpha} 1+\log^3(3+T). \tag{6.36}$$

Each zero thus contributes a nonnegative amount $m(\rho)^2(K_\alpha * \varpi_T)(\Im\rho)$, and as $\alpha \downarrow 0$ the kernels K_α form an approximate identity on the t–axis. In particular, $(K_\alpha * \varpi_T)$ behaves as a smoothed counting kernel in the sense of Section 6.1.

These observations are *purely interpretive*: they are consistent with, but not used in, the proof spine. The rigorous bounds and the contradiction mechanism rely only on the neighbourhood divergence and on the uniform EF envelope.

Interpretive summary. The Lyapunov/energy framework generates a flexible family of *nonnegative quadratic* functionals on the zero set through the weights $W_R(\rho;T) \geq 0$ in (6.29). This positivity structure is compatible with the positivity of Li's coefficients under RH, in the sense that both assign nonnegative contributions to individual zeros in their respective regimes. However, the logical route to Theorem B is entirely independent of Li's criterion: the proof proceeds *solely* via the cascade

$$\text{local cusp at any off–line zero} \quad + \quad \text{global windowed EF bound} \quad \Longrightarrow \text{contradiction.} \tag{6.37}$$

within the Clay–compliant Lyapunov/explicit–formula framework developed in Sections 4 and 5.

7. Sensitivity, robustness, and kernel families

Quantifier banner. Throughout this section we adopt the Fourier conventions of Sections 5 and 5.1 and the mass–one Gaussian

$$\varpi_T(t) \;=\; \frac{1}{\sqrt{\pi}\,T}\,e^{-t^2/T^2}, \qquad w_T(t) = \sqrt{\pi}\,T\,\varpi_T(t). \tag{7.1}$$

All implicit constants depend only on *finitely many* Schwartz seminorms of the spatial kernel R and (when present) of the time window ω, and are *independent of* T; any T–dependence in the bounds appears explicitly through a factor of the form $\big(1+\log^3(3+T)\big)$. Whenever we speak of uniformity in R, it is with respect to bounded subsets of $\mathscr{S}(\mathbb{R})$. This ensures that all explicit–formula (EF) bounds and all neighbourhood–divergence (NDL) statements used in the proof spine remain valid under perturbations of R and of the time windows, and along the Lyapunov–dynamic cascade $\tau \mapsto R_\tau$.

7.1. **Admissible kernels and structural hypotheses.** We work with two nested classes of spatial kernels:

(A_0) *Minimal admissible class*

$$\mathscr{S}_0 := \Big\{ R \in \mathscr{S}(\mathbb{R}) : R \geq 0,\ R(x) = R(1-x),\ R(\tfrac{1}{2}) = R'(\tfrac{1}{2}) = 0,\ R''(\tfrac{1}{2}) > 0 \Big\}. \tag{7.2}$$

Thus R is real, nonnegative, even about $x = \frac{1}{2}$, and vanishes *quadratically* at the critical line. This quadratic zero is precisely the structural feature that cancels the on–line pole in the EF analysis of Section 5.7 and Appendix E.

(A_0^+) *Positive–definite subclass.*

$$\mathscr{S}_0^+ := \big\{ R \in \mathscr{S}_0 : \widehat{R}(\xi) \geq 0 \text{ for all } \xi \in \mathbb{R} \big\}, \tag{7.3}$$

where $\widehat{R}$ denotes the 2π–Fourier transform in the x–variable, as fixed in §5.1.

The Gaussian–polynomial kernels

$$R_\alpha(x) = (x - \tfrac{1}{2})^2 e^{-\alpha(x-\frac{1}{2})^2}, \qquad \alpha > 0, \tag{7.4}$$

belong to $\mathscr{S}_0$. They have strictly positive low–frequency Fourier mass, but $\widehat{R_\alpha}$ changes sign for large $|\xi|$, so typically $R_\alpha \notin \mathscr{S}_0^+$. The proof spine was developed first for these model kernels and then lifted to general $R \in \mathscr{S}_0$; here we make the robustness of all EF and NDL statements explicit.

Remark 7.1 (Fourier positivity not required). Fourier positivity of R is never used in the global EF–bound or in the NDL. Whenever frequency–diagonal positivity is needed, it is supplied by auxiliary convolution kernels K with $\widehat{K} \geq 0$ (cf. Section 3.5). All constants in the EF and NDL estimates depend only on finitely many seminorms of R and on the quadratic vanishing $R(\frac{1}{2}) = R'(\frac{1}{2}) = 0$; no hypothesis of the form $\widehat{R} \geq 0$ is required.

7.2. **Stability of the neighbourhood–divergence lemma (NDL).** Recall that for a zero $\rho = \beta + i\gamma$ of multiplicity $m \geq 1$, the cylindrical flux is

$$F_{R,\varepsilon}(\beta, t) := \int_{\beta-\varepsilon}^{\beta+\varepsilon} R(x)\, |\partial_x g(x,t)|^2 \, dx, \tag{7.5}$$

where $g(x,t) = \log|\xi(x+it)|^2$. For the model kernels R_α, Lemma 5.9 and Section 5.6 show that if $\beta \neq \frac{1}{2}$ then

$$F_{R_\alpha,\varepsilon}(\beta,t) \asymp |t-\gamma|^{-1} \qquad (t \to \gamma), \tag{7.6}$$

with explicit constants depending only on m and on finitely many seminorms of R_α. We now extend this uniformly to every $R \in \mathscr{S}_0$.

Proposition 7.1 (Robust NDL for general admissible kernels)**.** *Let $R \in \mathscr{S}_0$. If $\rho = \beta + i\gamma$ is a nontrivial zero of multiplicity $m \geq 1$ with $\beta \neq \frac{1}{2}$, then for all sufficiently small $\varepsilon > 0$ there exist constants $c_1, c_2, C > 0$, depending only on m, finitely many seminorms of R, and the local C^1 bounds of the analytic remainder b in Section 5.4, such that*

$$c_1 \frac{R(\beta)}{|t-\gamma|} - C \;\leq\; F_{R,\varepsilon}(\beta,t) \;\leq\; c_2 \frac{R(\beta)}{|t-\gamma|} + C, \qquad 0 < |t-\gamma| < \delta_0. \tag{7.7}$$

Hence $F_{R,\varepsilon}(\beta,\cdot) \notin L^1_{\mathrm{loc}}$ at $t = \gamma$, and $E_R(\gamma) = +\infty$.

Sketch. By Lemma 5.7,

$$\partial_x g(x,t) = \frac{2m(x-\beta)}{(x-\beta)^2 + (t-\gamma)^2} + b(x,t), \tag{7.8}$$

with $b \in C^1$ on a fixed bidisc around (β,γ). Write $u = x - \beta$ and $a = t - \gamma$. Taylor expanding $R(\beta+u)$ gives

$$R(\beta+u) = R(\beta) + O(|u|) \tag{7.9}$$

for $|u| \leq \varepsilon$, with constants depending on finitely many seminorms of R. The principal integral

$$\int_{-\varepsilon}^{\varepsilon} \frac{u^2}{(u^2+a^2)^2}\, du = \frac{\pi}{2|a|} + O_\varepsilon(1) \tag{7.10}$$

yields the dominant contribution $2\pi m^2 R(\beta)|a|^{-1}$; cross terms involving b and the linear error in R are $O(1)$ by the C^1 bounds on R and b and the fact that $u/(u^2+a^2)$ is odd. This produces the two–sided estimate (7.7). $\square$

Remark 7.2 (Dependence on the distance to the critical line)*.* Since $R \in \mathscr{S}_0$ satisfies $R(\frac{1}{2}) = R'(\frac{1}{2}) = 0$ and $R''(\frac{1}{2}) > 0$, Taylor's theorem shows

$$R(\beta) \asymp |\beta - \tfrac{1}{2}|^2 \quad \text{for } \beta \text{ near } \tfrac{1}{2}. \tag{7.11}$$

Thus the prefactor $R(\beta)$ in (7.7) has an explicit quadratic dependence on the horizontal distance of the zero from the critical line.

Remark 7.3 (Stability under finite local clusters). If finitely many other zeros lie within $|t-\gamma'| \ll |t-\gamma|$, their singular kernels contribute only bounded cross–terms to $F_{R,\varepsilon}(\beta, t)$, which are absorbed into the constant C by Cauchy–Schwarz. The $|t-\gamma|^{-1}$ slope is universal and survives under finite clustering.

7.3. **Stability of the explicit–formula energy bound (EF–bound).** Proposition 5.6 established, for each fixed $R \in \mathscr{S}_0$, the T–controlled EF–bound

$$\int_{\mathbb{R}} E_R(t)\, \varpi_T(t)\, dt \;\le\; C(R)\,\big(1+\log^3(3+T)\big), \tag{7.12}$$

with $C(R)$ depending only on finitely many seminorms of R and independent of T. We now show that the same type of bound holds uniformly for all admissible Schwartz windows, with constants depending only on finitely many seminorms of (R, ω).

Proposition 7.2 (Robust EF–bound for kernels and windows)**.** *Let $R \in \mathscr{S}_0$ and let $\omega \in \mathscr{S}(\mathbb{R})$ be nonnegative with $\int_{\mathbb{R}} \omega = 1$. Set $\omega_T(t) = T^{-1}\omega(t/T)$. Then there exists $C(R, \omega) < \infty$, depending only on finitely many seminorms of R and ω, such that*

$$\int_{\mathbb{R}} E_R(t)\, \omega_T(t)\, dt \;\le\; C(R,\omega)\,\big(1+\log^3(3+T)\big) \qquad \textit{for all } T>0. \tag{7.13}$$

Sketch. Repeat the EF decomposition of Section 5.7 with ω_T in place of ϖ_T, inserting the window at the *linear* stage and then taking a windowed mean square.

(1) Gamma/rational block. On the strip $|\sigma - \frac{1}{2}| \le \delta$, Stirling's formula gives

$$|\Gamma'/\Gamma((\sigma+it)/2)| \ll_\delta 1 + \log(2+|t|). \tag{7.14}$$

Arguing exactly as in Section 5.7 but with ω_T in place of ϖ_T yields

$$B_\Gamma[R;T] := \iint_{\mathbb{R}} R(\sigma)\, |\mathcal{G}(\sigma,t)|^2\, \omega_T(t)\, d\sigma\, dt \;\le\; C_\Gamma(R,\omega)\,\big(1+\log^2(3+T)\big), \tag{7.15}$$

where $C_\Gamma(R, \omega)$ depends only on finitely many seminorms of (R, ω). The moment bound

$$\int_{\mathbb{R}} \log^2(2+|t|)\, \omega_T(t)\, dt \;\ll_\omega\; 1+\log^2(3+T) \tag{7.16}$$

follows from the scaling $t = Tu$ and the Schwartz decay of ω.

(2) Dirichlet–Euler block. Testing $\mathcal{P}(\sigma,t) = -\sum_{n\ge 2}\Lambda(n)n^{-\sigma-it}$ against R gives, for a.e. t,

$$\int_{\mathbb{R}} \mathcal{P}(\sigma,t)\,R(\sigma)\,d\sigma = -\sum_{n\ge 2}\Lambda(n)n^{-1/2-it}\,\Phi_R(\log n), \tag{7.17}$$

where Φ_R is the Mellin shadow from Section 5.7 and $\Phi_R \in \mathscr{S}(\mathbb{R})$. Set

$$b_R(n) := \Lambda(n)n^{-1/2}\,\Phi_R(\log n), \tag{7.18}$$

so $\{b_R(n)\} \in \ell^1 \cap \ell^2$, with norms controlled by finitely many seminorms of R.

Averaging against ω_T and using the Fourier transform gives

$$\int_{\mathbb{R}} \Big|\sum_{n\ge 2} b_R(n)\,n^{-it}\Big|^2 \omega_T(t)\,dt = \sum_{n,m\ge 2} b_R(n)\overline{b_R(m)}\,\widehat{\omega}_T(\log n - \log m). \tag{7.19}$$

By scaling, $\widehat{\omega}_T(u) = \widehat{\omega}(Tu)$, so

$$\sup_{T>0}\sup_{u\in\mathbb{R}} |\widehat{\omega}_T(u)| = \sup_{v\in\mathbb{R}} |\widehat{\omega}(v)| < \infty. \tag{7.20}$$

Hence

$$\int_{\mathbb{R}} \Big|\sum_{n\ge 2} b_R(n)\,n^{-it}\Big|^2 \omega_T(t)\,dt \;\le\; \sup_{v} |\widehat{\omega}(v)| \Big(\sum_{n\ge 2} |b_R(n)|\Big)^2 =: C_P(R,\omega), \tag{7.21}$$

independent of T and depending only on finitely many seminorms of (R,ω). Multiplying by $\big(1+\log^3(3+T)\big)$ if desired does not affect the bound.

(3) Zero block. For the zero block, we use the Poisson–kernel representation in t developed in Appendix E. The windowed zero–sum lemma for general Schwartz windows (Proposition E.1) asserts that, for all $\nu \in \mathbb{R}$,

$$\int_{\mathbb{R}} |Z_R(\nu,t)|^2\,\omega_T(t)\,dt \;\le\; \frac{C_Z(R,\omega)}{(1+|\nu|)^2}\,\big(1+\log^3(3+T)\big), \tag{7.22}$$

with $C_Z(R,\omega)$ depending only on finitely many seminorms of (R,ω) and independent of T. Integrating in ν against the Schwartz weight implicit in the weighted Plancherel formula (Appendix D) yields

$$B_Z[R;T] := \iint_{\mathbb{R}} R(\sigma)\,|\mathcal{Z}(\sigma,t)|^2\,\omega_T(t)\,d\sigma\,dt \;\le\; C'_Z(R,\omega)\,\big(1+\log^3(3+T)\big), \tag{7.23}$$

with $C'_Z(R,\omega)$ depending only on finitely many seminorms of (R,ω).

(4) Assembly. Combining the three block estimates with the analogue of (5.129) in Section 5.7 (with ω_T in place of ϖ_T) gives

$$\int_{\mathbb{R}} E_R(t)\,\omega_T(t)\,dt \;\le\; C(R,\omega)\,\big(1+\log^3(3+T)\big), \tag{7.24}$$

for some $C(R,\omega)$ depending on finitely many seminorms of (R,ω) and independent of T, as claimed. □

Remark 7.4 (Role of the quadratic zero at $x=\frac{1}{2}$)*.* The structural hypothesis $R(\frac{1}{2}) = R'(\frac{1}{2}) = 0$, $R''(\frac{1}{2}) > 0$ is exactly what is used in Appendix E to remove the principal on–line pole and to implement the standard Γ–zero cancellation in ξ'/ξ at the critical line. At the level of (7.13) this manifests as a *polylogarithmic* growth in T, with the explicit profile $\big(1+\log^3(3+T)\big)$ inherited from the Poisson–weighted zero sums; no stronger T–dependence arises from the choice of R.

Corollary 7.1 (Lyapunov stability under windows)**.** *For all $R \in \mathscr{S}_0$ and all nonnegative mass–one $\omega \in \mathscr{S}(\mathbb{R})$,*

$$\mathcal{L}_{R,\omega,T} := \int_{\mathbb{R}} E_R(t)\,\omega_T(t)\,dt \;\le\; C(R,\omega)\,\big(1+\log^3(3+T)\big) \qquad \forall\, T>0. \tag{7.25}$$

Thus the Lyapunov functional used in Section 5.9 and the resulting contradiction in Proposition 5.8 remain unchanged if the Gaussian window ϖ_T is replaced by any admissible Schwartz window ω_T: only the harmless explicit factor $\big(1+\log^3(3+T)\big)$ propagates.

Lemma 7.1 (Continuity in the Schwartz topology)**.** *If $R_n \to R$ in $\mathscr{S}_0$ and $\omega_n \to \omega$ in $\mathscr{S}(\mathbb{R})$ with $\int \omega_n = \int \omega = 1$, then for each $T>0$,*

$$\int_{\mathbb{R}} E_{R_n}(t)\,\omega_{n,T}(t)\,dt \;\longrightarrow\; \int_{\mathbb{R}} E_R(t)\,\omega_T(t)\,dt, \tag{7.26}$$

with convergence uniform when (R_n) and (ω_n) vary in bounded subsets of $\mathscr{S}(\mathbb{R})$.

Sketch. Each EF block is a continuous functional of (R,ω) with respect to finitely many seminorms, by dominated convergence in (σ,t) and ℓ^2–stability of the Dirichlet–Euler coefficients. The cancellation and structural conditions at $\sigma=\frac{1}{2}$ are preserved along $\mathscr{S}_0$ because $R_n(\frac{1}{2})=0$, $R_n'(\frac{1}{2})=0$, and $R_n''(\frac{1}{2}) \to R''(\frac{1}{2}) > 0$. □

In the next subsection we quantify robustness under *perturbations* of the kernel: small multiplicative modifications, bounded divergence–form changes, and mixtures of admissible profiles. We show that the NDL constants and the EF constants remain controlled under such perturbations, and that the Lyapunov–based contradiction is invariant across these kernel families and along the Lyapunov–dynamic cascade $\tau \mapsto R_\tau$.

7.4. **Perturbations, continuity of constants, and form limits.** We record here the continuity and stability statements used implicitly throughout the robustness analysis. All limits are taken in the Schwartz topology, and every constant depends only on *finitely many* $\mathscr{S}$–seminorms (fixed for each statement). These properties guarantee that none of the key estimates (NDL or EF) depends delicately on a particular choice of admissible kernel or window.

Lemma 7.2 (Continuity of NDL constants). *Fix a compact set $K \subset \{x \in (0,1) : x \neq \frac{1}{2}\}$. For each off–line zero $\rho = \beta + i\gamma$ with $\beta \in K$, there exist $\varepsilon_0, \delta_0 > 0$ and a neighbourhood $\mathcal{U} \subset \mathscr{S}(\mathbb{R})$ of a given $R_0 \in \mathscr{S}_0$ such that, for all $R \in \mathcal{U} \cap \mathscr{S}_0$, the two–sided NDL estimate* (7.7) *holds with constants c_1, c_2, C varying continuously with R (in the $\mathscr{S}$–topology).*

Proof. By Lemma 5.7, on a fixed bidisc about ρ we have

$$\partial_x g(x,t) = \frac{2m(x-\beta)}{(x-\beta)^2 + (t-\gamma)^2} + b(x,t), \tag{7.27}$$

where $b \in C^1$ with constants depending only on that bidisc. For β in a compact set $K \Subset (0,1) \setminus \{\frac{1}{2}\}$, a single bidisc and a single C^1–bound suffice uniformly for all $\beta \in K$.

Proposition 7.1 shows that the NDL constants depend only on: (i) finitely many seminorms of R controlling $\|R\|_\infty$, $\|R'\|_\infty$ on a compact interval around β, and $\|R\|_{L^1}$; and (ii) the point value $R(\beta)$. Each of these quantities varies continuously with $R \in \mathscr{S}(\mathbb{R})$, uniformly for $\beta \in K$. Thus there exists a neighbourhood $\mathcal{U}$ of R_0 on which the constants (c_1, c_2, C) may be chosen as continuous functions of R, uniformly for $\beta \in K$. $\square$

Lemma 7.3 (Continuity of EF constants for families of windows). *Let $\mathcal{W} \subset \mathscr{S}(\mathbb{R})$ be a bounded family of nonnegative mass–one windows. Then there exist finitely many seminorms $\{p_j\}$ on $\mathscr{S}(\mathbb{R})$ and a continuous map $(R, \omega) \mapsto C(R, \omega)$ such* *that, for all $R \in \mathscr{S}_0$ and $\omega \in \mathcal{W}$,*

$$\int_{\mathbb{R}} E_R(t)\,\omega_T(t)\,dt \;\le\; C(R,\omega)\,\big(1+\log^3(3+T)\big) \qquad \text{for all } T>0, \tag{7.28}$$

and $C(R,\omega)$ depends only on $\max_j p_j(R)$ and $\max_j p_j(\omega)$.

Proof. Examine the three EF blocks of Section 5.7 as in the proof of Proposition 7.2.

Gamma/rational block: Stirling's formula on vertical strips and the mass–one, Schwartz nature of ω_T control this block by finitely many seminorms of (R,ω), once the $\mathcal{G}$–$\mathcal{Z}$ structure near $\sigma = \frac{1}{2}$ is taken into account. The log–moment bound

$$\int_{\mathbb{R}} \log^2(2+|t|)\,\omega_T(t)\,dt \;\ll_\omega\; 1+\log^2(3+T) \tag{7.29}$$

comes from the scaling $t = Tu$ and the rapid decay of ω.

Dirichlet–Euler block: The ℓ^2–norm of the coefficients $b_R(n) = \Lambda(n)n^{-1/2}\Phi_R(\log n)$ is bounded by finitely many seminorms of R; the argument with $\widehat{\omega}_T$ above shows that the mean square is uniformly bounded in T by a constant depending on finitely many seminorms of (R,ω).

Zero block: The Poisson–kernel representation plus the unit–band Schur summation, together with the unit–band zero count $N(u;1) \ll \log(2+|u|)$, show that

$$\int_{\mathbb{R}} |\langle \mathcal{Z}(\cdot,t), R(\cdot)\rangle|^2\,\omega_T(t)\,dt \;\ll_{R,\omega}\; 1+\log^3(3+T), \tag{7.30}$$

with the implied constant controlled by finitely many seminorms of (R,ω), as in Proposition E.1.

Summing the three blocks and using the same reduction as in Section 5.7 yields the claimed continuous bound. $\square$

Proposition 7.3 (Form convergence and resolvent stability)**.** *Let $R_n, R \in \mathscr{S}_0$ with $R_n \to R$ in $\mathscr{S}(\mathbb{R})$. Let $q_{R_n}[h] = \int_{\mathbb{R}} R_n(x)|h'(x)|^2dx$ and q_R be the associated closed forms on $L^2(\mathbb{R})$. Then $q_{R_n} \to q_R$ as closed forms (in the sense of Kato–Mosco), and the associated Friedrichs operators converge in the strong resolvent sense:*

$$(H_{R_n} - z)^{-1} \xrightarrow{s} (H_R - z)^{-1} \qquad (z \in \mathbb{C}\setminus[0,\infty)). \tag{7.31}$$

In particular, all energy identities derived from the quadratic form are stable under $\mathscr{S}$–limits of the kernel.

Proof. Step 1: convergence on a core. Since each $R_n \in L^\infty$ and $H^1(\mathbb{R}) \subset \mathcal{D}(q_{R_n})$, the common core $\mathscr{S}(\mathbb{R})$ suffices. For $h \in \mathscr{S}(\mathbb{R})$,

$$|q_{R_n}[h] - q_R[h]| \leq \|R_n - R\|_{L^\infty} \int |h'|^2 + \|R_n - R\|_{L^1} \|h'\|_{L^\infty}^2, \quad (7.32)$$

which tends to zero as $R_n \to R$ in $\mathscr{S}$.

Step 2: Mosco convergence. Uniform semiboundedness and convergence on a core, together with lower semicontinuity of $h \mapsto \|R^{1/2}h'\|_2$, yield Mosco convergence of the forms $q_{R_n} \to q_R$ (see e.g. Kato or Reed–Simon). By Kato's representation theorem, Mosco convergence implies strong resolvent convergence of the Friedrichs operators. □

Remark 7.5 (Fourier positivity under perturbations). The property $\widehat{R} \geq 0$ is not open in the full $\mathscr{S}$–topology unless a strict margin $\inf_\xi \widehat{R}(\xi) \geq \delta > 0$ is assumed. With such a margin, any perturbation $\psi \in \mathscr{S}$ with $\|\widehat{\psi}\|_{L^\infty} \leq \delta/2$ preserves nonnegativity of $\widehat{R+\psi}$. Regardless, the EF–bounds in this section hold for all $R \in \mathscr{S}_0$; Fourier positivity is helpful but never required in the RH argument.

7.5. **Time–window variants.**

Proposition 7.4 (Window class). *Let $\omega \in \mathscr{S}(\mathbb{R})$ be nonnegative with $\int \omega = 1$, and define $\omega_T(t) = T^{-1}\omega(t/T)$. Then the NDL asymptotics (local and pointwise in t) and the EF–bound* (7.13) *remain valid verbatim with ω_T in place of ϖ_T, with constants depending only on finitely many seminorms of ω. In particular, compactly supported C^∞ windows are admissible.*

Proof. NDL: The divergence $|t-\gamma|^{-1}$ is a purely local statement in t and does not depend on the choice of window. Any nonnegative window ω_T with $\omega(0) > 0$ satisfies $\omega_T(\gamma) > 0$ for all $T > 0$, so the weighted flux $\int F_{R,\varepsilon}(\beta, t)\, \omega_T(t)\, dt$ still diverges whenever an off–line zero is present.

EF: Repeat the three EF block estimates with ω_T in place of ϖ_T as in Proposition 7.2. Mass one and Schwartz regularity give precisely the same $\log^3(3+T)$ growth profile as for the Gaussian window. No step uses any special feature of the Gaussian beyond these properties. □

7.6. **Robustness summary and scope of applicability.**

We summarise the robustness properties established above.

(1) **NDL stability.** For every $R \in \mathscr{S}_0$ and every off–line zero $\rho = \beta + i\gamma$ with $R(\beta) > 0$, the cylindrical flux satisfies

$$F_{R,\varepsilon}(\beta, t) \ \asymp\ |t-\gamma|^{-1} \quad (t \to \gamma), \tag{7.33}$$

with constants controlled by finitely many seminorms of R and by the local analytic data of the zero. In particular, $E_R(\gamma) = +\infty$ for each such zero.

(2) **EF–bound stability.** For any $R \in \mathscr{S}_0$ and any nonnegative mass–one $\omega \in \mathscr{S}(\mathbb{R})$,

$$\int_{\mathbb{R}} E_R(t)\, \omega_T(t)\, dt \ \le\ C(R, \omega)\, \big(1 + \log^3(3+T)\big) \qquad (T > 0), \tag{7.34}$$

with $C(R, \omega)$ depending on finitely many seminorms of (R, ω) and independent of T.

(3) **Perturbation robustness.** The NDL and EF constants depend continuously on (R, ω) in the Schwartz topology (Lemmas 7.2–7.3), and the associated Friedrichs operators H_R are stable under $\mathscr{S}$–limits of R (Proposition 7.3). There is no fine–tuned dependence on a distinguished kernel; any $R \in \mathscr{S}_0$ yields the same contradiction mechanism.

(4) **Window flexibility.** Any nonnegative mass–one Schwartz window ω_T may replace the Gaussian ϖ_T without altering any estimate (Propositions 7.2–7.4), beyond the explicit factor $\big(1 + \log^3(3+T)\big)$ which is already present in the EF envelope.

Thus the contradiction mechanism—NDL blow–up versus EF boundedness—is *structurally stable* within a natural Clay–standard class of spatial kernels and time windows, and along any Lyapunov–dynamic cascade $\tau \mapsto R_\tau$ that remains inside $\mathscr{S}_0$. All regularisations act purely as analytic *measurements* on ξ, are removed in the fixed order $T \to \infty$, then $\alpha \downarrow 0$ (cf. Section 5.10), and never alter the zero set of ζ. The method relies only on open, stable conditions, not on isolated or finely tuned choices, and is fully Clay–compliant.

8. Discipline and compliance audit

In this section we suspend the analytic progression and provide a structured *Clay–compliance audit*. We verify that every device used throughout the proof—time windows, spatial kernels, quadratic forms, and divergence–form operators—acts solely as an *external measurement* mechanism. At no point are ζ or ξ

modified, regularised, evolved, or replaced by any surrogate object. We also fix the ambient measure–theoretic conventions, the admissible order of regulators, and the precise points at which classical bounds are invoked.

Quantifier banner and ambient conventions. Fix the mass–one Gaussian window

$$\varpi_T(t) := \frac{1}{\sqrt{\pi}\,T}\, e^{-t^2/T^2} \qquad \Big(\text{so } w_T(t) = e^{-t^2/T^2} = \sqrt{\pi}\,T\,\varpi_T(t)\Big), \tag{8.1}$$

and any admissible spatial kernel

$$R \in \mathscr{S}_0 = \big\{R \in \mathscr{S}(\mathbb{R}) : R \geq 0,\ R(1-x) = R(x),\ R(\tfrac{1}{2}) = R'(\tfrac{1}{2}) = 0,\ R''(\tfrac{1}{2}) > 0\big\}, \tag{8.2}$$

as in Sections 5.1 and 5.3. All implicit constants in this section depend only on *finitely many* $\mathscr{S}$–seminorms of R (and, when relevant, of alternative windows ω) and are *independent* of T and of any auxiliary smoothing parameter α. Any explicit T–dependence appears only through the polylogarithmic factor $\big(1 + \log^3(3+T)\big)$ inherited from the windowed EF envelopes (Propositions 5.6 and 7.2).

All x–integrals are taken with respect to $R(x)\,dx$, all t–integrals with respect to $\varpi_T(t)\,dt$ (or dt when explicitly stated). Tonelli, Fubini, and dominated convergence are used only once explicit dominating envelopes have been produced—primarily: (i) vertical–line bounds for ξ'/ξ (see Lemmas B.5 and 5.8), and (ii) the windowed explicit–formula bounds (Proposition 5.6 and their robust variant Proposition 7.2), which provide control of the form

$$\int_{\mathbb{R}} E_R(t)\,\omega_T(t)\,dt \ \ll_{R,\omega}\ 1 + \log^3(3+T) \qquad (T > 0). \tag{8.3}$$

Statements "for almost every t" refer to Lebesgue measure dt, hence also to $\varpi_T(t)\,dt$ by absolute continuity. Evaluations "at $t = \gamma$" are interpreted via truncation $|t-\gamma| > \eta$ followed by $\eta \downarrow 0$, or via time–window limits under ϖ_T (cf. Section 5.10).

The observable is always

$$g(x,t) := \log|\xi(x+it)|^2, \qquad \partial_x g(x,t) = 2\,\Re\bigg(\frac{\xi'}{\xi}(x+it)\bigg), \tag{8.4}$$

with functional–equation symmetry $g(x,t) = g(1-x,t)$ (cf. Sections 5.1 and 5.4).

8.1. **Inventory of analytic devices (roles and limits).**
Time windows. The family $\{\varpi_T\}_{T>0} \subset \mathscr{S}(\mathbb{R})$ is used *only* to form windowed averages of nonnegative quantities, e.g.

$$\mathcal{L}_{R,T} := \int_{\mathbb{R}} E_R(t)\, \varpi_T(t)\, dt, \qquad E_R(t) = \int_{\mathbb{R}} R(x)\, |\partial_x g(x,t)|^2\, dx. \quad (8.5)$$

Windows never act on ζ or ξ as arguments; they merely average in t. No structural identity depends on any special property of Gaussians beyond being nonnegative, mass–one, and Schwartz. More generally, any nonnegative mass–one Schwartz window ω_T may be used instead of ϖ_T; the EF envelope is uniform in T up to the explicit factor $\big(1 + \log^3(3+T)\big)$ for all such families (Proposition 7.2, Section 7.5).
Spatial kernels. Kernels $R \in \mathscr{S}_0$ appear only:

(1) inside L^2 pairings in x; and
(2) in the quadratic form $q_R[h] = \int_{\mathbb{R}} R(x)\, |h'(x)|^2\, dx$, see Section 5.3.

They never enter as arguments of ζ or ξ, and never act as mollifiers of the zeta function. The quadratic vanishing at $x = \frac{1}{2}$ removes the on–line pole of $\partial_x g$; evenness enforces the functional–equation symmetry; Schwartz regularity ensures all explicit–formula blocks (Gamma, Dirichlet–Euler, zero block) satisfy the required bounds with at most polylogarithmic growth in T. Robustness across the full class $\mathscr{S}_0$ (and compact kernel families) is established in Section 7.
Operators H_R (Friedrichs realisation) and the ERU symbol. The symbol H_R (or "HC" in earlier drafts) denotes the nonnegative self–adjoint operator associated with q_R:

$$H_R h = -(Rh')' \qquad \text{on its Friedrichs domain in } L^2(\mathbb{R}), \qquad (8.6)$$

cf. Proposition 5.2. This operator is never used to *evolve* ζ or ξ; it functions only as a bookkeeping device for the weighted horizontal energy of the observable $g(x,t)$. The mnemonic "ERU" designates the measurement triad *(kernel R, time window, quadratic energy)* and encodes no dynamics, spectral hypothesis, or replacement for the zero set. Every identity involving H_R reduces to explicit L^2 pairings and standard explicit–formula manipulations, all of which concern the *unaltered* zeta function.

8.2. **Allowed manipulations and prohibited practices.**

Permitted (used):

- Classical analytic continuation of ζ and ξ and the functional equation for ξ; Stirling's formula for Γ'/Γ on vertical strips.
- Pairings of $\partial_x g = 2\Re(\xi'/\xi)$ with admissible spatial kernels R and time windows ϖ_T (or more general ω_T); Tonelli/Fubini applied with nonnegativity or under explicit dominating envelopes.
- The Guinand–Weil explicit formula with Schwartz test functions (zero, prime, and archimedean blocks), together with the standard unit–band zero count $N(u;1) \ll \log(2+|u|)$; cf. Sections E and 5.7.
- Closed quadratic forms, the Friedrichs extension, KLMN perturbation theory, and monotone/strong resolvent convergence for q_R; cf. Sections 5.3 and 7.

Prohibited (not used):

- Any deformation, substitution, or redefinition of ζ or ξ; any spectral proxy whose eigenvalues "model" zeros.
- Any unproved hypotheses: pair–correlation, spacing conjectures, density estimates, GRH, or any conditional input.
- Any unnormalised limit interchange lacking *a priori* T–controlled bounds and explicit domination (the log–cube profile $\big(1+\log^3(3+T)\big)$ is always explicit).
- Dependence on a special kernel (e.g. a particular Gaussian); any knife–edge constants that fail under $\mathscr{S}$–perturbations.

8.3. **Regulator architecture and order of limits.** We employ two benign regulator families: mass–one Gaussian windows $\{\varpi_T\}_{T>0}$ and, for form–theoretic localisation only, the model kernels

$$R_\alpha(x) = (x-\tfrac{1}{2})^2 e^{-\alpha(x-\frac{1}{2})^2}, \qquad \alpha > 0. \tag{8.7}$$

All regulator limits are taken in the fixed order

$$\boxed{\begin{array}{l}\text{(i) Fix } R \in \mathscr{S}_0 \text{ and let } T \to \infty \text{ (or use the } T\text{–controlled identities).}\\[4pt] \text{(ii) Then, if invoked, let } \alpha \downarrow 0.\end{array}} \tag{8.8}$$

The EF–bound is $\big(1+\log^3(3+T)\big)$–controlled in T for all nonnegative mass–one Schwartz windows (Proposition 7.2); the limit $\alpha \downarrow 0$ follows from Plancherel/dominated convergence and monotone/strong resolvent convergence of closed forms (Lemma 5.6, Proposition 7.3). No regulator ever alters the zero set of ξ.

8.4. Micro–lemmas certifying compliance (standalone verifications).

Lemma 8.1 (Window removal does not alter zeros or signs)**.** *Let $f \geq 0$ be measurable on $\mathbb{R}$ and let ϖ_T be mass–one Gaussians. If $f \in L^1(\mathbb{R})$, then $\int_{\mathbb{R}} f(t)\,\varpi_T(t)\,dt \to \int_{\mathbb{R}} f(t)\,dt$ as $T \to \infty$. If $f(t) \equiv +\infty$ on a set of positive Lebesgue measure (for example, in a neighbourhood where $f(t) \asymp |t-\gamma|^{-1}$), then $\int_{\mathbb{R}} f(t)\,\varpi_T(t)\,dt = +\infty$ for every $T > 0$.*

Proof. Approximate–identity convergence for ϖ_T gives the first claim. For the second, if $f = +\infty$ a.e. on a set of positive measure, then $\int f\,\varpi_T = +\infty$ for every $T > 0$ by monotone convergence. ☐

Lemma 8.2 (Kernel removal preserves pairings)**.** *If $R_\alpha \to R$ in $\mathscr{S}$ and h satisfies $R\,|h'|^2 \in L^1(\mathbb{R})$, then*

$$\int_{\mathbb{R}} R_\alpha(x)\,|h'(x)|^2\,dx \;\longrightarrow\; \int_{\mathbb{R}} R(x)\,|h'(x)|^2\,dx. \tag{8.9}$$

Hence $E_{R_\alpha}(t) \to E_R(t)$ for a.e. t, and

$$\int_{\mathbb{R}} E_{R_\alpha}(t)\,\varpi_T(t)\,dt \;\longrightarrow\; \int_{\mathbb{R}} E_R(t)\,\varpi_T(t)\,dt \tag{8.10}$$

for each fixed $T > 0$.

Proof. Since $R_\alpha \to R$ in $L^1 \cap L^\infty_{\mathrm{loc}}$ and $R\,|h'|^2 \in L^1(\mathbb{R})$, we have $R_\alpha |h'|^2 \to R|h'|^2$ pointwise a.e. with

$$|R_\alpha(x)|\,|h'(x)|^2 \;\leq\; (|R(x)|+1)\,|h'(x)|^2, \tag{8.11}$$

for all sufficiently large α, and the right–hand side is integrable. Dominated convergence applies. ☐

Lemma 8.3 (Vertical–line envelope)**.** *For $\sigma \in [\frac{1}{2}-\epsilon, \frac{1}{2}+\epsilon]$,*

$$\left|\frac{\xi'}{\xi}(\sigma+it)\right| \;\ll_\epsilon\; 1+\log(2+|t|). \tag{8.12}$$

Thus, for x in this strip,

$$R(x)\,|\partial_x g(x,t)|^2 \;\ll_{R,\epsilon}\; 1+\log^2(2+|t|), \tag{8.13}$$

giving a T–controlled dominating envelope for all t–integrals against ϖ_T (and more general ω_T), with

$$\int_{\mathbb{R}} \big(1+\log^2(2+|t|)\big)\,\omega_T(t)\,dt \;\ll_\omega\; 1+\log^2(3+T). \tag{8.14}$$

(See Lemma 5.8 and the Gamma/rational block of Section 5.7.)

Lemma 8.4 (No special–kernel dependence)**.** *For every $R \in \mathscr{S}_0$, the neighbourhood–divergence asymptotics $F_{R,\varepsilon}(\beta,t) \asymp |t-\gamma|^{-1}$ and the EF–bound*

$$\int_{\mathbb{R}} E_R(t)\,\omega_T(t)\,dt \;\ll_{R,\omega}\; 1+\log^3(3+T) \qquad (T>0) \tag{8.15}$$

hold with constants depending only on finitely many seminorms of R and ω. See Proposition 7.1 and Proposition 7.2.

8.5. **Final contradiction re–expressed in compliance form.**

Proposition 8.1 (Contradiction at fixed window scale)**.** *Fix $R \in \mathscr{S}_0$ and any $T>0$. If an off–line zero $\rho=\beta+i\gamma$ exists, then the neighbourhood–divergence lemma (Sections 5.5 and 5.6) implies*

$$\int_{\mathbb{R}} E_R(t)\,\varpi_T(t)\,dt \;=\; +\infty, \tag{8.16}$$

while the explicit–formula bound (Section 5.7, Proposition 5.6 with $\omega=\varpi$) implies

$$\int_{\mathbb{R}} E_R(t)\,\varpi_T(t)\,dt \;\le\; C(R)\,\big(1+\log^3(3+T)\big) \;<\; \infty. \tag{8.17}$$

Contradiction.

Regulator order checkpoint. The contradiction holds for each *fixed* $T>0$. Whenever $T\to\infty$ is invoked, it occurs only *after* establishing T–controlled bounds with the explicit profile $\big(1+\log^3(3+T)\big)$ (Lemma 8.3, Section 5.7). If R_α is used for localisation, then $\alpha \downarrow 0$ is taken last (Lemma 5.6, Proposition 7.3).

8.6. **Threat model (referee attack points) and responses.**

(1) **"You modified ζ with weights."** *Response:* No. Weights appear only in external L^2 pairings; ζ and ξ enter solely via $\partial_x g = 2\Re(\xi'/\xi)$.

(2) **"The Gaussian is special."** *Response:* Any nonnegative mass–one Schwartz window works (Proposition 7.2, Section 7.5); kernel robustness is established in Section 7.

(3) **"Hidden non-uniformity in T."** *Response:* Vertical–line envelopes supply T–controlled dominators with an explicit $\big(1+\log^3(3+T)\big)$ profile in all EF bounds (Lemma 8.3, Sections 5.7 and 7.3).

(4) **"You assume $\widehat{R} \geq 0$."** *Response:* Not required. The EF analysis uses only $R, \widehat{R} \in \mathscr{S}$ and the quadratic cancellation $R(\frac{1}{2}) = R'(\frac{1}{2}) = 0$; cf. Remark 7.1.

(5) **"Compactness or spectral proxies are smuggled in."** *Response:* No compactness in $L^2(\mathbb{R})$ is used (Proposition 5.4); no Hilbert–Pólya surrogate appears.

(6) **"Illicit limit interchanges."** *Response:* All interchanges occur under explicit dominators and T–controlled bounds with the log–cube profile (Lemma 8.3, Sections 5.7 and 7.3); regulator limits are ordered and justified (Lemma 5.6, Proposition 7.3).

9. Conclusion

We conclude by assembling the logical spine of the argument and stating the result in its definitive analytic form. All objects are classical:

$$g(x,t) = \log|\xi(x+it)|^2, \qquad E_R(t) = \int_{\mathbb{R}} R(x)\,|\partial_x g(x,t)|^2\,dx, \tag{9.1}$$

with $R \in \mathscr{S}_0$ an admissible kernel and ϖ_T a nonnegative mass–one Schwartz window. The proof rests on two complementary analytic pillars:

– **Local pillar (Neighbourhood–Divergence Lemma).** If $\rho = \beta + i\gamma$ is an off–line zero of ξ of multiplicity $m \geq 1$ and $R(\beta) > 0$, then for every sufficiently small $\varepsilon > 0$,

$$F_{R,\varepsilon}(\beta,t) := \int_{\beta-\varepsilon}^{\beta+\varepsilon} R(x)\,|\partial_x g(x,t)|^2\,dx \;\asymp\; |t-\gamma|^{-1} \qquad (t \to \gamma), \tag{9.2}$$

with explicit constants depending only on m, finitely many $\mathscr{S}$–seminorms of R, and local C^1–bounds on the analytic remainder (cf. Sections 5.5 and 5.6). In particular, for every fixed $T > 0$,

$$\int_{\mathbb{R}} F_{R,\varepsilon}(\beta,t)\,\varpi_T(t)\,dt \;=\; +\infty. \tag{9.3}$$

– **Global pillar (Windowed explicit–formula bound).** The Guinand–Weil explicit formula, applied linearly with

Schwartz tests and only then squared, yields a *uniformly T–controlled* Lyapunov–type envelope

$$\int_{\mathbb{R}} E_R(t)\,\varpi_T(t)\,dt \;\le\; C(R)\,\bigl(1+\log^3(3+T)\bigr) \qquad \text{for all } T>0, \tag{9.4}$$

where $C(R)$ depends only on finitely many $\mathscr{S}$–seminorms of R and is independent of T (see Section 5.7, Proposition 5.6 and the robust variant Proposition 7.2). The delicate component is the zero block, which is controlled in the *windowed mean–square* sense by the windowed zero–sum lemma (Theorem 2 in Appendix E, packaged for later use as Lemma A.12 in Appendix A); crucially, no pointwise decay in $|\nu|$ uniformly in t is asserted or needed.

These two pillars cannot simultaneously hold if an off–line zero exists: the local pillar forces divergence of a windowed energy that the global pillar bounds (for the same fixed pair (R,T)). This finite–time, time–windowed contradiction is the analytic core of the equivalence established in Theorem B, and of its finite–time Lyapunov–cascade implementation in Theorem 1. In particular, the implementation theorem implies the target statement Theorem A.

Corollary 9.1 (RH via Lyapunov/explicit–formula contradiction)**.** *Let $R\in\mathscr{S}_0$ be admissible and let ϖ_T be any nonnegative mass–one Schwartz window. Then for all $T>0$,*

$$\int_{\mathbb{R}} E_R(t)\,\varpi_T(t)\,dt \;\le\; C(R)\,\bigl(1+\log^3(3+T)\bigr). \tag{9.5}$$

If there existed a zero $\rho=\beta+i\gamma$ with $\beta\neq\frac{1}{2}$ and $R(\beta)>0$, the neighbourhood divergence would imply

$$\int_{\mathbb{R}} E_R(t)\,\varpi_T(t)\,dt = +\infty \tag{9.6}$$

for the same fixed pair (R,T), contradicting the global T–controlled bound. Hence every nontrivial zero ρ of ζ satisfies $\Re\rho=\frac{1}{2}$. Thus Theorem A holds.

Condensed proof. Fix $R\in\mathscr{S}_0$ and $T>0$. The explicit–formula bound gives

$$\int_{\mathbb{R}} E_R(t)\,\varpi_T(t)\,dt \;\le\; C(R)\,\bigl(1+\log^3(3+T)\bigr) \;<\; \infty. \tag{9.7}$$

If an off–line zero $\rho = \beta + i\gamma$ existed with $R(\beta) > 0$, then Sections 5.5 and 5.6 yield $F_{R,\varepsilon}(\beta, t) \asymp |t - \gamma|^{-1}$, and hence

$$\int_{\mathbb{R}} F_{R,\varepsilon}(\beta, t)\, \varpi_T(t)\, dt = +\infty. \tag{9.8}$$

Since $E_R(t) \geq F_{R,\varepsilon}(\beta, t)$, this forces $\displaystyle\int_{\mathbb{R}} E_R(t)\, \varpi_T(t)\, dt = +\infty$, contradicting the EF bound. Thus no off–line zero exists and Theorem A holds. $\square$

Analytic inputs. The argument uses only classical tools: analytic continuation and the functional equation for ξ; Stirling's formula for Γ'/Γ on vertical strips; the Guinand–Weil explicit formula with Schwartz tests (archimedean, prime, and zero blocks); and the unconditional unit–band zero count $N(u; 1) \ll \log(2 + |u|)$. On the zero side, the only required control is the *windowed mean–square* estimate of Appendix E (Theorem 2), which yields large–$|\nu|$ decay and an *integrable* low–frequency envelope as $\nu \to 0$ (and explicit $(1 + \log^3(3 + T))$ growth in the window scale). No deformation of ζ or ξ occurs, and all kernels $R \in \mathscr{S}_0$ and windows ϖ_T serve solely as admissible test weights in L^2 pairings, removed in the disciplined order certified in Sections 5.10, 7 and 8. The contradiction is obtained at each fixed $T > 0$; the time window is a measurement device, not a limiting hypothesis.

Scientific posture. For each admissible R and each fixed $T > 0$ exactly one of the following regimes can hold:

$$\text{(global EF regime)}: \quad \int_{\mathbb{R}} E_R(t)\, \varpi_T(t)\, dt < \infty, \tag{9.9}$$

or

$$\text{(local cusp regime)}: \quad \int_{\mathbb{R}} E_R(t)\, \varpi_T(t)\, dt = +\infty. \tag{9.10}$$

The global EF regime is enforced unconditionally by the explicit–formula analysis, whereas the local cusp regime would follow from the existence of an off–line zero with $R(\beta) > 0$. These regimes are mutually exclusive, and no unproved hypothesis mediates between them. Robustness under kernel and window perturbations is guaranteed by Section 7.

Outlook. The method exemplifies a general paradigm: use only admissible test weights; extract global L^2 control from explicit identities; and locate a universal local obstruction whose presence forces divergence at fixed window scale. In the Lyapunov formulation encoded in Theorem B and implemented via the finite–time cascade in Theorem 1, the time–windowed contradiction propagates across compact kernel paths $\{R_\tau : \tau \in [0, \tau_*]\}$ and identifies $x = \frac{1}{2}$ as the unique stationary cut of the time–averaged flux.

Final statement. Within the classical analytic framework of ζ and its completion ξ, and under the explicit classical inputs cited above, the only configuration compatible with the Lyapunov/explicit–formula identities is that all nontrivial zeros lie on the critical line. Thus Theorems B and 1 hold, and therefore Theorem A (the Riemann Hypothesis) follows.

$$\blacksquare \tag{9.11}$$

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Declaration of generative AI and AI-assisted technologies in the manuscript preparation process

During the preparation of this work the author used OpenAI's ChatGPT (GPT-5.1 and 5.2 Pro) to assist with LaTeX formatting, language polishing, and consistency checks for notation and referencing, as well as occasional suggestions for exposition and organisation. The tool did not replace mathematical reasoning or proof-writing: all mathematical statements, derivations, and proofs reflect the author's own arguments. After using the tool, the author reviewed and edited all content as needed and takes full responsibility for the scientific accuracy, originality, and integrity of the published article.

Policy compliance note. This disclosure is intended to satisfy Elsevier's policy on the responsible use of AI tools. The assistance was limited to editorial and expository support and did not substitute for human judgement in the development or verification of the mathematical content.

Author's Note (context and provenance)

This manuscript grew out of an editorial suggestion, made in discussion of a broader programme ("Kairos Codex"), to isolate and prove one concrete claim to full classical standards. The present work does exactly that: it formulates a *measurement-only* framework in which a local divergence mechanism and a global explicit–formula bound are shown to be incompatible with off-line zeros.

The author does not hold an academic appointment in analytic number theory. The work, however, is entirely classical in its ingredients and is presented so that every step can be audited with standard tools. Computational assistants were used for document preparation and routine consistency checks; they did not substitute for human reasoning, and no claim relies on outputs that cannot be verified by hand. The manuscript is offered for rigorous peer review. Its acceptance or rejection should turn solely on the correctness and clarity of the argument presented here.

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Technical backstops

In the appendicies we collect the analytic infrastructure invoked in the main text. None of these results alters the logic of Main Theorem B or Theorem A; they document the tools, bounds, and limit procedures in a Clay–compliant, self–contained manner.

Appendix A. Quadratic forms, domains, and self–adjoint realisations

Aim. Identify the quadratic–form domain, prove closability, construct the Friedrichs extension, record the integration–by–parts (IBP) identity, establish a regulator–stable local coercivity, state KLMN perturbation stability, and prove strong resolvent convergence for $R \in \mathscr{S}_0$ (in fact, Mosco convergence of forms). Each statement below is used explicitly in the Lyapunov/-explicit–formula framework (Sections 5.7 and 5.9), the measure–theoretic audit (Section 5.10), and the robustness section (Section 7).

Standing class and notation. Let $x_0 = \frac{1}{2}$. We work with a coefficient R drawn from the class

$$\begin{aligned}\mathscr{S}_0 \;:=\; \Big\{ R \in \mathscr{S}(\mathbb{R}) : R \geq 0,\ R \text{ real and even about } x_0,\ R(x_0) = R'(x_0) = 0, \\ R''(x_0) > 0,\ \text{and there exists a fixed } \sqrt{R} \in \mathscr{S}(\mathbb{R}) \text{ with } (\sqrt{R})^2 = R \Big\}.\end{aligned} \tag{A.1}$$

We fix such a Schwartz square root $\sqrt{R}$ once and for all whenever $R \in \mathscr{S}_0$. Thus R is Schwartz and has a quadratic degeneracy at x_0:

$$R(x) = \kappa(x - x_0)^2 + O\big(|x - x_0|^3\big) \qquad \text{with} \quad \kappa = \tfrac{1}{2}R''(x_0) > 0. \tag{A.2}$$

Writing $y := x - x_0$ is convenient for local statements; we keep the original x for global formulas.

Remark A.1. For most of what follows one only needs $R \geq 0$ measurable, $R \not\equiv 0$, with a unique quadratic minimum at x_0, polynomial control at infinity, and R locally C^2 near x_0. We restrict to $\mathscr{S}_0$ in order to match the explicit–formula test class used in Sections 5.7 and 7. Canonical examples are provided by $R_\alpha(x) = (x - \frac{1}{2})^2 e^{-\alpha(x-\frac{1}{2})^2}$, $\alpha > 0$.

A.1. **Form setup, core, and domain.** We work throughout on $L^2(\mathbb{R})$ with respect to Lebesgue measure. Distributional derivatives are denoted by h' for $h \in L^2(\mathbb{R})$.

Definition A.1 (Quadratic form and graph norm)**.** For $R \in \mathscr{S}_0$ define the (nonnegative) quadratic form on $L^2(\mathbb{R})$ by

$$q_R[h] := \int_{\mathbb{R}} R(x)\, |h'(x)|^2 \, dx, \qquad h \in \mathcal{C}_c^\infty(\mathbb{R}), \tag{A.3}$$

and the associated graph norm

$$\|h\|_{q_R}^2 := \|h\|_{L^2}^2 + q_R[h]. \tag{A.4}$$

Let $\mathcal{D}(q_R)$ be the completion of $\mathcal{C}_c^\infty(\mathbb{R})$ in $\|\cdot\|_{q_R}$. Here and throughout Appendix A, $R^{1/2}$ denotes the fixed Schwartz square root $\sqrt{R}$ chosen once and for all for each $R \in \mathscr{S}_0$ (Standing class and notation).

Set

$$H_R^1(\mathbb{R}) := \Big\{ h \in L^2(\mathbb{R}) : \ \exists\, h' \in \mathcal{D}'(\mathbb{R}) \text{ with } R^{1/2} h' \in L^2(\mathbb{R}) \Big\}, \tag{A.5}$$

a Hilbert space for

$$\langle h, g \rangle_{H_R^1} := \langle h, g \rangle_{L^2} + \langle R^{1/2} h', R^{1/2} g' \rangle_{L^2}. \tag{A.6}$$

Lemma A.1 (Closability and semiboundedness)**.** *The form q_R is densely defined, nonnegative, and closed on $L^2(\mathbb{R})$. In particular,*

$$\mathcal{D}(q_R) = H_R^1(\mathbb{R}), \qquad \|h\|_{q_R}^2 = \|h\|_{L^2}^2 + \|R^{1/2} h'\|_{L^2}^2 \quad \textit{for } h \in \mathcal{D}(q_R). \tag{A.7}$$

Proof. Density of $\mathcal{C}_c^\infty(\mathbb{R})$ in $L^2(\mathbb{R})$ is standard. Since $\|h\|_{L^2} \le \|h\|_{q_R}$ on $\mathcal{C}_c^\infty$, the completion $\mathcal{D}(q_R)$ embeds continuously into $L^2(\mathbb{R})$.

Closedness of q_R is built into the construction: we equip $\mathcal{D}(q_R)$ with $\|\cdot\|_{q_R}$ and extend the sesquilinear form $(h, g) \mapsto \int R h' \overline{g'}$ by continuity from $\mathcal{C}_c^\infty$. Equivalently, if $\{h_n\} \subset \mathcal{C}_c^\infty$ is Cauchy in $\|\cdot\|_{q_R}$ and $h_n \to 0$ in L^2, then $\|h_n\|_{q_R} \to 0$; this is the standard notion of closability (see Kato [21, §VI.1]).

To identify $\mathcal{D}(q_R)$ with $H_R^1(\mathbb{R})$, let $h \in \mathcal{D}(q_R)$. Then there exist $h_n \in \mathcal{C}_c^\infty(\mathbb{R})$ with $\|h_n - h\|_{q_R} \to 0$. In particular, $h_n \to h$ in L^2 and $R^{1/2} h_n'$ is Cauchy in L^2. Let $v := \lim_n R^{1/2} h_n' \in L^2(\mathbb{R})$.

We first define a distributional derivative h'. For every $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} h_n \, \overline{\phi'} \, dx \ \to \ \int_{\mathbb{R}} h \, \overline{\phi'} \, dx \qquad (n \to \infty), \tag{A.8}$$

since $h_n \to h$ in L^2 and $\phi' \in L^2$. Set

$$\langle h', \phi \rangle := -\int_{\mathbb{R}} h\,\overline{\phi'}\,dx, \qquad \phi \in \mathcal{C}_c^\infty(\mathbb{R}), \tag{A.9}$$

so $h' \in \mathcal{D}'(\mathbb{R})$ is the usual distributional derivative of h.

Next, we check that $R^{1/2}h' = v$ as distributions. For any $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} v(x)\,\overline{\psi(x)}\,dx = \lim_{n\to\infty} \int_{\mathbb{R}} R^{1/2}h_n'(x)\,\overline{\psi(x)}\,dx = \lim_{n\to\infty} \int_{\mathbb{R}} h_n'(x)\,\overline{R^{1/2}(x)\psi(x)}\,dx. \tag{A.10}$$

By integration by parts on the smooth compactly supported h_n,

$$\int_{\mathbb{R}} h_n'\,\overline{R^{1/2}\psi}\,dx = -\int_{\mathbb{R}} h_n\,\overline{(R^{1/2}\psi)'}\,dx. \tag{A.11}$$

Passing to the limit $n \to \infty$ and using $h_n \to h$ in L^2 gives

$$\int_{\mathbb{R}} v\,\overline{\psi}\,dx = -\int_{\mathbb{R}} h\,\overline{(R^{1/2}\psi)'}\,dx = \langle h', R^{1/2}\psi \rangle. \tag{A.12}$$

Since ψ was arbitrary, this is precisely the identity $\langle R^{1/2}h', \psi \rangle = \int_{\mathbb{R}} v\,\overline{\psi}\,dx$, i.e. $R^{1/2}h' = v$ in $\mathcal{D}'(\mathbb{R})$ and in $L^2(\mathbb{R})$. Thus $h \in H^1_R(\mathbb{R})$ and

$$\|h\|^2_{H^1_R} = \|h\|^2_{L^2} + \|R^{1/2}h'\|^2_{L^2} = \|h\|^2_{L^2} + \|v\|^2_{L^2} = \lim_{n\to\infty} \|h_n\|^2_{q_R} = \|h\|^2_{q_R}. \tag{A.13}$$

Conversely, suppose $h \in H^1_R(\mathbb{R})$, so $h \in L^2$ and $R^{1/2}h' \in L^2$. By Lemma A.2 below we can approximate h in the norm $\|h\|^2_{H^1_R} = \|h\|^2_{L^2} + \|R^{1/2}h'\|^2_{L^2}$ by functions $h_n \in \mathcal{C}_c^\infty(\mathbb{R})$. Hence $\mathcal{C}_c^\infty(\mathbb{R})$ is dense in $H^1_R(\mathbb{R})$ and the completion of $\mathcal{C}_c^\infty$ in $\|\cdot\|_{q_R}$ is exactly $H^1_R(\mathbb{R})$. Nonnegativity of q_R is immediate from $R \geq 0$. $\square$

Lemma A.2 (Core approximation)**.** *$\mathcal{C}_c^\infty(\mathbb{R})$ is a form core for q_R: for every $h \in \mathcal{D}(q_R)$ there exist $h_n \in \mathcal{C}_c^\infty(\mathbb{R})$ with $h_n \to h$ in L^2 and $R^{1/2}h_n' \to R^{1/2}h'$ in L^2.*

Proof. Let $\chi_n \in \mathcal{C}_c^\infty(\mathbb{R})$ be standard cutoffs with $\chi_n \equiv 1$ on $[-n, n]$, $0 \leq \chi_n \leq 1$, $\operatorname{supp}\chi_n \subset [-2n, 2n]$, and $\|\chi_n'\|_\infty \lesssim n^{-1}$. Set $h^{(n)} := \chi_n h$. Then $h^{(n)} \to h$ in L^2 by dominated convergence. Moreover,

$$(h^{(n)})' = \chi_n h' + \chi_n' h \tag{A.14}$$

in the distributional sense, so

$$R^{1/2}(h^{(n)})' - R^{1/2}h' = R^{1/2}(\chi_n - 1)h' + R^{1/2}\chi_n' h. \tag{A.15}$$

The first term tends to 0 in L^2 since $\chi_n \to 1$ pointwise and $|\chi_n - 1| \le 1$, while $R^{1/2}h' \in L^2$. For the second term, note that $\operatorname{supp}\chi_n' \subset \{|x| \sim n\}$ and

$$\int_{\mathbb{R}} R|\chi_n'|^2|h|^2\,dx \;\le\; \|\chi_n'\|_\infty^2 \sup_{\{|x|\sim n\}} R(x) \int_{\{|x|\sim n\}} |h(x)|^2\,dx. \qquad \text{(A.16)}$$

Since $R \in \mathscr{S}(\mathbb{R})$, $\sup_{\{|x|\sim n\}} R(x)$ decays faster than any power of n^{-1}, while the L^2 mass of h on $\{|x| \sim n\}$ tends to 0 as $n \to \infty$. Thus $\|R^{1/2}\chi_n' h\|_{L^2} \to 0$ as $n \to \infty$.

We have produced $h^{(n)} \in H^1_R \cap L^2$ with compact support and $\|h^{(n)} - h\|_{q_R} \to 0$. A standard mollification argument (convolution with a symmetric approximate identity) gives $h^{(n,\varepsilon)} \in \mathcal{C}_c^\infty(\mathbb{R})$ with $\|h^{(n,\varepsilon)} - h^{(n)}\|_{q_R} \to 0$ as $\varepsilon \to 0$. A diagonal extraction yields a sequence $h_k \in \mathcal{C}_c^\infty$ converging to h in the graph norm. □

Use in the main text. Lemmas A.1–A.2 justify that all x–pairings (e.g. $E_R(t) = q_R[g(\cdot,t)]$) are taken on a closed form with a concrete core, enabling IBP on cores in Section 5.7, passage to limits in Section 5.10, and the Lyapunov energy interpretation $t \mapsto E_R(t)$ based on the semigroup generated by H_R.

A.2. **Friedrichs realisation, weak operator identification, and IBP.**

Proposition A.1 (Friedrichs extension)**.** *There exists a unique self–adjoint operator $H_R \ge 0$ on $L^2(\mathbb{R})$ such that $\mathcal{D}(H_R^{1/2}) = \mathcal{D}(q_R)$ and $q_R[h] = \|H_R^{1/2}h\|_{L^2}^2$ for all $h \in \mathcal{D}(q_R)$.*

Proof. This is the representation theorem for closed, semibounded forms (Kato [21, Thm. VI.2.1], Reed–Simon [19, Thm. VIII.15]). □

Lemma A.3 (Integration by parts on the core)**.** *For $h \in \mathcal{C}_c^\infty(\mathbb{R})$,*

$$q_R[h] = \int_{\mathbb{R}} R(x)|h'(x)|^2\,dx = -\int_{\mathbb{R}} (Rh')'(x)\,\overline{h(x)}\,dx. \qquad \text{(A.17)}$$

Proof. Direct computation, using that $R \in C^1(\mathbb{R})$ and h has compact support:

$$\int_{\mathbb{R}} R|h'|^2\,dx = \int_{\mathbb{R}} Rh'\,\overline{h'}\,dx = -\int_{\mathbb{R}} (Rh')'\,\overline{h}\,dx, \qquad \text{(A.18)}$$

with no boundary terms at infinity. □

Lemma A.4 (Weak operator identity)**.** *Let H_R be as above. For $h \in \mathcal{D}(H_R)$ and $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$,*

$$\langle H_R h, \phi\rangle_{L^2} = q_R[h,\phi] = \int_{\mathbb{R}} R(x)\, h'(x)\, \overline{\phi'(x)}\, dx = -\int_{\mathbb{R}} (Rh')'(x)\, \overline{\phi(x)}\, dx, \tag{A.19}$$

so $H_R h = -(Rh')'$ in the sense of distributions on $\mathbb{R}$.

Proof. By Proposition A.1, for all $\phi \in \mathcal{D}(q_R)$, $\langle H_R h, \phi\rangle_{L^2} = q_R[h,\phi]$. In particular this holds for $\phi \in \mathcal{C}_c^\infty(\mathbb{R}) \subset \mathcal{D}(q_R)$. On the core, Lemma A.3 and polarisation yield

$$q_R[h,\phi] = \int_{\mathbb{R}} Rh'\, \overline{\phi'}\, dx = -\int_{\mathbb{R}} (Rh')'\, \overline{\phi}\, dx, \tag{A.20}$$

which is exactly the distributional identity $H_R h = -(Rh')'$. □

Lemma A.5 (Flux continuity across the degeneracy)**.** *If $h \in \mathcal{D}(H_R)$ then $Rh' \in H^1_{\mathrm{loc}}\big(\mathbb{R} \setminus \{x_0\}\big)$ and $(Rh')' \in L^2(\mathbb{R})$. Consequently, Rh' has a (finite) trace from the left and right at x_0, and these traces agree:*

$$\lim_{x\uparrow x_0} (Rh')(x) \;=\; \lim_{x\downarrow x_0} (Rh')(x). \tag{A.21}$$

Proof. For $h \in \mathcal{D}(H_R)$, Lemma A.4 yields $(Rh')' = -H_R h \in L^2(\mathbb{R})$ in the distributional sense. Hence $Rh' \in H^1_{\mathrm{loc}}(\mathbb{R})$ away from x_0, and Rh' admits one–sided limits at x_0. If there were a jump $J \neq 0$ at x_0, then the distributional derivative of Rh' would contain a term $J\,\delta_{x_0}$, which is not in $L^2(\mathbb{R})$. Thus the left and right traces must coincide. □

Use in the main text. The weak identification $H_R = -(Rh')'$ justifies viewing H_R as the Friedrichs realisation of the divergence–form operator. Flux continuity legitimises moving derivatives across the quadratic degeneracy in the EF linearisation (Section 5.7) and, via the semigroup e^{-sH_R}, feeds into the Lyapunov energy picture $t \mapsto E_R(t)$.

A.3. **Semi–coercivity with regulators, KLMN stability, and resolvent limits.**

Lemma A.6 (Local coercivity via compact regulators)**.** *Let $I \Subset \mathbb{R}$ be compact. Define the regulated kernel $R_{I,\varepsilon} := R + \varepsilon\, \mathbf{1}_I$ for $\varepsilon \in (0,1]$. Then there exists $c_I > 0$ (independent of ε) such that for all $h \in \mathcal{D}(q_{R_{I,\varepsilon}})$,*

$$\int_I |h'(x)|^2\, dx \;\le\; c_I\Big(q_{R_{I,\varepsilon}}[h] + \|h\|^2_{L^2(I)}\Big). \tag{A.22}$$

Consequently, by monotone convergence of forms as $\varepsilon \downarrow 0$,

$$\int_I |h'(x)|^2\,dx \;\le\; c_I\Big(q_R[h] + \|h\|^2_{L^2(I)}\Big) \qquad \text{for all } h \in \mathcal{D}(q_R). \tag{A.23}$$

Proof. Fix $I \Subset \mathbb{R}$. Since $R \in \mathscr{S}_0$, there exists a small interval $J \ni x_0$, $J \Subset I$, on which

$$c_J\,|x-x_0|^2 \;\le\; R(x) \;\le\; C_J\,|x-x_0|^2 \qquad (x \in J) \tag{A.24}$$

for some positive constants c_J, C_J; on the compact set $I \setminus J$, continuity and nonnegativity of R imply $R(x) \ge m_I > 0$.

Step 1: control on $I \setminus J$. On $I \setminus J$, $R_{I,\varepsilon}(x) \ge R(x) \ge m_I > 0$, so for all $h \in \mathcal{C}_c^\infty(\mathbb{R})$,

$$\int_{I\setminus J} |h'|^2\,dx \;\le\; m_I^{-1}\int_{I\setminus J} R_{I,\varepsilon}(x)|h'|^2\,dx \;\le\; m_I^{-1} q_{R_{I,\varepsilon}}[h], \tag{A.25}$$

and the same inequality extends to $h \in \mathcal{D}(q_{R_{I,\varepsilon}})$ by density.

Step 2: control on J. On J, we exploit the quadratic structure. The one–dimensional Hardy–Poincaré inequality (applied separately on each side of x_0 and recombined) yields

$$\int_J |h - \langle h\rangle_J|^2\,dx \;\le\; C_{\mathrm{HP}}\int_J |x-x_0|^2|h'(x)|^2\,dx, \tag{A.26}$$

where $\langle h\rangle_J := |J|^{-1}\int_J h$. Using $R(x) \ge c_J|x-x_0|^2$ on J,

$$\int_J |x-x_0|^2|h'(x)|^2\,dx \;\le\; c_J^{-1}\int_J R(x)|h'(x)|^2\,dx \;\le\; c_J^{-1} q_{R_{I,\varepsilon}}[h]. \tag{A.27}$$

We also have the trivial bound $\int_J |\langle h\rangle_J|^2\,dx \le |J|^{-1}\|h\|^2_{L^2(J)}$. Combining these yields

$$\int_J |h|^2\,dx \;\le\; 2\int_J |h-\langle h\rangle_J|^2\,dx + 2\int_J |\langle h\rangle_J|^2\,dx \;\le\; C_J'\Big(q_{R_{I,\varepsilon}}[h] + \|h\|^2_{L^2(J)}\Big), \tag{A.28}$$

with C_J' independent of ε.

A standard scaling argument on the bounded interval J now gives

$$\int_J |h'|^2\,dx \;\le\; C_J''\Big(q_{R_{I,\varepsilon}}[h] + \|h\|^2_{L^2(J)}\Big), \tag{A.29}$$

where C_J'' depends only on J and finitely many $\mathscr{S}$–seminorms of R, but not on $\varepsilon \in (0,1]$.

Step 3: combine. Adding the bounds on $I \setminus J$ and on J gives

$$\int_I |h'|^2\,dx \;\le\; c_I\Big(q_{R_{I,\varepsilon}}[h] + \|h\|^2_{L^2(I)}\Big) \tag{A.30}$$

for some $c_I > 0$ depending only on I and finitely many $\mathscr{S}$–seminorms of R, and independent of ε.

For the $\varepsilon \downarrow 0$ statement, note that $R_{I,\varepsilon} \downarrow R$ pointwise and $q_{R_{I,\varepsilon}}[h] \downarrow q_R[h]$ for each fixed h. The monotone convergence theorem for closed forms (Kato [21, Thm. VIII.3.11]) then yields the same inequality with q_R in place of $q_{R_{I,\varepsilon}}$. □

Use in the main text. This local coercivity controls localisation errors in Sections 5.7 and 5.9, and justifies distributional manipulations near the quadratic degeneracy at x_0 in the finite–time Lyapunov framework.

Proposition A.2 (KLMN perturbations)**.** *Let V be a (possibly indefinite) form perturbation on $\mathcal{D}(q_R)$ with $|V[h]| \le a\, q_R[h] + b\,\|h\|_{L^2}^2$ for some $a < 1$, $b \ge 0$. Then $q_R + V$ is closed and semi-bounded on $\mathcal{D}(q_R)$, and its Friedrichs operator is self–adjoint.*

Proof. This is the KLMN theorem (Kato [21, Thm. X.17], Reed–Simon [19, Thm. X.12]). □

Use in the main text. This ensures stability of the EF decomposition and harmless lower–order corrections in Section 5.7, including those arising from decompositions of $g(\cdot, t)$ into regular and singular parts.

Proposition A.3 (Form convergence & strong resolvent limit)**.** *If $R_n, R \in \mathscr{S}_0$ with $R_n \to R$ in $\mathscr{S}(\mathbb{R})$, then $q_{R_n} \to q_R$ in the sense of Mosco. Consequently, the associated self–adjoint operators H_{R_n} converge to H_R in the strong resolvent sense, and*

$$e^{-tH_{R_n}} \;\to\; e^{-tH_R} \quad \text{strongly on } L^2(\mathbb{R}) \text{ for each } t \ge 0. \tag{A.31}$$

Proof (Mosco). We recall the two Mosco conditions for forms q_{R_n} on $L^2(\mathbb{R})$:

(M1) If $h_n \rightharpoonup h$ weakly in L^2 and $\sup_n q_{R_n}[h_n] < \infty$, then $q_R[h] \le \liminf_{n\to\infty} q_{R_n}[h_n]$.

(M2) For every $h \in \mathcal{D}(q_R)$ there exist $h_n \in \mathcal{D}(q_{R_n})$ with $h_n \to h$ in L^2 and $\limsup_{n\to\infty} q_{R_n}[h_n] \le q_R[h]$.

(M1: liminf). Suppose $h_n \rightharpoonup h$ in $L^2(\mathbb{R})$ and $\sup_n q_{R_n}[h_n] < \infty$. Then $\{R_n^{1/2} h_n'\}$ is bounded in $L^2(\mathbb{R})$, so (up to a subsequence) $R_n^{1/2} h_n' \rightharpoonup w$ in L^2. Since $h_n \rightharpoonup h$ in L^2, we have for every $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} h_n\, \overline{\phi'}\, dx \to \int_{\mathbb{R}} h\, \overline{\phi'}\, dx. \tag{A.32}$$

Define the distributional derivative h' by $\langle h', \phi\rangle := -\int_{\mathbb{R}} h\, \overline{\phi'}\, dx$.

To identify w with $R^{1/2}h'$, let $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$. Then

$$\langle w, \psi\rangle = \lim_{n\to\infty} \int_{\mathbb{R}} R_n^{1/2} h_n' \, \overline{\psi}\, dx = \lim_{n\to\infty} \int_{\mathbb{R}} h_n' \, \overline{R_n^{1/2}\psi}\, dx = -\lim_{n\to\infty} \int_{\mathbb{R}} h_n \, \overline{(R_n^{1/2}\psi)'}\, dx. \tag{A.33}$$

Since $R_n^{1/2} \to R^{1/2}$ in $\mathscr{S}(\mathbb{R})$, we have $(R_n^{1/2}\psi)' \to (R^{1/2}\psi)'$ in L^2, and $\{h_n\}$ is bounded in L^2 by weak convergence. Hence

$$-\lim_{n\to\infty} \int_{\mathbb{R}} h_n \, \overline{(R_n^{1/2}\psi)'}\, dx = -\int_{\mathbb{R}} h \, \overline{(R^{1/2}\psi)'}\, dx = \langle h', R^{1/2}\psi\rangle = \langle R^{1/2}h', \psi\rangle. \tag{A.34}$$

Thus $w = R^{1/2}h' \in L^2(\mathbb{R})$, and $h \in \mathcal{D}(q_R)$. By weak lower semicontinuity of the L^2–norm,

$$q_R[h] = \|R^{1/2}h'\|_{L^2}^2 = \|w\|_{L^2}^2 \le \liminf_{n\to\infty} \|R_n^{1/2}h_n'\|_{L^2}^2 = \liminf_{n\to\infty} q_{R_n}[h_n]. \tag{A.35}$$

(M2: limsup). Let $h \in \mathcal{D}(q_R)$. By Lemma A.2 there exist $h^{(k)} \in \mathcal{C}_c^\infty(\mathbb{R})$ with $\|h^{(k)} - h\|_{q_R} \to 0$ as $k \to \infty$. Fix k. Since $R_n \to R$ in $\mathscr{S}(\mathbb{R})$ and $h^{(k)}$ is smooth with compact support, dominated convergence gives

$$q_{R_n}[h^{(k)}] = \int_{\mathbb{R}} R_n |h^{(k)\prime}|^2\, dx \;\longrightarrow\; \int_{\mathbb{R}} R|h^{(k)\prime}|^2\, dx = q_R[h^{(k)}] \quad (n\to\infty). \tag{A.36}$$

For each k pick n_k so that $|q_{R_{n_k}}[h^{(k)}] - q_R[h^{(k)}]| \le 2^{-k}$. Define a sequence $\{u_m\}$ by setting $u_m = h^{(k)}$ whenever $n_k \le m < n_{k+1}$. Then $u_m \to h$ in L^2 (because $h^{(k)} \to h$ in $\|\cdot\|_{q_R}$) and

$$\limsup_{m\to\infty} q_{R_m}[u_m] \;\le\; \limsup_{k\to\infty} q_{R_{n_k}}[h^{(k)}] \;\le\; \limsup_{k\to\infty} \big(q_R[h^{(k)}] + 2^{-k}\big) = q_R[h]. \tag{A.37}$$

This verifies (M2).

By Kato's Mosco–convergence theorem (Kato [21, Thm. VIII.3.11 & Cor. VIII.3.12], Reed–Simon [19, Thm. VIII.25]), (M1)–(M2) imply strong resolvent convergence $H_{R_n} \to H_R$ and strong convergence of the associated semigroups. □

Use in the main text. This justifies the robustness claims in Section 7 (e.g. passage $R_\alpha \to R$ after proving T–controlled bounds), and underpins the "measure, not modify" regulator removal in the Lyapunov/EF framework: the EF–bank bound is proved for regularised kernels and then passed to the limit.

A.4. **Auxiliary comparisons, compactness, and local identities.** We collect routine but repeatedly used facts.

Lemma A.7 (Form monotonicity and domains). *If $0 \le R_1 \le R_2$ a.e., then for all $h \in \mathcal{C}_c^\infty(\mathbb{R})$, $q_{R_1}[h] \le q_{R_2}[h]$. Moreover, $\mathcal{D}(q_{R_2}) \subset \mathcal{D}(q_{R_1})$ and $q_{R_1}[h] \le q_{R_2}[h]$ for all $h \in \mathcal{D}(q_{R_2})$.*

Proof. The inequality on $\mathcal{C}_c^\infty$ is immediate from the definitions. The domain inclusion and extension of the inequality follow by completion and Fatou's lemma. $\square$

Lemma A.8 (Local Poincaré–Hardy control). *Let $I \Subset \mathbb{R}$ and let $\psi \in \mathcal{C}_c^\infty(I)$ be a cutoff. Then for all $h \in \mathcal{D}(q_R)$,*

$$\int_I |h-\langle h\rangle_I|^2\,dx \;\lesssim\; \int_I |x-x_0|^2|h'(x)|^2\,dx \;\lesssim\; q_R[\psi h]+\|h\|^2_{L^2(I)}, \tag{A.38}$$

where $\langle h\rangle_I := |I|^{-1}\int_I h$ and the implicit constants depend on I and R only through finitely many $\mathscr{S}$–seminorms.

Proof. The first inequality is the 1D Hardy–Poincaré inequality, applied on each side of x_0 and recombined. For the second, note that on I we have $R(x) \sim \kappa|x-x_0|^2$ and $\psi \equiv 1$ on a slightly smaller subinterval, so

$$\int_I |x-x_0|^2|h'(x)|^2\,dx \;\lesssim\; \int_{\mathbb{R}} R(x)|(\psi h)'(x)|^2\,dx+\|h\|^2_{L^2(I)} = q_R[\psi h]+\|h\|^2_{L^2(I)}, \tag{A.39}$$

using Lemma A.6 to control the term involving $\psi' h$. $\square$

Lemma A.9 (Caccioppoli–type estimate). *Let $h \in \mathcal{D}(H_R)$ solve $H_R h = f \in L^2_{\mathrm{loc}}(\mathbb{R})$ in the weak sense. Then for any $\eta \in \mathcal{C}_c^\infty(\mathbb{R})$,*

$$\int_{\mathbb{R}} \eta^2 R|h'|^2\,dx \;\lesssim\; \int_{\mathbb{R}} R|(\eta h)'|^2\,dx+\int_{\mathbb{R}}(\eta')^2 R\,|h|^2\,dx \;\lesssim\; \|\eta f\|^2_{L^2}+\|h\|^2_{L^2(\operatorname{supp}\eta)}, \tag{A.40}$$

where the last inequality uses $q_R[\eta h] = \langle f, \eta^2 h\rangle$ and Cauchy–Schwarz/Young.

Proof. Expanding $(\eta h)'$ gives

$$R|(\eta h)'|^2 = R\eta^2|h'|^2 + 2R\eta\eta'\,\Re(h'\overline{h}) + R(\eta')^2|h|^2. \tag{A.41}$$

Integrate and absorb the cross term using Young's inequality to obtain the first inequality. For the second, use the weak formulation $\langle H_R h, \eta^2 h\rangle = q_R[h, \eta^2 h] = \langle f, \eta^2 h\rangle$, then apply Cauchy–Schwarz and Young, and bound $\|\eta h\|_{L^2}$ by $\|h\|_{L^2(\operatorname{supp}\eta)}$. $\square$

Lemma A.10 (Rellich compactness on compacts). *If $I \Subset \mathbb{R}$, the embedding*

$$\left\{\, h \in \mathcal{D}(q_R) :\; \|h\|^2_{L^2} + q_R[h] \le 1 \,\right\} \;\hookrightarrow\; L^2(I) \tag{A.42}$$

is compact.

Proof. By Lemma A.6, the set $\{h : \ \|h\|_{L^2}^2 + q_R[h] \leq 1\}$ has uniformly bounded $\int_I |h'|^2\,dx$, hence is bounded in $H^1(I)$. Rellich–Kondrachov on the bounded interval I implies that the embedding into $L^2(I)$ is compact. □

A.5. **Characterisation of the operator domain.**

Proposition A.4 (Weak/strong domain characterisation)**.** *Let H_R be the Friedrichs operator associated with q_R. Then*

$$\mathcal{D}(H_R) = \left\{ h \in H^1_R(\mathbb{R}) \ : \ \exists\, f \in L^2(\mathbb{R}) \text{ with } \int_{\mathbb{R}} Rh'\,\overline{\phi'}\,dx = \int_{\mathbb{R}} f\,\overline{\phi}\,dx \ \ \forall\, \phi \in H^1_R(\mathbb{R}) \right\}. \tag{A.43}$$

For $h \in \mathcal{D}(H_R)$, $Rh' \in H^1_{\mathrm{loc}}(\mathbb{R})$ with $(Rh')' = -f \in L^2(\mathbb{R})$, and the flux is continuous at x_0 (Lemma A.5). Conversely, any $h \in L^2$ with $h \in H^1_R(\mathbb{R})$, $Rh' \in H^1_{\mathrm{loc}}(\mathbb{R})$ and $(Rh')' \in L^2(\mathbb{R})$ belongs to $\mathcal{D}(H_R)$ with $H_R h = -(Rh')'$.

Proof. The first characterisation is the standard variational definition of the operator associated to a closed form (Proposition A.1 and Kato [21, §VI.2]): given $h \in \mathcal{D}(q_R) = H^1_R(\mathbb{R})$, we have $h \in \mathcal{D}(H_R)$ iff there exists $f \in L^2(\mathbb{R})$ such that

$$q_R[h, \phi] = \langle f, \phi \rangle_{L^2} \quad \text{for all } \phi \in \mathcal{D}(q_R) = H^1_R(\mathbb{R}), \tag{A.44}$$

in which case $H_R h = f$.

If $h \in \mathcal{D}(H_R)$, Lemma A.4 shows that $(Rh')' = -f \in L^2(\mathbb{R})$ in the distributional sense, hence $Rh' \in H^1_{\mathrm{loc}}(\mathbb{R})$. Flux continuity at x_0 is Lemma A.5.

Conversely, suppose $h \in L^2(\mathbb{R})$ satisfies $h \in H^1_R(\mathbb{R})$, $Rh' \in H^1_{\mathrm{loc}}(\mathbb{R})$ and $(Rh')' = -f \in L^2(\mathbb{R})$. Then for any $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} Rh'\,\overline{\phi'}\,dx = -\int_{\mathbb{R}} (Rh')'\,\overline{\phi}\,dx = \int_{\mathbb{R}} f\,\overline{\phi}\,dx. \tag{A.45}$$

By density of $\mathcal{C}_c^\infty(\mathbb{R})$ in $H^1_R(\mathbb{R})$, this identity extends to all $\phi \in H^1_R(\mathbb{R})$, so $h \in \mathcal{D}(H_R)$ with $H_R h = f$. □

A.6. **Typical perturbations covered by KLMN.** We record a convenient sufficient condition for the form–smallness hypothesis used in Section 5.7.

Lemma A.11 (Local potentials are small after localisation)**.** *Let $V \in L^1_{\mathrm{loc}}(\mathbb{R}) + L^\infty(\mathbb{R})$. Then for every compact $I \Subset \mathbb{R}$ and $\delta > 0$ there is $C_{I,\delta}$ such that for all $h \in \mathcal{D}(q_R)$,*

$$\left| \int_I V(x) |h(x)|^2\,dx \right| \ \leq \ \delta\, q_R[h] \ + \ C_{I,\delta}\, \|h\|_{L^2(I)}^2. \tag{A.46}$$

Hence, by a partition of unity and Lemma A.6, any potential V with $V_- \in L^1_{\mathrm{loc}}(\mathbb{R})$ sufficiently small on each piece and $V_+ \in L^\infty(\mathbb{R})$ is KLMN–admissible.

Proof. Fix $I \Subset \mathbb{R}$ and decompose $V = V_1 + V_\infty$ with $V_1 \in L^1_{\mathrm{loc}}(\mathbb{R})$, $V_\infty \in L^\infty(\mathbb{R})$. On I,

$$\Big|\int_I V_\infty |h|^2\,dx\Big| \;\le\; \|V_\infty\|_{L^\infty(I)}\|h\|^2_{L^2(I)}. \tag{A.47}$$

For V_1, Hölder's inequality gives

$$\int_I |V_1||h|^2\,dx \;\le\; \|V_1\|_{L^1(I)}\,\|h\|^2_{L^\infty(I)}. \tag{A.48}$$

Using the Sobolev embedding $H^1(I) \hookrightarrow L^\infty(I)$ and Lemma A.6, we may bound $\|h\|^2_{L^\infty(I)}$ by $C\big(\int_I |h'|^2 + \|h\|^2_{L^2(I)}\big)$, which in turn is bounded by $C'\big(q_R[h] + \|h\|^2_{L^2(I)}\big)$. Absorbing the $q_R[h]$–term into $\delta q_R[h]$ by choosing the localisation scale (and hence $\|V_1\|_{L^1(I)}$) sufficiently small, and adjusting the coefficient in front of $\|h\|^2_{L^2(I)}$, yields

$$\Big|\int_I V|h|^2\,dx\Big| \;\le\; \delta q_R[h] + C_{I,\delta}\|h\|^2_{L^2(I)}. \tag{A.49}$$

For the global KLMN statement, cover $\mathbb{R}$ by a finite–overlap partition of unity $\{\psi_j\}$ subordinate to compact intervals I_j, apply the above estimate to each $\psi_j h$, and sum over j. The assumption that the negative part V_- is small on each piece ensures that the resulting form perturbation has relative bound < 1 with respect to q_R, so Proposition A.2 applies. ☐

Lemma A.12 (Windowed zero–sum envelope)**.** *Let $R \in \mathscr{S}_0$ and $T > 0$. With $\varpi_T(t) = (\sqrt{\pi}\,T)^{-1}e^{-t^2/T^2}$ and $\mathcal{Z}_R(\nu,t)$ as in Appendix E, one has for all $\nu \in \mathbb{R}$,*

$$\int_{\mathbb{R}} |\mathcal{Z}_R(\nu,t)|^2\,\varpi_T(t)\,dt \;\le\; \frac{C_Z(R)}{(1+|\nu|)^2}\,\big(1+\log^3(3+T)\big)\,\Big(1+\log^2\!\big(2+|\nu|^{-1}\big)\Big), \tag{A.50}$$

with $C_Z(R)$ depending on finitely many $\mathscr{S}$–seminorms of R (and on the fixed cutoff ϑ) and independent of T. For $\nu = 0$ we interpret $\log(2+|\nu|^{-1})$ as 0. If R ranges over a compact $K \subset \mathscr{S}_0$, the same bound holds with a constant $C_Z(K)$ uniform in $R \in K$.

In particular,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |\mathcal{Z}_R(\nu,t)|^2\,\varpi_T(t)\,dt\,d\nu \;\le\; C'_Z(R)\,\big(1+\log^3(3+T)\big), \tag{A.51}$$

for some $C'_Z(R) < \infty$ depending only on R (and ϑ) but independent of T, since $\int_{\mathbb{R}}(1+|\nu|)^{-2}\big(1+\log^2(2+|\nu|^{-1})\big)\,d\nu < \infty$.

Proof. The first displayed inequality is exactly the conclusion of Theorem 2 in Appendix E, with the same convention at $\nu = 0$.

For the integrated bound, note that the integrand is nonnegative, so Tonelli's theorem applies. Integrating the bound in ν gives

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |\mathcal{Z}_R(\nu,t)|^2\,\varpi_T(t)\,dt\,d\nu \;\le\; C_Z(R)\,\Big(1+\log^3(3+T)\Big)\int_{\mathbb{R}} \frac{1+\log^2(2+|\nu|^{-1})}{(1+|\nu|)^2}\,d\nu. \tag{A.52}$$

The ν–integral is finite: split into $|\nu| \ge 1$ and $|\nu| \le 1$. For $|\nu| \ge 1$ one has $\log(2+|\nu|^{-1}) \le \log 3$, so the integrand is $\ll (1+|\nu|)^{-2}$, which is integrable. For $|\nu| \le 1$ the denominator is $\asymp 1$ and the integrand is $\ll 1+\log^2(2+|\nu|^{-1})$; the latter is integrable near 0 since $\int_0^1 \log^2(2+\nu^{-1})\,d\nu < \infty$ (e.g. by the substitution $\nu = e^{-u}$, which yields an exponentially decaying weight e^{-u}).

Thus the ν–integral equals a finite absolute constant, and we may set

$$C'_Z(R) := C_Z(R)\int_{\mathbb{R}} \frac{1+\log^2(2+|\nu|^{-1})}{(1+|\nu|)^2}\,d\nu, \tag{A.53}$$

which is independent of T. If R ranges over a compact $K \subset \mathscr{S}_0$, then $C_Z(R)$ may be chosen uniformly (by Theorem 2), hence the same definition yields a uniform constant $C'_Z(K)$. $\square$

Remark A.2 (Domain of applicability). Lemma D.1 is a purely Fourier–in–x identity for the quadratic form q_R and is used *pointwise* in t only... at those ordinates for which $E_R(t) = q_R[g(\cdot,t)]$ is finite. In particular, no global L^2 assumption is made on the map $t \mapsto h(\cdot,t)$, and we never apply Plancherel in the t–variable. In the main text, the explicit–formula bounds for the windowed energy $\int_{\mathbb{R}} E_R(t)\,\varpi_T(t)\,dt$ are obtained by inserting the Gaussian window at the *linear* level in ξ'/ξ and then using Cauchy–Schwarz in (σ,t). The weighted Plancherel identity serves only as an x–side bookkeeping device for $E_R(t)$ and is never invoked a priori in any regime where $E_R(t)$ would be infinite.

Connections back to the proof.

– *Lyapunov functional and kernel–side energy:* $E_R(t) = q_R[g(\cdot,t)]$ is well–posed on the closed form (Lemmas A.1, A.2), and the semigroup e^{-sH_R} provides a canonical kernel–side evolution compatible with the finite–time Lyapunov framework.

- *EF linearisation and IBP:* Weak identification $H_R = -(Rh')'$, flux continuity, and core IBP (Lemmas A.4, A.3, A.5) justify moving derivatives off g even across the degeneracy (Section 5.7), in a way compatible with the windowed explicit–formula decomposition in Theorem B.
- *Local analysis near $x = \frac{1}{2}$:* Regulator–stable coercivity and Poincaré–Hardy control (Lemmas A.6, A.8) are used in the neighbourhood–divergence analysis and to bound localisation errors (Section 5.9), uniformly in the time–window scale.
- *Perturbations and limits:* KLMN stability (Proposition A.2) covers lower–order EF corrections; Mosco/strong resolvent convergence (Proposition A.3) implements the R–perturbation and $R_\alpha \to R$ limits (Section 7), with strong semigroup convergence for the kernel–side evolutions.
- *Zero–block envelope:* Lemma A.12 packages the windowed zero–sum bound from Appendix E in a form directly used in the EF energy bound, making explicit that the zero–block contribution has integrable low–frequency behaviour (via the harmless factor $1+\log^2(2+|\nu|^{-1})$), decays like $(1+|\nu|)^{-2}$ for large $|\nu|$, and grows at most like $1 + \log^3(3 + T)$ in the window scale, with constants depending only on finitely many $\mathscr{S}$–seminorms of R (uniformly on compact kernel families).

Bibliographic anchors. We invoke only standard results: Kato [21] (form representation, KLMN, monotone/Mosco convergence, resolvents/semigroups) and Reed–Simon [19] (complementary operator–theoretic statements).

This completes Appendix A.

Appendix B. Vertical–line envelopes and dominated convergence

Aim. Record vertical–line bounds for Γ'/Γ and ξ'/ξ on compact strips around the critical line, with explicit dependence on $|t|$; give a *local pole decomposition* for ξ'/ξ near zeros; and state dominated–convergence/Fubini criteria for time–averaged pairings against the mass–one Gaussian window used throughout the paper. All statements align with the truncation convention at zero ordinates from Section 5.10, and are used only as *fixed–T* backstops for the finite–time Lyapunov/EF framework (Sections 5.7 and 5.9). They are formulated so that envelopes and dominated–convergence arguments remain stable under compact kernel families $K \subset \mathscr{S}_0$ and auxiliary parameters (e.g. α, τ), with dependence only on finitely many Schwartz seminorms of R.

Window convention. We work with the mass–one Gaussian from the main text,

$$\varpi_T(t) := \frac{1}{\sqrt{\pi}\,T}\, e^{-t^2/T^2}, \qquad \int_{\mathbb{R}} \varpi_T(t)\,dt = 1, \tag{B.1}$$

and note that the arguments below apply (with identical proofs) to any nonnegative mass–one window $\omega_T(t) := T^{-1}\omega(t/T)$ with $\omega \in \mathscr{S}(\mathbb{R})$; cf. Proposition 7.2. In particular, all dominated–convergence statements and Fubini/Tonelli interchanges in this appendix are *window–robust.* The genuinely *T–uniform* bounds used in the Lyapunov/explicit–formula analysis come from the blockwise EF estimates in Section 5.7, not from the crude envelopes recorded here.

B.1. Uniform Stirling on compact strips.

Lemma B.1 (Uniform Stirling)**.** *Fix $\epsilon \in (0, \frac{1}{2})$. For $\sigma \in [\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon]$ and all $t \in \mathbb{R}$,*

$$\Big|\frac{\Gamma'}{\Gamma}\Big(\frac{\sigma + it}{2}\Big)\Big| \ll_\epsilon \log\big(2 + |t|\big), \tag{B.2}$$

with an implicit constant uniform in σ on the strip.

Proof. Let $z = \frac{\sigma+it}{2}$. On any sector $|\arg z| \le \pi - \delta$ the classical Stirling expansion gives $\log\Gamma(z) = (z - \frac{1}{2})\log z - z + O_\delta(1)$ and hence $\Gamma'/\Gamma(z) = \log z + O_\delta(1/|z|)$. As σ ranges in a fixed compact interval about $\frac{1}{2}$, there exists $\delta(\epsilon) \in (0, \pi)$ so that z remains in such a sector for all $t \neq 0$. Then $|\log z| \asymp \log(2 + |t|)$

and $|z|^{-1} \ll 1$. For bounded t, Γ'/Γ is smooth on compacta; enlarging the implicit constant covers all t. □

B.2. ξ'/ξ**: vertical–line envelope and local pole decomposition.** Recall

$$\xi(s) = \tfrac{1}{2}\, s(s-1)\, \pi^{-s/2}\, \Gamma\Big(\frac{s}{2}\Big)\, \zeta(s), \qquad \frac{\xi'}{\xi}(s) = \frac{1}{s} + \frac{1}{s-1} - \tfrac{1}{2}\log\pi + \tfrac{1}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{s}{2}\Big) + \frac{\zeta'}{\zeta}(s). \tag{B.3}$$

Lemma B.2 (Vertical–line envelope for ξ'/ξ)**.** *Fix $\epsilon \in (0, \frac{1}{2})$. For $\sigma \in [\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon]$ and all $t \in \mathbb{R}$ with $\xi(\sigma + it) \neq 0$,*

$$\left|\frac{\xi'}{\xi}(\sigma + it)\right| \ll_\epsilon \; 1 + \log\big(2 + |t|\big). \tag{B.4}$$

Proof. The rational terms $1/s$ and $1/(s-1)$ are $O(1)$ uniformly on the strip. By Lemma B.1, the gamma term is $\ll_\epsilon \log(2 + |t|)$. On fixed strips around $\frac{1}{2}$, the classical bound $\zeta'/\zeta(\sigma + it) = O_\epsilon(\log(2 + |t|))$ holds at nonzeros (see Titchmarsh–Heath-Brown [1, Chs. III–IV] or Ivić [2, §8.2]). Summing the contributions gives the claim. □

Lemma B.3 (Local pole decomposition)**.** *Fix $\epsilon \in (0, \frac{1}{2})$. There exists $C_\epsilon > 0$ such that for $\sigma \in [\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon]$ and $|t| \geq 2$,*

$$\frac{\xi'}{\xi}(\sigma + it) \;=\; \sum_{|\gamma - t| \leq 1} \frac{1}{\sigma + it - \rho} \;+\; O_\epsilon\big(\log(2 + |t|)\big), \tag{B.5}$$

where the sum is over nontrivial zeros $\rho = \beta + i\gamma$ of ζ, counted with multiplicity.

Proof. Starting from the Hadamard product for ξ and differentiating $\log \xi$, one obtains the usual partial fraction expansion for ζ'/ζ on fixed strips about $\frac{1}{2}$ (see [1, Ch. IV]):

$$\frac{\zeta'}{\zeta}(s) = \sum_{|\gamma - t| \leq 1} \frac{1}{s - \rho} \;+\; O_\epsilon\big(\log(2 + |t|)\big), \qquad s = \sigma + it, \tag{B.6}$$

valid for $|t| \geq 2$ and s away from zeros. Inserting this into the expression for ξ'/ξ and using Lemma B.1 for the gamma term, together with boundedness of $1/s$ and $1/(s-1)$ on the strip, shows that the additional contributions are $O_\epsilon(\log(2 + |t|))$, giving (B.5). □

Remark B.1 (A.E. interpretation and truncation). Both Lemma B.2 and (B.5) hold for a.e. t (with respect to dt). At ordinates γ where $\sigma + i\gamma$ is a zero of multiplicity m, the left-hand side is meromorphic with principal part $m/(s-\rho) = m/(\sigma-\beta+i(t-\gamma))$. Values "at $t=\gamma$" are taken by truncation $|t-\gamma|>\eta$, $\eta\downarrow 0$ (cf. Section 5.10).

B.3. **Dominating envelopes for the horizontal derivative.** Let $g(x,t)=\log|\xi(x+it)|^2$. Then $\partial_x g(x,t) = 2\,\Re\big(\xi'/\xi(x+it)\big)$.

Corollary B.1 (Envelope away from zero ordinates)**.** *Fix $\epsilon\in(0,\frac{1}{2})$ and $R\in\mathscr{S}_0$. For $x\in[\epsilon,1-\epsilon]$ and for a.e. $t\in\mathbb{R}$,*

$$R(x)\,|\partial_x g(x,t)|^2 \ \ll_{R,\epsilon}\ \big(1+\log^2(2+|t|)\big). \tag{B.7}$$

If $K\subset\mathscr{S}_0$ is compact, the implied constant may be chosen uniformly for $R\in K$, depending only on ϵ and on finitely many Schwartz seminorms of R as R ranges over K. In particular, for each fixed $T>0$ the right-hand side is integrable against $\varpi_T(t)\,dt$.

Proof. By Lemma B.2, $|\partial_x g(x,t)|\ll_\epsilon 1+\log(2+|t|)$ off poles. Since R is bounded on $[\epsilon,1-\epsilon]$, squaring yields the stated envelope. If R ranges over a compact $K\subset\mathscr{S}_0$, then

$$M_{K,\epsilon} := \sup_{R\in K}\ \sup_{x\in[\epsilon,1-\epsilon]} |R(x)| < \infty, \tag{B.8}$$

so the constant may be chosen uniformly on K. Integrability against ϖ_T follows from the Gaussian tails and Lemma B.4 below. □

Remark B.2 (What is and is not T–uniform). Corollary B.1 guarantees *existence* of the t–integrals at each fixed T, but does not claim uniformity as T varies: $\int\log^2(2+|t|)\,\varpi_T(t)\,dt$ grows slowly with T and is not uniformly bounded in T. All genuinely *T–uniform* bounds used in the paper (e.g. for $\int E_R\,\varpi_T$ or for the EF energies of the separate blocks) come from the *explicit–formula, blockwise analysis* in Section 5.7, not from this crude envelope.

Lemma B.4 (Gaussian–log moments)**.** *For each $k\ge 1$ there exists $C_k<\infty$ such that, for all $T>0$,*

$$\int_{\mathbb{R}} \log^k\!\big(2+|t|\big)\,\varpi_T(t)\,dt \ \le\ C_k\Big(1+\log^k\big(3+T\big)\Big). \tag{B.9}$$

Proof. Write $t=Tu$; then with respect to $\varpi_T(t)\,dt$, the variable u has density $(\sqrt{\pi})^{-1}e^{-u^2}$. We have

$$\log(2+|t|) = \log(2+T|u|) \le \log\big((2+T)(1+|u|)\big) \le \log(3+T)+\log(1+|u|). \tag{B.10}$$

By $(a+b)^k \ll_k a^k + b^k$,

$$\log^k(2+|t|) \ll_k \log^k(3+T) + \log^k(1+|u|). \tag{B.11}$$

Integrating against $(\sqrt{\pi})^{-1}e^{-u^2}\,du$ and using $\int_{\mathbb{R}} \log^k(1+|u|)\,e^{-u^2}\,du < \infty$ gives the claim. $\square$

B.4. **Dominated convergence and Fubini/Tonelli criteria (fixed T).** We now formulate a dominated–convergence/Fubini schema adapted to the way parameters appear in the main text: kernels R in a compact family $K \subset \mathscr{S}_0$, and auxiliary parameters such as α (spatial regulator) or τ (kernel path parameter). The key point is that the dominating envelope depends only on t (and on K through finitely many seminorms), never on T.

Lemma B.5 (DC/Fubini schema at fixed window scale). *Fix $\epsilon \in (0, \frac{1}{2})$ and $T > 0$, and let U be an arbitrary parameter set (e.g. $U = K \times (0, \alpha_0] \times [0, \tau_*]$). For each $u \in U$ let*

$$F_u : [\epsilon, 1-\epsilon] \times \mathbb{R} \to [0, \infty] \tag{B.12}$$

be measurable. Assume there exists $D \in L^1\big(\mathbb{R}, \varpi_T(t)\,dt\big)$ such that

$$F_u(x,t) \;\le\; D(t) \qquad \text{for all } x \in [\epsilon, 1-\epsilon],\; t \in \mathbb{R},\; u \in U. \tag{B.13}$$

Then for any compact $K \subset \mathscr{S}_0$:

(i) For every $u \in U$ and every $R \in K$,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} F_u(x,t)\,R(x)\,dx\,\varpi_T(t)\,dt < \infty, \tag{B.14}$$

and the integral is absolutely convergent. In particular, Tonelli and Fubini apply.

(ii) If $(F_{u,\lambda})_\lambda$ is a family with $F_{u,\lambda}(x,t) \to F_u(x,t)$ pointwise for all (x,t,u) and $F_{u,\lambda}(x,t) \le D(t)$ for all λ, then for all $R \in K$,

$$\iint F_{u,\lambda}(x,t)\,R(x)\,dx\,\varpi_T(t)\,dt \;\longrightarrow\; \iint F_u(x,t)\,R(x)\,dx\,\varpi_T(t)\,dt \tag{B.15}$$

as $\lambda \to \lambda_0$, uniformly in $u \in U$.

(iii) If $F_u(x,t) = \sum_n F_{u,n}(x,t)$ with $F_{u,n} \ge 0$ and $F_{u,n}(x,t) \le D(t)$ for all n, u, x, t, then for every $R \in K$,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \sum_n F_{u,n}(x,t)\,R(x)\,dx\,\varpi_T(t)\,dt \;=\; \sum_n \int_{\mathbb{R}}\int_{\mathbb{R}} F_{u,n}(x,t)\,R(x)\,dx\,\varpi_T(t)\,dt, \tag{B.16}$$

i.e. the t–integral and x–integral may be interchanged with the sum (Tonelli/Fubini), uniformly in $u \in U$ and $R \in K$.

Proof. Since R is bounded on $[\epsilon, 1-\epsilon]$ and $K \subset \mathscr{S}_0$ is compact,

$$M_{K,\epsilon} := \sup_{R\in K} \sup_{x\in[\epsilon,1-\epsilon]} |R(x)| < \infty. \tag{B.17}$$

For any $u \in U$ and $R \in K$,

$$\iint F_u(x,t)\,|R(x)|\,dx\,\varpi_T(t)\,dt \;\le\; M_{K,\epsilon}\,\mathrm{length}([\epsilon,1-\epsilon]) \int_{\mathbb{R}} D(t)\,\varpi_T(t)\,dt < \infty, \tag{B.18}$$

which gives absolute integrability and (i).

For (ii), apply dominated convergence with dominating function $M_{K,\epsilon}D(t)$; the domination and pointwise convergence are uniform in $u \in U$, so the convergence of integrals is uniform in u.

For (iii), let $S_N(x,t) := \sum_{n\le N} F_{u,n}(x,t)$. Each S_N is dominated by D and nonnegative, so Tonelli's theorem gives

$$\iint S_N(x,t)\,R(x)\,dx\,\varpi_T(t)\,dt = \sum_{n\le N} \iint F_{u,n}(x,t)\,R(x)\,dx\,\varpi_T(t)\,dt. \tag{B.19}$$

As $N \to \infty$, $S_N \uparrow \sum_n F_{u,n}$ pointwise, and monotone convergence yields the full statement. ☐

Remark B.3 (Truncation at zero ordinates). When F_u carries the horizontal derivative $R(x)\,|\partial_x g(x,t)|^2$, the envelope in Corollary B.1 applies for a.e. t. At $t = \gamma$ the integrand may diverge; all statements are interpreted by truncation $|t-\gamma| > \eta$, $\eta \downarrow 0$, exactly as in Section 5.10. The DC/Fubini schema is applied only to these truncated integrals, and the envelope D is uniform in u and $R \in K$.

B.5. **Proof detail for the local decomposition (tail estimate).** For completeness we indicate how the $O_\epsilon(\log(2+|t|))$ tail in (B.5) arises.

Let $s = \sigma + it$ with $\sigma \in [\frac{1}{2}-\epsilon, \frac{1}{2}+\epsilon]$ and $|t| \ge 2$. Split

$$\sum_\rho \frac{1}{s-\rho} \;=\; \sum_{|\gamma-t|\le 1} \frac{1}{s-\rho} \;+\; \sum_{|\gamma-t|>1} \frac{1}{s-\rho}. \tag{B.20}$$

For the tail sum, use the Riemann–von Mangoldt formula

$$N(T) = \#\{\rho:\ 0<\gamma\le T\} = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T) \tag{B.21}$$

and a dyadic decomposition in $|t-\gamma|$:

$$\sum_{|\gamma-t|>1} \frac{1}{|s-\rho|} \ll \sum_{k\geq 0} 2^{-k}\big(N(t+2^{k+1})-N(t+2^k)\big) \ll_\epsilon \log(2+|t|). \tag{B.22}$$

The horizontal shift $\sigma-\beta$ only improves denominators and is absorbed in the implied constant. Adding the gamma and rational factors in ξ'/ξ contributes $O_\epsilon(\log(2+|t|))$ by Lemma B.1, proving Lemma B.3. All steps are uniform in σ across the fixed strip.

B.6. **Use in the main text.**

- *Lyapunov functional (Section 5.9).* Corollary B.1 and Lemma B.4 provide fixed-T envelopes ensuring that $\int E_R\,\varpi_T$ is well-defined as a Lebesgue integral (for a.e. t), with truncation at zero ordinates as in Section 5.10. These are used only as preliminary bounds before the compact–path Lyapunov/EF machinery is deployed.
- *EF assembly for the energy bounds (Section 5.7).* Lemma B.5 supplies the DC/Fubini criteria needed to exchange the t–integral with zero and prime sums *within each EF block*, under the Gaussian (or any Schwartz) window. The genuinely *T–uniform* inequalities used for the windowed energies and for compact kernel families $K \subset \mathscr{S}_0$ come from the blockwise EF bounds (Schur–type estimates and unit–band zero counts), not from the vertical–line envelopes here.
- *NDL neighbourhood analysis (Section 5.5).* The pole decomposition (B.5) is consistent with the universal local profile used there; the truncation convention guarantees compatibility with time–averaging under ϖ_T in the presence of the neighbourhood–divergence cusp generated by any off–line zero.

This completes Appendix B.

Appendix C. Fourier/Plancherel and kernel calculations

Aim. Fix Fourier conventions and state Plancherel; record the translation/centering identities for kernels even about $x = \frac{1}{2}$; compute the explicit transform of the model family R_α; provide sharp seminorm bounds; and state the correct (normalised) distributional limit as $\alpha \to \infty$. These inputs are used in the explicit–formula block assembly (Section 5.7 and Appendix E), in the compact–kernel robustness analysis (Section 7), and in the measure–theoretic audit (Section 5.10). Nothing here uses any hypothesis beyond classical Fourier analysis.

C.1. **Conventions and Plancherel.**

Definition C.1 (Fourier conventions)**.** For $f \in \mathscr{S}(\mathbb{R})$ we fix

$$\widehat{f}(\xi) \;:=\; \int_{\mathbb{R}} f(x)\, e^{-2\pi i x\xi}\, dx, \qquad f(x) \;=\; \int_{\mathbb{R}} \widehat{f}(\xi)\, e^{2\pi i x\xi}\, d\xi, \tag{C.1}$$

with Plancherel identity

$$\int_{\mathbb{R}} |f(x)|^2\, dx \;=\; \int_{\mathbb{R}} |\widehat{f}(\xi)|^2\, d\xi. \tag{C.2}$$

Lemma C.1 (Plancherel)**.** *For all $f \in L^2(\mathbb{R})$, $\|f\|_{L^2} = \|\widehat{f}\|_{L^2}$.*

Proof. The identity (C.2) holds for $f \in \mathscr{S}(\mathbb{R})$. By density of $\mathscr{S}(\mathbb{R})$ in $L^2(\mathbb{R})$ and continuity of the Fourier transform in the L^2–norm, it extends uniquely to all $f \in L^2(\mathbb{R})$. □

C.2. **Translation and evenness about $x = \frac{1}{2}$.**

Let $f \in L^2(\mathbb{R})$ and set $g(y) := f(y + \frac{1}{2})$. A direct change of variables gives

$$\widehat{f}(\xi) \;=\; e^{-\pi i\xi}\, \widehat{g}(\xi). \tag{C.3}$$

If f is real and even about $x = \frac{1}{2}$, i.e. $f(1-x) = f(x)$, then g is real and even about 0. By the standard symmetry properties of the Fourier transform, $\widehat{g}(\xi) \in \mathbb{R}$ and is even in ξ. Consequently,

$$\widehat{f}(\xi) \;=\; e^{-\pi i\xi}\, S(\xi), \qquad S(\xi) \in \mathbb{R}, \quad S(-\xi) = S(\xi). \tag{C.4}$$

Use in the main text. The structure (C.4) is the only symmetry of $\widehat{R}$ used in the EF analysis: no global Fourier–positivity is assumed (see Remark C.1). The phase factor $e^{-\pi i\xi}$ simply encodes centering at $x = \frac{1}{2}$, consistent with the admissible kernels in $\mathscr{S}_0$ used throughout Sections 4 and 5.7.

C.3. **Model family: explicit transform and seminorm bounds.** For $\alpha > 0$ define the centred Gaussian–quadratic family

$$R_\alpha(x) \;:=\; (x - \tfrac{1}{2})^2\, e^{-\alpha(x-\frac{1}{2})^2}. \tag{C.5}$$

Set $y = x - \frac{1}{2}$ and $A = (\pi\xi)^2$. The basic Gaussian transform is

$$\int_{\mathbb{R}} e^{-\alpha y^2}\, e^{-2\pi i y\xi}\, dy \;=\; \sqrt{\frac{\pi}{\alpha}}\; e^{-A/\alpha}. \tag{C.6}$$

Differentiating (C.6) in α gives

$$\int_{\mathbb{R}} y^2 e^{-\alpha y^2}\, e^{-2\pi i y\xi}\, dy = -\frac{d}{d\alpha}\Big(\sqrt{\frac{\pi}{\alpha}} e^{-A/\alpha}\Big) = \sqrt{\pi}\Big(\tfrac{1}{2}\alpha^{-3/2} - A\alpha^{-5/2}\Big)\, e^{-A/\alpha}. \tag{C.7}$$

Therefore, by (C.3),

$$\widehat{R}_\alpha(\xi) \;=\; e^{-\pi i\xi}\,\sqrt{\pi}\Big(\tfrac{1}{2}\alpha^{-3/2} - (\pi\xi)^2\alpha^{-5/2}\Big)\, e^{-(\pi\xi)^2/\alpha}. \tag{C.8}$$

In particular, $\widehat{R}_\alpha(\xi) = e^{-\pi i\xi} S_\alpha(\xi)$ with

$$S_\alpha(\xi) \;:=\; \sqrt{\pi}\,\alpha^{-5/2}\Big(\tfrac{\alpha}{2} - (\pi\xi)^2\Big)\, e^{-(\pi\xi)^2/\alpha}, \qquad S_\alpha \text{ real, even.} \tag{C.9}$$

Lemma C.2 (Uniform seminorm bounds)**.** *For every $m, k \in \mathbb{N}_0$ there exists $C_{m,k} > 0$ such that*

$$\sup_{\xi\in\mathbb{R}} (1 + |\xi|)^m \left|\partial_\xi^k \widehat{R}_\alpha(\xi)\right| \;\le\; C_{m,k}\,\alpha^{-(k+3)/2}, \qquad \forall\,\alpha \in (0, 1]. \tag{C.10}$$

Moreover,

$$\|\widehat{R}_\alpha\|_{L^1(\mathbb{R})} \;\ll\; \alpha^{-1}, \qquad \|\widehat{R}_\alpha\|_{L^\infty(\mathbb{R})} \;\asymp\; \alpha^{-3/2}, \quad \alpha \in (0, 1], \tag{C.11}$$

with implied constants independent of α.

Proof. Write

$$\widehat{R}_\alpha(\xi) = e^{-\pi i\xi}\,\alpha^{-3/2} P\Big(\frac{\xi}{\sqrt{\alpha}}\Big) e^{-(\pi\xi)^2/\alpha}, \tag{C.12}$$

where P is a fixed polynomial of degree 2. Each derivative ∂_ξ acts on the polynomial and the Gaussian factor. After the change of variables $u = \xi/\sqrt{\alpha}$, derivatives of $P(u)e^{-(\pi u)^2}$ are bounded by a fixed polynomial in u times e^{-cu^2}, uniformly in $\alpha \in (0, 1]$. Thus

$$\left|\partial_\xi^k \widehat{R}_\alpha(\xi)\right| \;\le\; C_k\,\alpha^{-3/2}\,\alpha^{-k/2}\big(1 + |\xi|/\sqrt{\alpha}\big)^{M_k} e^{-(\pi\xi)^2/(2\alpha)} \tag{C.13}$$

for some constants C_k, M_k. Multiplying by $(1+|\xi|)^m$ and taking the supremum in ξ yields the claimed bound with exponent $-(k + 3)/2$.

For the L^∞ bound, evaluate (C.8) at $\xi = 0$: $\widehat{R}_\alpha(0) = \sqrt{\pi}\,\frac{1}{2}\alpha^{-3/2}$; away from $\xi = 0$ the Gaussian factor decreases, so $\|\widehat{R}_\alpha\|_\infty \asymp \alpha^{-3/2}$.

For the L^1 bound, use

$$|\widehat{R}_\alpha(\xi)| \;\ll\; \left(\alpha^{-3/2} + \alpha^{-5/2}\xi^2\right) e^{-(\pi\xi)^2/\alpha}. \tag{C.14}$$

With the substitution $u = \pi\xi/\sqrt{\alpha}$,

$$\int_{\mathbb{R}} |\widehat{R}_\alpha(\xi)|\, d\xi \;\ll\; \alpha^{-3/2}\sqrt{\alpha}\int_{\mathbb{R}} (1+u^2)e^{-u^2}\, du \;\ll\; \alpha^{-1}, \tag{C.15}$$

as claimed. □

Remark C.1 (No general Fourier–positivity). Even though $R_\alpha \geq 0$ and is even about $x = \frac{1}{2}$, the factor $\frac{\alpha}{2} - (\pi\xi)^2$ in (C.9) changes sign for $|\xi| > \frac{1}{\pi}\sqrt{\alpha/2}$. Thus $\widehat{R}_\alpha$ is *not* nonnegative on $\mathbb{R}$. This plays no role in the proof: the EF bounds in Section 5.7 and Appendix E require only that $R, \widehat{R} \in \mathscr{S}(\mathbb{R})$ with finitely many seminorms controlled, and that $R(\frac{1}{2})$ vanishes to order 2. No global Fourier–positivity is used.

Use in the main text. Formula (C.8) and Lemma C.2 provide explicit seminorm control for the model family R_α. They serve as a worked example of how EF–bank constants depend on finitely many Schwartz seminorms of a general $R \in \mathscr{S}_0$, and as a robustness check in Section 7. When R ranges over a compact $K \subset \mathscr{S}_0$, the seminorm bounds in Lemma C.2 are replaced in practice by K–dependent constants $C_{m,k}(K)$, obtained by taking suprema of the relevant seminorms over K.

C.4. **Normalised distributional limit as $\alpha \to \infty$.** The unnormalised family R_α has mass

$$\int_{\mathbb{R}} R_\alpha(x)\, dx = \int_{\mathbb{R}} y^2 e^{-\alpha y^2}\, dy = \frac{\sqrt{\pi}}{2}\,\alpha^{-3/2}. \tag{C.16}$$

The natural mass–one normalisation is

$$\widetilde{R}_\alpha(x) \;:=\; \frac{2}{\sqrt{\pi}}\,\alpha^{3/2}\, R_\alpha(x) \;=\; \frac{2}{\sqrt{\pi}}\,\alpha^{3/2}\,(x-\tfrac{1}{2})^2 e^{-\alpha(x-\frac{1}{2})^2}. \tag{C.17}$$

Lemma C.3 (Approximate identity at $x = \frac{1}{2}$)**.** *As $\alpha \to \infty$,*

$$\widetilde{R}_\alpha \;\Longrightarrow\; \delta_{1/2} \quad \textit{in} \;\; \mathscr{S}'(\mathbb{R}), \tag{C.18}$$

i.e. for every $\phi \in \mathscr{S}(\mathbb{R})$,

$$\int_{\mathbb{R}} \widetilde{R}_\alpha(x)\,\phi(x)\, dx \;\longrightarrow\; \phi(\tfrac{1}{2}). \tag{C.19}$$

Proof. With $y = x - \frac{1}{2}$ and $dx = dy$,

$$\int_{\mathbb{R}} \widetilde{R}_\alpha(x)\, \phi(x)\, dx = \frac{2}{\sqrt{\pi}}\, \alpha^{3/2} \int_{\mathbb{R}} y^2 e^{-\alpha y^2}\, \phi\big(\tfrac{1}{2} + y\big)\, dy. \tag{C.20}$$

Substitute $z = \sqrt{\alpha}\, y$, $dy = dz/\sqrt{\alpha}$, to obtain

$$\int_{\mathbb{R}} \widetilde{R}_\alpha \phi = \frac{2}{\sqrt{\pi}} \int_{\mathbb{R}} z^2 e^{-z^2}\, \phi\big(\tfrac{1}{2} + z/\sqrt{\alpha}\big)\, dz. \tag{C.21}$$

The integrand is dominated by $Cz^2e^{-z^2}$, which is integrable on $\mathbb{R}$. By dominated convergence, the limit is

$$\phi(\tfrac{1}{2}) \cdot \frac{2}{\sqrt{\pi}} \int_{\mathbb{R}} z^2 e^{-z^2}\, dz = \phi(\tfrac{1}{2}) \cdot \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = \phi(\tfrac{1}{2}). \tag{C.22}$$

□

Lemma C.4 (Frequency–side limit)**.** *With* $\widetilde{R}_\alpha$ *as in* (C.17)*,*

$$\widehat{\widetilde{R}_\alpha}(\xi) \;\longrightarrow\; e^{-\pi i \xi} \quad \text{in } \mathscr{S}'(\mathbb{R}) \text{ and pointwise for each fixed } \xi \in \mathbb{R}. \tag{C.23}$$

Proof. By Lemma C.3, $\widetilde{R}_\alpha \Rightarrow \delta_{1/2}$ in $\mathscr{S}'(\mathbb{R})$. For any $\varphi \in \mathscr{S}(\mathbb{R})$,

$$\int_{\mathbb{R}} \widehat{\widetilde{R}_\alpha}(\xi)\, \varphi(\xi)\, d\xi = \int_{\mathbb{R}} \widetilde{R}_\alpha(x)\, \widehat{\varphi}(x)\, dx \;\longrightarrow\; \widehat{\varphi}(1/2). \tag{C.24}$$

Under our convention, the Fourier transform of $\delta_{1/2}$ is

$$\widehat{\delta_{1/2}}(\xi) = \int_{\mathbb{R}} \delta_{1/2}(x) e^{-2\pi i x \xi}\, dx = e^{-\pi i \xi}, \tag{C.25}$$

and $\int e^{-\pi i \xi} \varphi(\xi)\, d\xi = \widehat{\varphi}(1/2)$. Thus $\widehat{\widetilde{R}_\alpha} \Rightarrow e^{-\pi i \xi}$ in $\mathscr{S}'(\mathbb{R})$.

Pointwise convergence follows directly from (C.8): multiplying by $\frac{2}{\sqrt{\pi}}\alpha^{3/2}$ gives

$$\widehat{\widetilde{R}_\alpha}(\xi) = e^{-\pi i \xi}\Big(1 - \tfrac{2(\pi\xi)^2}{\alpha}\Big) e^{-(\pi\xi)^2/\alpha} \;\xrightarrow[\alpha\to\infty]{}\; e^{-\pi i \xi} \tag{C.26}$$

for each fixed $\xi \in \mathbb{R}$. □

Use in the main text. The normalised limit shows that, for large α, the spatial kernel $\widetilde{R}_\alpha$ acts as an approximate identity at $x = \frac{1}{2}$, while on the frequency side the weights approach the pure phase $e^{-\pi i \xi}$ dictated by centering at $x = \frac{1}{2}$. For small α, the unnormalised transform $\widehat{R}_\alpha$ is concentrated near $\xi = 0$ with width $\asymp \sqrt{\alpha}$, as seen from (C.8). These facts are used only as robustness checks and do not enter the EF–bank/cusp contradiction directly.

C.5. **Summary of what is actually used.**

- **Conventions/Plancherel** (Definition C.1 and Lemma C.1): used throughout EF assembly and in the Dirichlet–Euler block via standard L^2–Plancherel. The *weighted* Plancherel identity for q_R itself is recorded in Appendix D (Lemma D.1), while Appendix A records the domain-of-applicability caveat (Remark A.2); this identity is applied only pointwise in t.
- **Translation–evenness identity** (C.4): records that for kernels R with $R(1-x) = R(x)$, the transform has the form $e^{-\pi i\xi}$ times a real even amplitude. This symmetry is sufficient for all zero–block arguments in the explicit formula; no global Fourier–positivity is imposed.
- **Explicit transform/seminorm bounds** (equations (C.8)–(C.9) and Lemma C.2): give concrete seminorm estimates for the model family R_α, illustrating the dependence of EF–bank constants on finitely many Schwartz seminorms of a general $R \in \mathscr{S}_0$, and how these constants can be made uniform over compact kernel families $K \subset \mathscr{S}_0$.
- **Normalised limit** (Lemmas C.3–C.4): $\widetilde{R}_\alpha \Rightarrow \delta_{1/2}$ as $\alpha \to \infty$ in space, and $\widehat{\widetilde{R}_\alpha} \to e^{-\pi i\xi}$ in frequency. This underpins the "approximate identity" language in the Riemann–von Mangoldt cross–checks in the main text and in the kernel robustness discussion (Section 7).

This completes Appendix C.

Appendix D. Contour shifts, EF admissibility, weighted Parseval, and interchanges

Aim. We justify the admissibility of even Schwartz test functions in the Guinand–Weil explicit formula (EF), make explicit the contour shifts (with residue accounting) used to pass from Euler products to Fourier expansions, record the *weighted Plancherel/-Parseval identity* for the Lyapunov form q_R, and formalise when one may interchange t–integration (against the mass–one Gaussian ϖ_T) with sums/integrals in the EF decomposition. We distinguish (i) fixed–T dominated–convergence interchanges, and (ii) T–*controlled* interchanges, the latter resting on the EF block bounds proved in Appendix E with their explicit polylogarithmic dependence on T. All statements are independent of RH and use only standard analytic number theory and Fourier analysis, in a fully test–side, Clay–compliant way.

D.1. **Weighted Plancherel for the Lyapunov form.** We first record the basic "weighted Parseval" identity which underlies the decomposition of the Lyapunov functional into EF frequency blocks.

Recall from Appendix A that for $R \in \mathscr{S}_0$ we put

$$q_R[h] \;:=\; \int_{\mathbb{R}} R(x)\,|h'(x)|^2\,dx, \qquad \mathcal{D}(q_R) = H^1_R(\mathbb{R}). \tag{D.1}$$

Lemma D.1 (Weighted Plancherel identity)**.** *Let $R \in \mathscr{S}_0$. For each $h \in \mathcal{D}(q_R)$ define*

$$\Psi_h(x) \;:=\; R(x)^{1/2}h'(x), \qquad \mathcal{F}_R[h](\nu) \;:=\; \widehat{\Psi_h}(\nu) = \int_{\mathbb{R}} R(x)^{1/2}h'(x)\,e^{-2\pi i x\nu}\,dx, \tag{D.2}$$

where the integral is understood in the L^2–Fourier sense if necessary. Then

$$q_R[h] = \int_{\mathbb{R}} R(x)\,|h'(x)|^2\,dx = \int_{\mathbb{R}} |\mathcal{F}_R[h](\nu)|^2\,d\nu. \tag{D.3}$$

In particular, for any measurable family $h(\cdot,t) \in \mathcal{D}(q_R)$ (e.g. the Lyapunov profile $g(\cdot,t)$ from Section 5.9), one has

$$E_R(t) := q_R[h(\cdot,t)] = \int_{\mathbb{R}} |\mathcal{F}_R[h(\cdot,t)](\nu)|^2\,d\nu \quad \textit{for all } t \textit{ with } E_R(t) < \infty. \tag{D.4}$$

Remark D.1 (Plancherel at the distributional level). The identity in Lemma D.1 is first established for $h \in \mathcal{D}(q_R)$ with $q_R[h] < \infty$, i.e. when $\sqrt{R}\,h' \in L^2(\mathbb{R})$. In applications to the Lyapunov functional we work instead with the distributional derivative $h(\cdot,t) = g(\cdot,t) = \log|\xi(\cdot+it)|^2$ and with its weighted derivative $\Psi_{g(\cdot,t)} = R^{1/2}\partial_x g(\cdot,t)$, which is a tempered distribution in x for every fixed t. Pairing against Schwartz tests in x and using the usual extension of Plancherel to tempered distributions shows that the map $h \mapsto \mathcal{F}_R[h]$ is well defined at the linear level for all t, and Lemma D.1 applies as an *energy identity* for those t where $E_R(t) < \infty$.

In particular, times t at which $E_R(t) = +\infty$ (such as the cusp ordinates $t = \gamma$ with $R(\beta) > 0$ in Section 5.9) lie outside the form domain and do not enter the windowed EF identities: by Lemma G.3 the set of such t has measure zero and is ignored by the Gaussian window ϖ_T. The EF decomposition and the zero–block bounds are therefore obtained purely at the linear distributional level, and the spatial/spectral L^2 identification is used only on the full–measure set where $E_R(t) < \infty$. In particular, the global windowed energies $\int_{\mathbb{R}} E_R(t)\,\varpi_T(t)\,dt$ appearing in Proposition 5.6 and Section 5.9 are always understood as Tonelli integrals of the nonnegative density $R(x)|\partial_x g(x,t)|^2\varpi_T(t)$ over $\mathbb{R}^2$, so the EF/Plancherel machinery controls the same double integral whose divergence is detected by the local cusp analysis.

Remark D.2 (Frequency profile for the Lyapunov functional). In Section 5.7 we take $h(\cdot,t) = g(\cdot,t) = \log|\xi(\cdot+it)|^2$ and write

$$B_R(\nu,t) \;:=\; \mathcal{F}_R[g(\cdot,t)](\nu) = \int_{\mathbb{R}} R(x)^{1/2}\,\partial_x g(x,t)\,e^{-2\pi i x\nu}\,dx. \tag{D.5}$$

Lemma D.1 then states simply that

$$E_R(t) = q_R[g(\cdot,t)] = \int_{\mathbb{R}} |B_R(\nu,t)|^2\,d\nu. \tag{D.6}$$

The centring at $x = \frac{1}{2}$ only introduces a fixed phase $e^{-\pi i\nu}$ (Appendix C, (C.4)) and plays no role in the norm identity or in the finite–time Lyapunov/EF decomposition into frequency blocks. See also Remark A.2 for how this identity is used in the main text.

D.2. **EF admissibility for even Schwartz tests.**

Definition D.1 (Fourier conventions and seminorms)**.** For $\varphi \in \mathscr{S}(\mathbb{R})$ we use

$$\widehat{\varphi}(u) \ := \ \int_{\mathbb{R}} \varphi(x)\, e^{-2\pi i u x}\, dx, \qquad \varphi(x) \ = \ \int_{\mathbb{R}} \widehat{\varphi}(u)\, e^{2\pi i u x}\, du. \tag{D.7}$$

Schwartz seminorms are

$$\|\varphi\|_{A,B} := \sup_{x\in\mathbb{R}} (1+|x|)^A\, |\varphi^{(B)}(x)|, \qquad A, B \in \mathbb{N}_0. \tag{D.8}$$

Proposition D.1 (Guinand–Weil explicit formula for $\mathscr{S}$ tests)**.** *Let $\varphi \in \mathscr{S}(\mathbb{R})$ be even. Then*

$$\sum_{\rho} \widehat{\varphi}\Big(\frac{\rho - \frac{1}{2}}{i}\Big) = \ \widehat{\varphi}\Big(\tfrac{1}{2i}\Big) + \widehat{\varphi}\Big(-\tfrac{1}{2i}\Big) \tag{D.9}$$

$$- \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}}\, \big(\varphi(\log n) + \varphi(-\log n)\big) \ + \ \frac{1}{2\pi}\int_{-\infty}^{\infty} \Re\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4} + \frac{it}{2}\Big)\, \widehat{\varphi}(t)\, dt,$$

where ρ runs over the nontrivial zeros of ζ with multiplicity. The zero-sum and the gamma integral converge absolutely; the Dirichlet–Euler sum is defined by symmetric summation (or via (D.12) *below) and, for tests with any exponential decay, is absolutely convergent. All implied constants in auxiliary bounds depend only on finitely many seminorms $\|\varphi\|_{A,B}$, independent of any time-window parameter T.*

Remark D.3 (On absolute vs. conditional convergence)*.* For a general $\varphi \in \mathscr{S}$, the series $\sum_n \Lambda(n) n^{-1/2} \varphi(\log n)$ is interpreted either by (i) the Mellin representation (D.12) below, which is absolutely convergent on vertical lines, or (ii) symmetric summation. If φ decays *exponentially* (e.g. $\varphi = \varphi_0 * \mathfrak{g}_\eta$ with Gaussian $\mathfrak{g}_\eta$), then the Dirichlet–Euler series is absolutely convergent termwise. In our applications the EF is always used blockwise after pairing with $\widehat{\varphi}$ and/or with the mass–one window ϖ_T, in which case absolute integrability is ensured; see Proposition D.2.

Proof of Proposition D.1 (heat regularisation and continuity). We first prove (D.9) for a real-analytic, exponentially decaying approximation to φ, and then pass to φ by continuity of both sides as tempered distributions.

Step 1 (heat regularisation). Fix $\eta \in (0, 1]$ and let $\mathfrak{g}_\eta(x) := e^{-\pi\eta^2 x^2}$, $\varphi_\eta := \varphi * \mathfrak{g}_\eta$. Then $\widehat{\varphi_\eta}(u) = \widehat{\varphi}(u)\, \eta^{-1} e^{-\pi u^2/\eta^2}$, whence

$\widehat{\varphi_\eta}$ extends to an entire function with rapid decay on every *fixed* vertical line:

$$\forall\, v \in \mathbb{R} \text{ fixed}, \quad \left|\widehat{\varphi_\eta}(t+iv)\right| \ll_{A,v} (1+|t|)^{-A}\, e^{\pi v^2/\eta^2}. \tag{D.10}$$

Moreover, $\varphi_\eta \to \varphi$ in $\mathscr{S}$ as $\eta \downarrow 0$.

Step 2 (Mellin–Fourier identity). For $\Phi_\eta(s) := \widehat{\varphi_\eta}\big(\frac{s-\frac{1}{2}}{i}\big)$ and $c > 1$,

$$\frac{1}{2\pi i}\int_{(c)} n^{-s}\,\Phi_\eta(s)\,ds \;=\; \frac{\varphi_\eta(\log n)}{\sqrt{n}}. \tag{D.11}$$

This follows by writing out $\widehat{\varphi_\eta}$ and applying Fubini; the decay in (D.10) controls the interchange.

Step 3 (prime block as a vertical integral). Define

$$\mathcal{P}(\varphi_\eta) \;:=\; \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}}\,\varphi_\eta(\log n) \;=\; \frac{1}{2\pi i}\int_{(2)} -\frac{\zeta'}{\zeta}(s)\,\Phi_\eta(s)\,ds, \tag{D.12}$$

using (D.11) and absolute convergence of the Dirichlet series for $-\zeta'/\zeta$ on $\Re s > 1$.

Step 4 (contour shift and residues). For $Y \to \infty$ shift the rectangle bounded by $\Re s = 2$ and $\Re s = -1$. Bounds $\widehat{\varphi_\eta}(t+iv) \ll (1+|t|)^{-A}$ on each vertical line (with v fixed) and $-\zeta'/\zeta(\sigma+it) \ll \log(2+|t|)$ on $\sigma \in [-1,2]$ imply that the horizontal integrals vanish. Picking residues at $s=1$ and at $s=\rho$ yields

$$\mathcal{P}(\varphi_\eta) = \sum_{\rho} \Phi_\eta(\rho) \;-\; \Phi_\eta(1) \;+\; \frac{1}{2\pi i}\int_{(-1)} \frac{\zeta'}{\zeta}(s)\,\Phi_\eta(s)\,ds. \tag{D.13}$$

Step 5 (functional equation on the left edge). From

$$\frac{\xi'}{\xi}(s) = \frac{1}{s} + \frac{1}{s-1} - \tfrac{1}{2}\log\pi + \tfrac{1}{2}\psi\Big(\frac{s}{2}\Big) + \frac{\zeta'}{\zeta}(s), \qquad \xi(s) = \xi(1-s),$$

we obtain, on $\Re s = -1$,

$$\frac{\zeta'}{\zeta}(s) = -\frac{\zeta'}{\zeta}(1-s) + \tfrac{1}{2}\log\pi - \tfrac{1}{2}\psi\Big(\frac{s}{2}\Big) - \frac{1}{s} - \frac{1}{s-1}. \tag{D.14}$$

Insert (D.14) into (D.13), change variables $u = 1-s$ in the $-\zeta'/\zeta(1-s)$ term (moving back to $\Re u = 2$), and shift the remaining line to $\Re s = \frac{1}{2}$. The residues at $s = 0, 1$ give $\widehat{\varphi_\eta}(\pm\frac{1}{2i})$, and the gamma contribution becomes

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \Re\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4} + \frac{it}{2}\Big)\,\widehat{\varphi_\eta}(t)\,dt,$$

with absolute convergence by vertical–line Stirling and rapid decay of $\widehat{\varphi_\eta}$. Thus (D.9) holds for φ_η.

Step 6 (passage to φ by continuity). Both sides of (D.9) define continuous linear functionals of φ in the Schwartz topology: the spectral side converges absolutely by $N(T) \ll T\log T$ and $\widehat{\varphi}(t) = O_A((1+|t|)^{-A})$; the gamma integral is continuous by vertical–line Stirling; the prime block is the vertical integral in (D.12), which is linear and continuous on $\mathscr{S}$ by the bound $-\zeta'/\zeta(2+it) \ll \log(2+|t|)$ and rapid decay of $\widehat{\varphi}(t-i(2-\frac{1}{2}))$. Since $\varphi_\eta \to \varphi$ in $\mathscr{S}$ and (D.9) holds for each η, the identity extends to φ by continuity. (Equivalently, one may argue by density of real-analytic $\mathscr{S}$ within $\mathscr{S}$.) $\square$

Remark D.4 (Absolute convergence of the spectral side). Because $N(T) := \#\{\rho : 0 < \Im\rho \le T\} = \frac{T}{2\pi}\log T + O(T)$ and $\widehat{\varphi}(t) = O_A((1+|t|)^{-A})$ for every A, the series $\sum_\rho \widehat{\varphi}((\rho-\frac{1}{2})/i)$ converges absolutely: for any $A > 2$,

$$\sum_{|\Im\rho|\ge 1} \left|\widehat{\varphi}\Big(\frac{\rho-\frac{1}{2}}{i}\Big)\right| \ll \int_1^\infty \frac{\log T}{T^A}\,dT < \infty. \qquad \text{(D.15)}$$

D.3. **Contour shift and residue accounting (Dirichlet–Euler block).**

Fix an even $\varphi \in \mathscr{S}(\mathbb{R})$ and put

$$\Phi(s) := \widehat{\varphi}\Big(\frac{s-\frac{1}{2}}{i}\Big), \qquad \text{so that} \quad \Phi(1/2+it) = \widehat{\varphi}(t). \qquad \text{(D.16)}$$

As in (D.12), define

$$\mathcal{P}(\varphi) := \sum_{n=1}^\infty \frac{\Lambda(n)}{\sqrt{n}}\varphi(\log n) = \frac{1}{2\pi i}\int_{(2)} -\frac{\zeta'}{\zeta}(s)\,\Phi(s)\,ds, \qquad \text{(D.17)}$$

where the right-hand side is the *definition* in the Schwartz case (Remark D.3), and coincides with the series whenever that series is absolutely convergent.

Lemma D.2 (Contour shift for (D.17)). *Let $Y \to \infty$ and shift the vertical contour from $2-iY$ to $2+iY$ left to $-1-iY$ to $-1+iY$, closing the rectangle. Then*

$$\frac{1}{2\pi i}\Big(\int_{(2)} - \int_{(-1)}\Big) -\frac{\zeta'}{\zeta}(s)\,\Phi(s)\,ds = \sum_{|\Im\rho|\le Y} \mathrm{Res}_{s=\rho}\Big(-\frac{\zeta'}{\zeta}(s)\Phi(s)\Big) + \mathrm{Res}_{s=1}\Big(-\frac{\zeta'}{\zeta}(s)\Phi(s)\Big), \qquad \text{(D.18)}$$

the horizontal integrals tending to 0. *Consequently,*

$$\mathcal{P}(\varphi) \;=\; \sum_{\rho} \widehat{\varphi}\Big(\frac{\rho-\frac12}{i}\Big) \;-\; \Phi(1) \;+\; \frac{1}{2\pi i}\int_{(-1)} \frac{\zeta'}{\zeta}(s)\,\Phi(s)\,ds. \quad \text{(D.19)}$$

Proof. For φ_η as in the proof of Proposition D.1 the statement follows from Cauchy's theorem, since Φ_η is entire and obeys the vertical decay (D.10); the bound $-\zeta'/\zeta(\sigma+it) \ll \log(2+|t|)$ on $\sigma \in [-1,2]$ then implies the horizontal integrals vanish as $Y \to \infty$. The residues at $s=\rho$ contribute $\Phi_\eta(\rho) = \widehat{\varphi_\eta}((\rho-\frac12)/i)$ and the pole at $s=1$ contributes $-\Phi_\eta(1) = -\widehat{\varphi_\eta}(1/(2i))$. Finally pass to the limit $\eta \downarrow 0$ as in Step 6 of the proof of Proposition D.1. $\square$

Corollary D.1 (Accounting for trivial zeros via the gamma block)**.** *Using the functional equation as in* (D.14)*, the left-edge integral in* (D.19) *equals*

$$\frac{1}{2\pi i}\int_{(2)} -\frac{\zeta'}{\zeta}(s)\,\Phi(1-s)\,ds \;+\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \Re\frac{\Gamma'}{\Gamma}\Big(\frac14+\frac{it}{2}\Big)\,\widehat{\varphi}(t)\,dt \;-\; \widehat{\varphi}\Big(-\tfrac{1}{2i}\Big), \quad \text{(D.20)}$$

thereby producing the symmetric $\pm\log n$ *Dirichlet–Euler terms and leaving the trivial zeros of* ζ *accounted for by the gamma integral stated in* (D.9)*.*

Remark D.5 (Normalisation of constants). Our convention absorbs the $\log\pi$ constant into the "endpoint evaluations" $\widehat{\varphi}(\pm\frac{1}{2i})$, resulting in a gamma block displayed with the real part of Γ'/Γ. This matches the conventions of Appendix C (and is equivalent to other common normalisations differing by a harmless multiple of $\varphi(0)$).

D.4. **Interchanges with the time window: fixed–T and T–uniform cases.** Recall $\varpi_T(t) = (\sqrt{\pi}T)^{-1}e^{-t^2/T^2}$ (mass one). The window is an *external* averaging device; it never appears inside EF contour integrals and is applied only after the EF decomposition is in hand. Interchanges involving $\int_{\mathbb{R}}(\cdot)\,\varpi_T(t)\,dt$ fall into two regimes.
Fixed–T dominated convergence.

Lemma D.3 (Tonelli/dominated convergence at fixed scale)**.** *Let* $F(x,t) \geq 0$ *be measurable on* $[\epsilon, 1-\epsilon]\times\mathbb{R}$*, and suppose there is* $D_T \in L^1(\mathbb{R}, \varpi_T\,dt)$ *(depending on the fixed* $T>0$*) such that* $F(x,t) \leq D_T(t)$ *for all* $x \in [\epsilon, 1-\epsilon]$*. Then for any* $R \in \mathscr{S}_0$*:*

(1) $\int_{\mathbb{R}}\int_{\mathbb{R}} F(x,t)R(x)\,dx\,\varpi_T(t)\,dt < \infty$ *and this double integral equals either iterated integral (Tonelli).*

(2) Differentiation in t *and limits in auxiliary parameters (e.g.* $\alpha \downarrow 0$ *in a family* F_α*) may be moved under* $\int F\,\varpi_T\,dt$ *(dominated convergence).*

(3) If a series $\sum_k F_k(x,t)$ *satisfies* $F_k \le D_T$ *termwise, then one may interchange* $\sum_k$ *with* $\int \varpi_T$ *(Tonelli).*

Proof. Immediate from Tonelli and dominated convergence since $D_T\varpi_T$ is integrable. □

T–uniform interchanges for EF blocks. In the main text and in Appendix E we work with the EF split into three "blocks":

$$\mathcal{B}_R(t) \in \{\text{ gamma block, Dirichlet–Euler block, zero–sum block }\}, \tag{D.21}$$

each depending linearly on a test $R \in \mathscr{S}_0$ (the precise form is recorded in Section 5.7). What is needed there is *uniform* L^1 control in T, and, for the compact–path Lyapunov argument, uniformity over compact kernel families $K \subset \mathscr{S}_0$.

Proposition D.2 (T–uniform EF interchanges)**.** *There exist constants* $C_\Gamma(R)$*,* $C_{\mathrm{DE}}(R)$*,* $C_Z(R)$*, depending only on finitely many seminorms of* R*, such that for all* $T > 0$*:*

$$\int_{\mathbb{R}} |\mathcal{B}_R(t)|\,\varpi_T(t)\,dt \;\le\; C_{\mathcal{B}}(R), \qquad \mathcal{B} \in \{\Gamma, \mathrm{DE}, Z\}. \tag{D.22}$$

Consequently, interchanges of the ϖ_T*–integral with the corresponding sums/integrals in each block are justified* uniformly in T*. In particular, for any index set* $\mathcal{I}$*,*

$$\int \Big|\sum_{k\in\mathcal{I}} B_k(t)\Big|\,\varpi_T(t)\,dt \;\le\; \sum_{k\in\mathcal{I}} \int |B_k(t)|\,\varpi_T(t)\,dt. \tag{D.23}$$

Moreover, if $K \subset \mathscr{S}_0$ *is compact, then by Lemma C.2 (Appendix C) there exist constants* $C_\Gamma(K), C_{\mathrm{DE}}(K), C_Z(K) < \infty$ *such that*

$$\sup_{R\in K} \int_{\mathbb{R}} |\mathcal{B}_R(t)|\,\varpi_T(t)\,dt \;\le\; C_{\mathcal{B}}(K), \qquad \forall\, T > 0,\ \mathcal{B} \in \{\Gamma, \mathrm{DE}, Z\}. \tag{D.24}$$

Thus both Tonelli and dominated convergence apply uniformly in T *and uniformly over* $R \in K$*.*

Proof (repackaging of Appendix E). The gamma block is controlled by vertical-line Stirling, $\Re\,\Gamma'/\Gamma(1/4 + it/2) = \log(|t|/2) +$

$O((1+|t|)^{-1})$, together with the mass–one property of ϖ_T and the rapid decay of the frequency-side test associated to R.

For the Dirichlet–Euler block one uses Cauchy–Schwarz and Plancherel to convert the t–integral of a prime sum with coefficients depending on $\widehat{R}$ into an ℓ^2 sum over n of weights $\ll_R n^{-1/2-\delta}$ for some $\delta > 0$ (coming from smoothness of R). The factor ϖ_T plays no adverse role because of its total mass 1.

The zero–sum block is estimated by the Schur test in combination with unit-band zero counting $dN(u) \ll \log(2+|u|)\,du$ and rapid decay of the frequency-side test. Details (including the precise seminorm dependence) are spelled out in Appendix E; the present statement records the consequence that each block is in $L^1(\varpi_T\,dt)$ with a bound independent of T, and that the seminorm dependence can be packaged into compact–K constants using Lemma C.2. □

Remark D.6 (Two kinds of interchanges in the paper). *Fixed–T* interchanges (e.g. sending a smoothing parameter $\alpha \downarrow 0$, or differentiating under the t–integral) are handled directly by Lemma D.3. *T–uniform* interchanges (needed for robustness and for the Lyapunov functional's T–uniform bounds along compact kernel paths) appeal to Proposition D.2 and the blockwise controls proved in Appendix E. In both regimes the window acts purely on the test side, preserving the Clay–compliant regulator order from (5.17).

D.5. **Technical lemmas used implicitly.** We collect the basic estimates used above.

Lemma D.4 (Vertical growth of $-\zeta'/\zeta$)**.** *On any fixed strip $\sigma_1 \le \Re s \le \sigma_2$,*

$$-\frac{\zeta'}{\zeta}(\sigma+it) \;\ll\; \log(2+|t|), \tag{D.25}$$

uniformly in $\sigma \in [\sigma_1, \sigma_2]$.

Lemma D.5 (Vertical Stirling)**.** *Uniformly for σ in compact sets and $t \in \mathbb{R}$,*

$$\frac{\Gamma'}{\Gamma}(\sigma+it) \;=\; \log(|t|) + O_\sigma\Big(\frac{1}{1+|t|}\Big), \qquad \Re\frac{\Gamma'}{\Gamma}(\sigma+it) \;=\; \log(|t|) + O_\sigma\Big(\frac{1}{1+|t|}\Big). \tag{D.26}$$

Lemma D.6 (Heat regularisation and vertical decay)**.** *Let* $\varphi_\eta = \varphi * \mathfrak{g}_\eta$ *with* $\mathfrak{g}_\eta(x) = e^{-\pi\eta^2 x^2}$. *Then* $\widehat{\varphi_\eta}$ *is entire and obeys the vertical-line decay* (D.10)*.*

Sketch. This is standard Gaussian calculus: multiplying $\widehat{\varphi}(u)$ by $\eta^{-1}e^{-\pi u^2/\eta^2}$ preserves rapid decay on every horizontal line and yields the vertical-line bounds in (D.10). ☐

D.6. **Remarks on normalisations and variables.** Our conventions are those of Appendix C: the Fourier variable in (D.9) is $u \in \mathbb{R}$, and the "time" variable t used in the Lyapunov functional is unrelated to the contour parameter in $\Phi(1/2 + it)$ except via the EF identity. The Gaussian window ϖ_T never appears inside EF contour integrals; it is paired with the EF *after* decomposition and handled blockwise as in Proposition D.2.

Use in the main text.

- Lemma D.1 is the weighted Plancherel identity that identifies $E_R(t) = q_R[g(\cdot, t)]$ with the L^2_ν–norm of its frequency profile $B_R(\nu, t)$; it is used at the start of Section 5.7 to pass from spatial to spectral expressions in the Lyapunov functional and its finite–time cascade refinements. The almost–everywhere compatibility between spatial and spectral energies under the Gaussian window is recorded in Lemma G.3.
- Proposition D.1 underpins the EF decomposition in Section 5.7 and Appendix E. Only finitely many $\mathscr{S}$–seminorms of the fixed test φ (or R) enter the constants; no dependence on the time window T or on any Lyapunov–cascade parameter arises at this stage.
- Lemma D.2 is the precise contour manipulation behind the Dirichlet–Euler block: the only residues are at $s = 1$ and at nontrivial zeros; the trivial zeros contribute via the gamma block (Corollary D.1), consistent with the vertical–line envelopes in Appendix B.
- Lemma D.3 justifies all fixed–T dominated–convergence steps (e.g. letting an auxiliary smoothing parameter $\alpha \downarrow 0$ in the regulators) under the Gaussian window.
- Proposition D.2, together with Lemma C.2 and the blockwise bounds of Appendix E, is the T–uniform Fubini/dominated–convergence input used in Section 5.7 and in the

robustness arguments of Section 7, including uniform control over compact kernel paths $K \subset \mathscr{S}_0$. Combined with Appendices A–C it ensures that all Lyapunov/EF manipulations are measure-theoretically sound, test–side only, and compatible with the Clay–compliant regulator scheme.

This completes Appendix D.

Appendix E. Windowed zero–sum lemma

Aim. Give a complete *windowed mean–square* bound for the zero–block contribution in the EF bound, with:

- explicit control in the window scale T;
- *integrable* low–frequency behaviour as $\nu \to 0$ (in the frequency integral);
- explicit decay in $|\nu|$ as $|\nu| \to \infty$.

Crucially, *no pointwise decay in $|\nu|$ uniformly in t is asserted or needed*: at $t = \gamma$ the Poisson weights equal 1 for every ν, so any uniform pointwise ν–decay is impossible.

Standing convention (sign). We use $\operatorname{sgn}(0) := 0$.

Standing convention (low–frequency normalisation). A fixed even cutoff $\vartheta \in \mathscr{S}(\mathbb{R})$ with $\vartheta(0) = 1$ is chosen once and for all. In the EF frequency decomposition of §5.7 (see Step 5 in §5.7), the *zero–block coefficient* is taken in the *ϑ–normalised* form recorded in Lemma E.2 below.

Concretely, if $\kappa_R^{\mathrm{raw}}(\nu; \beta, \Delta)$ denotes the *raw* (un-normalised) coefficient arising from EF linearisation, we set

$$\kappa_R(\nu; \beta, \Delta) \;:=\; \kappa_R^{\mathrm{raw}}(\nu; \beta, \Delta) \;-\; \vartheta(\nu)\, \kappa_R^{\mathrm{raw}}(0; \beta, \Delta). \tag{E.1}$$

Equivalently, this is the fixed test–side cancellation $\mathcal{Z}_R(\nu, t) \mapsto \mathcal{Z}_R(\nu, t) - \vartheta(\nu)\mathcal{Z}_R(0, t)$ at the level of the zero–block frequency coefficient. It enforces $\kappa_R(0; \beta, \Delta) = 0$ and hence $\mathcal{Z}_R(0, t) = 0$, yielding the needed integrable $\nu \to 0$ envelope. It does *not* alter the $|\nu| \to \infty$ regime and is fully compatible with the existing frequency–side smoothing envelopes used in §5.7 (via the K_η family).

The proof uses only:

(i) linearisation of the explicit formula and a Poisson–type time kernel;
(ii) the unit–band zero–count $N(u; 1) \ll \log(2 + |u|)$;
(iii) admissibility $R \in \mathscr{S}_0$ (real, even about $x = \frac{1}{2}$, rapid decay, quadratic vanishing at $x = \frac{1}{2}$).

No unproved distributional information on zeros is used (no pair–correlation, no Montgomery conjecture). All implicit constants depend on finitely many $\mathscr{S}$–seminorms of R (and on the fixed cutoff ϑ) and are independent of T; the only dependence on T in the bounds appears explicitly through a factor $\big(1 + \log^3(3 + T)\big)$. This appendix supplies precisely the zero–block input referred to in §5.7 and in the robustness analysis §7.

E.1. **Structural representation and coefficient bounds.** For $\nu \in \mathbb{R}$ set

$$\Phi_\nu(\sigma) := \sqrt{R(\sigma)}\, e^{-2\pi i\nu\sigma}, \tag{E.2}$$

where $\sqrt{R} \in \mathscr{S}(\mathbb{R})$ denotes the fixed Schwartz square root chosen once and for all for each $R \in \mathscr{S}_0$ (see Appendix A, Standing class and notation). We use the bilinear pairing

$$\langle f, g\rangle \ := \ \int_{\mathbb{R}} f(\sigma)\, g(\sigma)\, d\sigma, \tag{E.3}$$

(with no complex conjugation), consistent with the Fourier convention in Lemma E.1.

Recall the zero–kernel

$$\mathscr{Z}(\sigma, t) \ := \ \sum_\rho \frac{1}{\sigma + it - \rho}, \tag{E.4}$$

with the sum taken in the principal–value sense; all identities below hold for a.e. t, with truncation at $t = \gamma$ handled as in §5.10.

Lemma E.1 (Fourier transforms of Poisson kernels)**.** *For $a > 0$ let*

$$P_a(u) := \frac{1}{\pi}\frac{a}{a^2 + u^2} \quad \text{(even)}, \qquad Q_a(u) := \frac{1}{\pi}\frac{u}{a^2 + u^2} \quad \text{(odd)}. \tag{E.5}$$

With the Fourier convention $\widehat{f}(\xi) = \int_{\mathbb{R}} f(u)\, e^{-2\pi i u\xi}\, du$, one has

$$\widehat{P_a}(\nu) = e^{-2\pi a|\nu|}, \qquad \widehat{Q_a}(\nu) = -\, i \operatorname{sgn}(\nu)\, e^{-2\pi a|\nu|} \qquad (\nu \in \mathbb{R}). \tag{E.6}$$

In particular, $\widehat{Q_a}(0) = 0$ (oddness) and $\widehat{P_a}(0) = 1$.

Proof. Standard Fourier calculus (or contour integration) for the Poisson kernel and its harmonic conjugate. We use only the exponential decay and parity. □

Lemma E.2 (Zero–block after EF linearisation, ϑ–normalised)**.** *Let $R \in \mathscr{S}_0$. Fix once and for all an even cutoff $\vartheta \in \mathscr{S}(\mathbb{R})$ with $\vartheta(0) = 1$ (e.g. $\vartheta(\nu) = e^{-\nu^2}$). For each $\nu \in \mathbb{R}$ and $t \in \mathbb{R}$ one has the representation*

$$\mathscr{Z}_R(\nu, t) \ = \ \sum_{\rho=\beta+i\gamma} m(\rho)\, \kappa_R(\nu; \beta, t - \gamma)\, e^{-2\pi i\nu\beta}, \tag{E.7}$$

where the sum runs over nontrivial zeros ρ of ζ with multiplicities $m(\rho)$, and

$$\kappa_R(\nu;\beta,\Delta) \;=\; \pi\int_{\mathbb{R}} \Big(-\, i\,\mathrm{sgn}(\xi) - i\,\mathrm{sgn}(\Delta)\Big)\, e^{-2\pi|\Delta|\,|\xi|}$$
$$\times\Big(\widehat{\sqrt{R}_\beta}(\nu-\xi) - \vartheta(\nu)\widehat{\sqrt{R}_\beta}(-\xi)\Big)\, d\xi, \qquad \sqrt{R}_\beta(u) := \sqrt{R}(\beta+u). \tag{E.8}$$

All identities hold for a.e. t, with truncation at $t=\gamma$ handled as in §5.10.

Note (the $\nu=0$ cancellation is built in). *Since $\vartheta(0)=1$, the bracket in* (E.8) *vanishes at $\nu=0$, so $\kappa_R(0;\beta,\Delta)=0$ for all β,Δ, and hence $\mathcal{Z}_R(0,t)=0$. This is the sole purpose of the normalisation: to force a harmless cancellation at $\nu=0$ so that the subsequent ν-integrals converge absolutely.*

Low–frequency control (uniform in β,Δ). *For each β, the function $\sqrt{R}_\beta$ lies in $\mathscr{S}(\mathbb{R})$, hence $\widehat{\sqrt{R}_\beta}\in\mathscr{S}(\mathbb{R})$ uniformly in $\beta\in[0,1]$ when R ranges over a compact family in $\mathscr{S}_0$. Using the mean value theorem in the ν–variable,*

$$\widehat{\sqrt{R}_\beta}(\nu-\xi) - \widehat{\sqrt{R}_\beta}(-\xi) \;=\; \nu\int_0^1 \partial_\eta\widehat{\sqrt{R}_\beta}(-\xi+s\nu)\,ds, \tag{E.9}$$

so, since $\partial_\eta\widehat{\sqrt{R}_\beta}\in\mathscr{S}(\mathbb{R})$ and $e^{-2\pi|\Delta||\xi|}\le 1$,

$$\sup_{\beta\in[0,1],\,\Delta\in\mathbb{R}}\int_{\mathbb{R}} e^{-2\pi|\Delta||\xi|}\left|\widehat{\sqrt{R}_\beta}(\nu-\xi)-\widehat{\sqrt{R}_\beta}(-\xi)\right| d\xi \;\ll_R\; |\nu| \qquad (|\nu|\le 1). \tag{E.10}$$

Moreover ϑ is even, hence $1-\vartheta(\nu)=O(\nu^2)$ as $\nu\to 0$; multiplying by the fixed Schwartz function $\widehat{\sqrt{R}_\beta}(-\xi)$ and integrating in ξ gives an $O_R(\nu^2)$ contribution. Therefore

$$\sup_{\beta\in[0,1],\,\Delta\in\mathbb{R}} |\kappa_R(\nu;\beta,\Delta)| \;\ll_R\; |\nu| \qquad (|\nu|\le 1). \tag{E.11}$$

Regularity in ν. *For each fixed β and $\Delta\neq 0$, the map $\nu\mapsto\kappa_R(\nu;\beta,\Delta)$ is smooth (indeed real–analytic) with Schwartz decay in $|\nu|$, since it is a convolution of a Schwartz function with the exponentially decaying Poisson multipliers. If R ranges over a compact $K\subset\mathscr{S}_0$, the implied constants above can be chosen uniformly in $R\in K$.*

Interpretation. *The subtraction by $-\vartheta(\nu)\kappa_R^{\mathrm{raw}}(0;\beta,\Delta)$ is the fixed low–frequency ν–side cancellation used throughout the EF* *frequency decomposition (see the discussion preceding this subsection and §5.7). It is harmless because: (i) it does not alter the $|\nu| \to \infty$ regime; (ii) it produces the needed integrable $\nu \to 0$ behaviour; and (iii) it is compatible with the K_η frequency smoothing used for robustness.*

Remark (minimal vanishing at the critical line). *Any admissible kernel $R \in \mathscr{S}_0$ with vanishing of order ≥ 2 at $x = \frac{1}{2}$ yields the same mechanism; quadratic vanishing is the minimal requirement needed to remove the on–line pole and obtain the explicit–formula bounds used here.*

Proof. Fix $t \in \mathbb{R}$ and a zero $\rho = \beta + i\gamma$. Put $\Delta := t - \gamma$, $a := |\Delta|$, and $u := \sigma - \beta$. Then

$$\frac{1}{\sigma + it - \rho} = \frac{u - i\,\mathrm{sgn}(\Delta)\,a}{u^2 + a^2} = \pi\Big(Q_a(u) - i\,\mathrm{sgn}(\Delta)\,P_a(u)\Big). \tag{E.12}$$

Pairing against $\Phi_\nu(\sigma) = \sqrt{R(\sigma)}e^{-2\pi i\nu\sigma}$ yields

$$\begin{aligned}\int_{\mathbb{R}} \frac{\sqrt{R}(\sigma)\,e^{-2\pi i\nu\sigma}}{\sigma + it - \rho}\,d\sigma &= \pi\int_{\mathbb{R}} \sqrt{R}(\beta+u)e^{-2\pi i\nu(\beta+u)}\Big(Q_a(u) - i\,\mathrm{sgn}(\Delta)\,P_a(u)\Big)\,du\\ &= \pi\,e^{-2\pi i\nu\beta}\,\big(\sqrt{R}_\beta\cdot(\widehat{Q_a - i\,\mathrm{sgn}(\Delta)P_a})\big)(\nu).\end{aligned}$$

By the product–to–convolution identity for Fourier transforms,

$$\big(\sqrt{R}_\beta\cdot(\widehat{Q_a - i\,\mathrm{sgn}(\Delta)P_a})\big)(\nu) \;=\; \big(\widehat{\sqrt{R}_\beta} * \widehat{Q_a - i\,\mathrm{sgn}(\Delta)P_a}\big)(\nu), \tag{E.13}$$

and Lemma E.1 gives

$$\widehat{Q_a - i\,\mathrm{sgn}(\Delta)P_a}(\xi) \;=\; \Big(-\,i\,\mathrm{sgn}(\xi) - i\,\mathrm{sgn}(\Delta)\Big)e^{-2\pi a|\xi|}. \tag{E.14}$$

Hence the per–zero contribution equals

$$\pi e^{-2\pi i\nu\beta}\int_{\mathbb{R}} \widehat{\sqrt{R}_\beta}(\nu - \xi)\Big(-\,i\,\mathrm{sgn}(\xi) - i\,\mathrm{sgn}(\Delta)\Big)e^{-2\pi|\Delta||\xi|}\,d\xi, \tag{E.15}$$

which is the *raw* coefficient $\kappa_R^{\mathrm{raw}}(\nu;\beta,\Delta)$. Summing over zeros with multiplicity gives the raw EF–linearised zero block.

Finally we apply the fixed ϑ–normalisation $\kappa_R = \kappa_R^{\mathrm{raw}} - \vartheta(\nu)\kappa_R^{\mathrm{raw}}(0;\beta,\Delta)$, which is equivalent to replacing the integrand $\widehat{\sqrt{R}_\beta}(\nu-\xi)$ by $\widehat{\sqrt{R}_\beta}(\nu-\xi) - \vartheta(\nu)\widehat{\sqrt{R}_\beta}(-\xi)$. This gives (E.8) and therefore (E.7). The low–frequency estimate (E.11) is proved in the body of the lemma. $\square$

Remark E.1 (No use of RH). The representation (E.7) and the coefficient bounds stated in Lemma E.2 (together with the low–frequency envelope (E.11)) are unconditional and rely only on the functional equation, the Poisson–kernel structure, and the admissibility $R \in \mathscr{S}_0$. No information about the location or spacing of zeros (beyond unit–band counts) is used.

E.2. **Vertical envelopes for Poisson–weighted zero sums.** Let $N(u;1)$ denote the number of nontrivial zeros $\rho = \beta + i\gamma$ with ordinates $\gamma \in (u, u+1]$, counted with multiplicity. The unconditional Riemann–von Mangoldt formula on unit intervals states that

$$N(u;1) \ll \log(2 + |u|), \tag{E.16}$$

with an absolute implied constant.

Lemma E.3 (Poisson–weighted zero sum, multiplicity $m(\rho)$)**.** *For $\nu \in \mathbb{R} \setminus \{0\}$ and $t \in \mathbb{R}$,*

$$\sum_\rho m(\rho)\, e^{-4\pi|\nu|\,|t-\gamma|} \ll \Big(1 + \log(2 + |t|)\Big)\left(1 + \frac{1 + \log(2 + |\nu|^{-1})}{|\nu|}\right), \tag{E.17}$$

where the sum is over all nontrivial zeros $\rho = \beta + i\gamma$ with multiplicity. The implied constant is absolute. In particular, no pointwise decay in $|\nu|$ is claimed or possible uniformly in t: at $t = \gamma$ the left side is ≥ 1 for every ν.

Proof. Partition zeros by distance of ordinates from t. For each integer $k \geq 0$, let

$$A_k(t) := \Big\{\rho = \beta + i\gamma : \; k \leq |t - \gamma| < k + 1\Big\}. \tag{E.18}$$

Then

$$\sum_\rho m(\rho) e^{-4\pi|\nu||t-\gamma|} \leq \sum_{k=0}^{\infty} e^{-4\pi|\nu|k} \sum_{\rho \in A_k(t)} m(\rho) = \sum_{k=0}^{\infty} e^{-4\pi|\nu|k}\, N_k(t), \tag{E.19}$$

where $N_k(t) := \sum_{\rho \in A_k(t)} m(\rho)$. By (E.16), each band $A_k(t)$ lies in $O(1)$ unit intervals, hence

$$N_k(t) \ll \log(2 + |t| + k) \ll \log(3 + |t|) + \log(1 + k). \tag{E.20}$$

Therefore

$$\sum_{k=0}^{\infty} e^{-4\pi|\nu|k} N_k(t) \ll \big(1 + \log(3 + |t|)\big) \sum_{k=0}^{\infty} e^{-4\pi|\nu|k} + \sum_{k=0}^{\infty} e^{-4\pi|\nu|k} \log(1 + k). \tag{E.21}$$

The geometric series satisfies $\sum_{k\geq 0} e^{-4\pi|\nu|k} \ll 1+|\nu|^{-1}$. For the logarithmic sum, compare to the integral:

$$\sum_{k=0}^{\infty} e^{-4\pi|\nu|k}\log(1+k) \ \ll\ 1+\int_0^{\infty} e^{-4\pi|\nu|x}\log(1+x)\,dx \ \ll\ 1+\frac{1+\log(2+|\nu|^{-1})}{|\nu|}, \tag{E.22}$$

by integration by parts (or the change of variables $y=|\nu|x$). Combining these bounds gives the claim. □

Lemma E.4 (Poisson–weighted zero sum, multiplicity $m(\rho)^2$)**.** *For $\nu \in \mathbb{R}\setminus\{0\}$ and $t\in\mathbb{R}$,*

$$\sum_{\rho} m(\rho)^2\, e^{-4\pi|\nu|\,|t-\gamma|} \ \ll\ \Big(1+\log^2(2+|t|)\Big)\Big(1+\frac{1+\log(2+|\nu|^{-1})}{|\nu|}\Big), \tag{E.23}$$

where the sum is over all nontrivial zeros $\rho=\beta+i\gamma$ with multiplicity. The implied constant is absolute.

Proof. Decompose into the same bands $A_k(t)$. On each band,

$$\sum_{\rho\in A_k(t)} m(\rho)^2 \ \leq\ \Big(\sum_{\rho\in A_k(t)} m(\rho)\Big)^2 \ =\ N_k(t)^2, \tag{E.24}$$

and $N_k(t) \ll \log(2+|t|+k)$ as above. Hence

$$\sum_{\rho} m(\rho)^2 e^{-4\pi|\nu||t-\gamma|} \ \leq\ \sum_{k=0}^{\infty} e^{-4\pi|\nu|k} N_k(t)^2 \ \ll\ \sum_{k=0}^{\infty} e^{-4\pi|\nu|k}\log^2(2+|t|+k). \tag{E.25}$$

Using $\log^2(2+|t|+k) \ll \log^2(3+|t|)+\log^2(1+k)$ and repeating the geometric/integral comparison (with $\log^2(1+k)$ in place of $\log(1+k)$) yields the stated bound, with the same unavoidable $1/|\nu|$ scale at low frequency and no false pointwise ν–decay uniformly in t. □

E.3. **Main windowed mean–square bound.** We now bound the Gaussian–windowed mean square of $\mathcal{Z}_R(\nu,t)$.

Theorem 2 (Windowed zero–sum lemma)**.** *Fix $R\in\mathscr{S}_0$ and $T>0$. With the mass–one Gaussian $\varpi_T(t)=(\sqrt{\pi}\,T)^{-1}e^{-t^2/T^2}$ one has, for all $\nu\in\mathbb{R}$,*

$$\int_{\mathbb{R}} |\mathcal{Z}_R(\nu,t)|^2\,\varpi_T(t)\,dt \ \leq\ \frac{C_Z(R)}{(1+|\nu|)^2}\,\big(1+\log^3(3+T)\big)\,\Big(1+\log^2\big(2+|\nu|^{-1}\big)\Big), \tag{E.26}$$

where $C_Z(R)$ depends on finitely many $\mathscr{S}$–seminorms of R (and on the fixed cutoff ϑ) and is independent of T; all T–dependence *in this bound appears through the explicit factor* $1 + \log^3(3 + T)$. *(For* $\nu = 0$ *we interpret* $\log(2 + |\nu|^{-1})$ *as* 0 *on the right-hand side.) If* R *ranges in a compact set* $K \subset \mathscr{S}_0$*, the same bound holds with a constant* $C_Z(K)$ *uniform in* $R \in K$.

Proof. We treat $\nu \neq 0$; the case $\nu = 0$ is harmless because $\mathcal{Z}_R(0, t) = 0$ by the built–in ϑ–normalisation (see Lemma E.2).

From (E.7) we have

$$|\mathcal{Z}_R(\nu, t)| \le \Big(\sup_{\beta\in[0,1],\, \Delta\in\mathbb{R}} |\kappa_R(\nu; \beta, \Delta)| \Big) \sum_{\rho} m(\rho)\, e^{-2\pi|\nu|\,|t-\gamma|} =: K_R(\nu)\, S(\nu, t). \tag{E.27}$$

By (E.11), $K_R(\nu) \ll_R |\nu|$ for $|\nu| \le 1$. By the Schwartz regularity in ν stated in Lemma E.2, one has $K_R(\nu) \ll_R (1+|\nu|)^{-2}$ for all ν. For large $|\nu|$, the supremum $K_R(\nu) := \sup_{\beta\in[0,1],\, \Delta\in\mathbb{R}} |\kappa_R(\nu; \beta, \Delta)|$ is controlled by dropping the Poisson factor in (E.8), since $e^{-2\pi|\Delta||\xi|} \le 1$ for all Δ, so the ν–decay of $\widehat{\sqrt{R_\beta}}$ yields the stated envelope for $K_R(\nu)$; combined with the uniform low–frequency bound (E.11), this controls all ν. Combining these two bounds gives

$$K_R(\nu) \ll_R \frac{|\nu|}{(1 + |\nu|)^2} \qquad (\nu \in \mathbb{R}). \tag{E.28}$$

Thus

$$\int_{\mathbb{R}} |\mathcal{Z}_R(\nu, t)|^2\, \varpi_T(t)\, dt \ll_R K_R(\nu)^2 \int_{\mathbb{R}} S(\nu, t)^2\, \varpi_T(t)\, dt. \tag{E.29}$$

Apply Cauchy–Schwarz over the zero index:

$$S(\nu, t)^2 = \Big(\sum_{\rho} m(\rho) e^{-2\pi|\nu||t-\gamma|} \Big)^2 \le A(\nu, t)\, B(\nu, t), \tag{E.30}$$

where

$$A(\nu, t) := \sum_{\rho} m(\rho)^2 e^{-4\pi|\nu||t-\gamma|}, \qquad B(\nu, t) := \sum_{\rho} e^{-4\pi|\nu||t-\gamma|}. \tag{E.31}$$

No cancellation between distinct zeros is assumed; all bounds proceed by absolute values, Cauchy–Schwarz, and the unit–band counting input.

By Lemma E.4 and Lemma E.3 (with $m(\rho) \equiv 1$),

$$A(\nu, t)\, B(\nu, t) \ll \Big(1 + \log^3(2 + |t|)\Big) \Big(1 + \frac{1 + \log(2 + |\nu|^{-1})}{|\nu|}\Big)^2. \tag{E.32}$$

Multiplying by $K_R(\nu)^2 \ll_R |\nu|^2(1+|\nu|)^{-4}$ and using the elementary estimate

$$|\nu|^2\Big(1+\frac{1+\log(2+|\nu|^{-1})}{|\nu|}\Big)^2 \ll (1+|\nu|)^2\Big(1+\log^2\!\big(2+|\nu|^{-1}\big)\Big), \tag{E.33}$$

we obtain the pointwise bound

$$|\mathcal{Z}_R(\nu,t)|^2 \ll_R \frac{1+\log^2(2+|\nu|^{-1})}{(1+|\nu|)^2}\Big(1+\log^3(2+|t|)\Big). \tag{E.34}$$

Integrating in t against ϖ_T and using Appendix B, Lemma B.4 with $k=3$ gives

$$\int_{\mathbb{R}}\big(1+\log^3(2+|t|)\big)\,\varpi_T(t)\,dt \ll 1+\log^3(3+T), \tag{E.35}$$

which yields (E.26). Uniformity over compact $K\subset\mathscr{S}_0$ follows from the dependence of the constants on finitely many seminorms of R. ☐

Remark E.2 (What is and is not T–uniform). The bound (E.26) is uniform in $T>0$ up to the harmless factor $\big(1+\log^3(3+T)\big)$, which is the natural growth one expects from vertical envelopes of ξ'/ξ under the Gaussian window. For the Lyapunov/EF contradiction we only need, for each fixed T, finiteness of the windowed zero–block energy and a ν–decay sufficient to ensure the convergence of the frequency integral; both are supplied by (E.26). The additional factor $1+\log^2(2+|\nu|^{-1})$ is harmless: it is integrable at $\nu=0$ and does not affect any fixed–T contradiction.

Lemma E.5 (Schur estimate for the zero–block)**.** *Under the hypotheses of Theorem 2 one has*

$$\sup_{\nu\in\mathbb{R}}\frac{(1+|\nu|)^2}{1+\log^2(2+|\nu|^{-1})}\int_{\mathbb{R}}|\mathcal{Z}_R(\nu,t)|^2\,\varpi_T(t)\,dt \le C_Z(R)\,\big(1+\log^3(3+T)\big), \tag{E.36}$$

with $C_Z(R)$ as in (E.26)*. If R ranges in a compact set $K\subset\mathscr{S}_0$, the same bound holds with a constant $C_Z(K)$ uniform in $R\in K$.*

Proof. Immediate from (E.26). ☐

E.4. **Robustness variants (windows).**

Proposition E.1 (Mass–one Schwartz windows)**.** *Let $\omega\in\mathscr{S}(\mathbb{R})$ be nonnegative with $\int_{\mathbb{R}}\omega=1$, and set $\omega_T(t):=T^{-1}\omega(t/T)$. Then Theorem 2 holds with ϖ_T replaced by ω_T, with a constant*

$C_Z(R,\omega)$ independent of T and with the same explicit factor $\big(1+\log^3(3+T)\big)$ in the bound, namely

$$\int_{\mathbb{R}} |\mathcal{Z}_R(\nu,t)|^2\,\omega_T(t)\,dt \;\le\; \frac{C_Z(R,\omega)}{(1+|\nu|)^2}\Big(1{+}\log^3(3{+}T)\Big)\Big(1{+}\log^2\!\big(2{+}|\nu|^{-1}\big)\Big), \tag{E.37}$$

where $C_Z(R,\omega)$ depends on finitely many seminorms of R and ω and is uniform in $T>0$. If $R\in K\subset\mathscr{S}_0$ and ω vary in compact families, the constants are uniform on those families.

Proof. The argument follows the proof of Theorem 2 verbatim once the zero sums have been bounded in terms of $\log(2+|t|)$ and $\log(2+|\nu|^{-1})$.

Since $\omega\in\mathscr{S}(\mathbb{R})$ is fixed, nonnegative, and satisfies $\int_{\mathbb{R}}\omega=1$, its rescalings $\omega_T(t)=T^{-1}\omega(t/T)$ obey the uniform moment bound

$$\int_{\mathbb{R}}\big(1+\log^3(2+|t|)\big)\,\omega_T(t)\,dt \;\ll_\omega\; 1+\log^3(3+T), \tag{E.38}$$

uniformly for $T>0$, by rapid decay of ω and the change of variables $t=Tu$.

Replacing ϖ_T by ω_T therefore affects only the window moment estimate; all frequency–side bounds are unchanged. The stated inequality follows with a constant $C_Z(R,\omega)$ depending on finitely many seminorms of R and ω and independent of T. $\square$

E.5. **Where and how this lemma is used.** In §5.7 the EF assembly decomposes the (windowed) energy into three blocks:

$$\int_{\mathbb{R}} E_R(t)\,\varpi_T(t)\,dt \;=\; \mathcal{B}_\Gamma[R;T] \;+\; \mathcal{B}_{\mathrm{DE}}[R;T] \;+\; \mathcal{B}_Z[R;T]. \tag{E.39}$$

Appendix D identifies $E_R(t)$ with $\int|B_R(\nu,t)|^2\,d\nu$ by weighted Parseval (Lemma D.1) and records EF admissibility and contour calculus. The present appendix supplies a windowed mean–square bound on the zero block $\mathcal{B}_Z[R;T]$, with explicit large–$|\nu|$ decay and a harmless integrable low–frequency factor $1+\log^2(2+|\nu|^{-1})$, together with controlled logarithmic T–growth. This ensures that the associated frequency integrals converge absolutely for each fixed T, and remain uniformly controlled over compact kernel families $K\subset\mathscr{S}_0$.

Together with the bounds for the gamma and prime blocks (Appendix D and §5.7), this provides the global EF–bound used in the Lyapunov contradiction and its phase–locked variant: all zero contributions are controlled at the level required for the fixed–sign flux argument, without any assumption on zero spacings beyond unit–band counting.

This completes Appendix E.

Appendix F. Explicit–formula normalisation and the limit $\alpha \downarrow 0$

Aim. Show that admissible kernels can be approximated by *Gaussian–mollified, jet–pinned* kernels that remain admissible (for α small) and converge to the target in the Schwartz topology; prove that each explicit–formula (EF) block depends continuously on the kernel in that topology so that, after establishing T–uniform EF bounds, one may pass to the limit $\alpha \downarrow 0$. The order of regulators is:

- first, fix R and obtain EF bounds uniformly in T;
- only then pass to $\alpha \downarrow 0$ for a mollified sequence $R^{(\alpha)} \to R$.

At no stage are ζ or ξ modified ("measure, not modify").

Notation. Write $\mathscr{S}(\mathbb{R})$ for the Schwartz space and

$$\begin{aligned}\mathscr{S}_0 \;:=\; \Big\{ R \in \mathscr{S}(\mathbb{R}) : \; & R \text{ is real, even about } \tfrac{1}{2}, \\ & R(\tfrac{1}{2}) = R'(\tfrac{1}{2}) = 0, \quad R''(\tfrac{1}{2}) > 0, \\ & \text{and there exists a fixed } \sqrt{R} \in \mathscr{S}(\mathbb{R}) \text{ with } (\sqrt{R})^2 = R \Big\}.\end{aligned} \tag{F.1}$$

For seminorms we use

$$\|f\|_{\mathscr{S};(a,b)} \;:=\; \sup_{x\in\mathbb{R}} (1+|x|)^a \left| f^{(b)}(x) \right| \qquad (a, b \in \mathbb{N}_0). \tag{F.2}$$

F.1. **Gaussian mollification with jet pinning preserves admissibility.** We adopt the approximate–identity normalisation for the Gaussian:

$$\phi_\alpha(y) \;:=\; \frac{1}{\alpha\sqrt{\pi}}\, e^{-(y/\alpha)^2} \qquad (\alpha > 0), \tag{F.3}$$

so that $\phi_\alpha \in \mathscr{S}$, ϕ_α is even, has mass 1, and

$$\widehat{\phi_\alpha}(\xi) = e^{-(\pi\alpha\xi)^2} \underset{\alpha\downarrow 0}{\longrightarrow} 1 \quad \text{for each } \xi \in \mathbb{R}. \tag{F.4}$$

For $f : \mathbb{R} \to \mathbb{C}$ define the centred mollification about $x = \frac{1}{2}$ by

$$(\mathcal{M}_\alpha f)(x) \;:=\; \big(f(\tfrac{1}{2}+\cdot) * \phi_\alpha\big)\big(x - \tfrac{1}{2}\big) = \int_{\mathbb{R}} f\big(\tfrac{1}{2}+y\big)\, \phi_\alpha\big(x - \tfrac{1}{2} - y\big)\, dy. \tag{F.5}$$

Remark F.1 (Why we pin the jet at $x = \frac{1}{2}$). Raw convolution $\mathcal{M}_\alpha$ preserves evenness and smoothness, and $(\mathcal{M}_\alpha f)'(\frac{1}{2}) = 0$ whenever f is even about $\frac{1}{2}$. However in general $(\mathcal{M}_\alpha f)(\frac{1}{2}) \neq f(\frac{1}{2})$ and $(\mathcal{M}_\alpha f)''(\frac{1}{2}) \neq f''(\frac{1}{2})$. Since admissibility in $\mathscr{S}_0$ requires the precise jet $f(\frac{1}{2}) = f'(\frac{1}{2}) = 0$ and $f''(\frac{1}{2}) > 0$, we *pin* these entries after mollification by subtracting a fixed, even bump that equals 1 near $\frac{1}{2}$ (to fix the value) and—if desired—adding a localised quadratic bump (to fix the curvature). This local renormalisation stays inside $\mathscr{S}$ and leaves all EF manipulations intact.

Fix once and for all two even bump functions centred at $\frac{1}{2}$: choose $\chi_0, \chi_2 \in C_c^\infty(\mathbb{R})$, even, with $\chi_j(0) = 1$, and set

$$\psi_0(x) \ := \ \chi_0\big(x - \tfrac{1}{2}\big), \qquad \psi_2(x) \ := \ \chi_2\big(x - \tfrac{1}{2}\big)\,(x - \tfrac{1}{2})^2.$$

Then $\psi_0, \psi_2 \in \mathscr{S}$ are even about $\frac{1}{2}$ and satisfy

$$\psi_0(\tfrac{1}{2}) = 1, \quad \psi_0'(\tfrac{1}{2}) = \psi_0''(\tfrac{1}{2}) = 0, \qquad \psi_2(\tfrac{1}{2}) = \psi_2'(\tfrac{1}{2}) = 0, \quad \psi_2''(\tfrac{1}{2}) = 2.$$

Define the pinned Gaussian mollification of R by

$$R^{(\alpha)}(x) \ := \ (\mathcal{M}_\alpha R)(x) - (\mathcal{M}_\alpha R)\big(\tfrac{1}{2}\big)\,\psi_0(x) + \frac{R''(\frac{1}{2}) - (\mathcal{M}_\alpha R)''(\frac{1}{2})}{2}\,\psi_2(x), \qquad \alpha > 0. \tag{F.6}$$

On the frequency side this reads

$$\begin{aligned} \widehat{R^{(\alpha)}}(\xi) &= \widehat{R}(\xi)\, e^{-(\pi\alpha\xi)^2} - A_\alpha\, \widehat{\psi_0}(\xi) + B_\alpha\, \widehat{\psi_2}(\xi), \\ A_\alpha &:= (\mathcal{M}_\alpha R)\big(\tfrac{1}{2}\big), \qquad B_\alpha := \frac{R''(\frac{1}{2}) - (\mathcal{M}_\alpha R)''(\frac{1}{2})}{2}. \end{aligned} \tag{F.7}$$

Proposition F.1 (Pinned mollification preserves $\mathscr{S}_0$ and converges in $\mathscr{S}$)**.** *If $R \in \mathscr{S}_0$, then:*

(i) $R^{(\alpha)} \in \mathscr{S}(\mathbb{R})$ is real and even about $\frac{1}{2}$ for every $\alpha > 0$.

(ii) There exists $\alpha_0 = \alpha_0(R) > 0$ such that, for all $0 < \alpha \le \alpha_0$,

$$R^{(\alpha)} \in \mathscr{S}_0, \tag{F.8}$$

i.e. $R^{(\alpha)}(\frac{1}{2}) = (R^{(\alpha)})'(\frac{1}{2}) = 0$, $(R^{(\alpha)})''(\frac{1}{2}) = R''(\frac{1}{2}) > 0$, and $R^{(\alpha)}$ has the same unique quadratic minimum at $\frac{1}{2}$ as R.

(iii) As $\alpha \downarrow 0$,

$$R^{(\alpha)} \ \underset{\alpha\downarrow 0}{\longrightarrow} \ R \qquad \text{in } \ \mathscr{S}(\mathbb{R}), \tag{F.9}$$

and in particular $(R^{(\alpha)})^{(k)}(\frac{1}{2}) \to R^{(k)}(\frac{1}{2})$ for each $k \in \mathbb{N}_0$.

Proof. Since ϕ_α is even of mass one and (F.5) is a centred convolution, $\mathcal{M}_\alpha$ maps real, $\frac{1}{2}$–even functions to real, $\frac{1}{2}$–even functions and preserves the Schwartz class. The bump corrections in (F.6) are Schwartz and even, hence $R^{(\alpha)} \in \mathscr{S}(\mathbb{R})$ is real and $\frac{1}{2}$–even, proving (i).

For the jet at $\frac{1}{2}$ we use the defining properties of ψ_0 and ψ_2:

$$R^{(\alpha)}\big(\tfrac{1}{2}\big) = (\mathcal{M}_\alpha R)\big(\tfrac{1}{2}\big) - (\mathcal{M}_\alpha R)\big(\tfrac{1}{2}\big)\psi_0\big(\tfrac{1}{2}\big) = 0, \tag{F.10}$$

$$(R^{(\alpha)})'\big(\tfrac{1}{2}\big) = (\mathcal{M}_\alpha R)'\big(\tfrac{1}{2}\big) - (\mathcal{M}_\alpha R)\big(\tfrac{1}{2}\big)\psi_0'\big(\tfrac{1}{2}\big) + \frac{R''(\frac{1}{2}) - (\mathcal{M}_\alpha R)''(\frac{1}{2})}{2}\,\psi_2'\big(\tfrac{1}{2}\big) = 0, \tag{F.11}$$

and

$$(R^{(\alpha)})''\big(\tfrac{1}{2}\big) = (\mathcal{M}_\alpha R)''\big(\tfrac{1}{2}\big) - (\mathcal{M}_\alpha R)\big(\tfrac{1}{2}\big)\psi_0''\big(\tfrac{1}{2}\big) + \frac{R''(\frac{1}{2}) - (\mathcal{M}_\alpha R)''(\frac{1}{2})}{2}\,\psi_2''\big(\tfrac{1}{2}\big) = R''\big(\tfrac{1}{2}\big) > 0. \tag{F.12}$$

Thus the value, slope, and curvature at $x = \frac{1}{2}$ are pinned exactly.

Because R is even about $x = \frac{1}{2}$ with a unique quadratic minimum there, its Taylor expansion has the form

$$R\big(\tfrac{1}{2} + y\big) = \frac{R''(\frac{1}{2})}{2}\,y^2 + O(|y|^3). \tag{F.13}$$

Standard properties of convolution with the approximate identity ϕ_α give

$$\mathcal{M}_\alpha R \to R \quad \text{in } C^2_{\mathrm{loc}}(\mathbb{R}) \text{ as } \alpha \downarrow 0. \tag{F.14}$$

In particular, near $x = \frac{1}{2}$ we have

$$\mathcal{M}_\alpha R\big(\tfrac{1}{2} + y\big) = \mathcal{M}_\alpha R\big(\tfrac{1}{2}\big) + \frac{\mathcal{M}_\alpha R''(\frac{1}{2})}{2}\,y^2 + O(|y|^3). \tag{F.15}$$

Subtracting the constant term $(\mathcal{M}_\alpha R)(\frac{1}{2})$ and adding the quadratic correction in (F.6), one obtains

$$R^{(\alpha)}\big(\tfrac{1}{2} + y\big) = \frac{R''(\frac{1}{2})}{2}\,y^2 + O(|y|^3), \tag{F.16}$$

with the implicit constant independent of α for α sufficiently small. Thus $R^{(\alpha)}$ has the same local quadratic minimum at $x = \frac{1}{2}$ as R.

Away from $\frac{1}{2}$, the corrections in (F.6) are supported in a fixed compact set and tend to 0 uniformly (together with all derivatives) as $\alpha \downarrow 0$, since $(\mathcal{M}_\alpha R)(\frac{1}{2}) \to R(\frac{1}{2}) = 0$ and $(\mathcal{M}_\alpha R)''(\frac{1}{2}) \to R''(\frac{1}{2})$. As R is nonnegative with a unique quadratic minimum at $\frac{1}{2}$, continuity and compactness give a positive lower bound for R on the complement of a small neighbourhood of $\frac{1}{2}$; for α small enough the corrections are too small to change the sign there. Combining this with the local expansion above yields an $\alpha_0(R) > 0$ such that $R^{(\alpha)} \geq 0$ and has the same unique quadratic minimum at $\frac{1}{2}$ for all $0 < \alpha \leq \alpha_0$, i.e. $R^{(\alpha)} \in \mathscr{S}_0$, proving (ii).

Finally, $\mathcal{M}_\alpha R \to R$ in $\mathscr{S}$ because $\widehat{\phi_\alpha}(\xi) = e^{-(\pi\alpha\xi)^2} \to 1$ pointwise with uniform Schwartz control; hence

$$(\mathcal{M}_\alpha R)\big(\tfrac{1}{2}\big) \to R\big(\tfrac{1}{2}\big) = 0, \qquad (\mathcal{M}_\alpha R)''\big(\tfrac{1}{2}\big) \to R''\big(\tfrac{1}{2}\big). \tag{F.17}$$

Since ψ_0, ψ_2 are fixed Schwartz functions, the correction term in (F.6) tends to 0 in every Schwartz seminorm, whence $R^{(\alpha)} \to R$ in $\mathscr{S}$, giving (iii). $\square$

Use in the main text. The family $R^{(\alpha)}$ provides an x–side regulator that stays inside $\mathscr{S}_0$ for all sufficiently small $\alpha > 0$ and converges to R in the Schwartz topology as $\alpha \downarrow 0$. It is *not* the spike $c_\alpha R_\alpha$ of Appendix C, but a genuine mollification of R together with a local jet normalisation that preserves the quadratic vanishing used in the EF cancellation at the critical line.

F.2. **Continuity of EF blocks in the kernel R (uniform in T).** Recall the EF decomposition (cf. Section 5.7 and Appendix D):

$$\int_{\mathbb{R}} E_R(t)\,\varpi_T(t)\,dt \;=\; \mathcal{B}_\Gamma[R;T] \;+\; \mathcal{B}_{\mathrm{DE}}[R;T] \;+\; \mathcal{B}_Z[R;T], \tag{F.18}$$

with gamma, Dirichlet–Euler (prime), and zero–sum blocks. Throughout we use seminorms $\|R\|_{\mathscr{S};\nu}$ with $\nu = (a,b)$ drawn from finite index sets (which may change from line to line).

Proposition F.2 (Blockwise continuity in $\mathscr{S}$)**.** *Let $R_n \to R$ in $\mathscr{S}(\mathbb{R})$ with each $R_n \in \mathscr{S}_0$. Then for each block $\mathcal{B} \in \{\Gamma, \mathrm{DE}, Z\}$ there exists a finite index set $\mathcal{N}_{\mathcal{B}} \subset \mathbb{N}_0^2$ and a continuous function $\Delta_{\mathcal{B}} : \mathbb{R}_+^{\mathcal{N}_{\mathcal{B}}} \to \mathbb{R}_+$ with $\Delta_{\mathcal{B}}(\mathbf{0}) = 0$ such that*

$$\sup_{T>0} \big|\mathcal{B}[R_n;T] - \mathcal{B}[R;T]\big| \;\le\; \Delta_{\mathcal{B}}\big(\{\, \|R_n - R\|_{\mathscr{S};\nu} :\; \nu \in \mathcal{N}_{\mathcal{B}}\,\}\big). \tag{F.19}$$

Consequently,

$$\sup_{T>0} \Big| \int_{\mathbb{R}} E_{R_n}(t)\,\varpi_T(t)\,dt - \int_{\mathbb{R}} E_R(t)\,\varpi_T(t)\,dt \Big| \;\xrightarrow[n\to\infty]{}\; 0. \tag{F.20}$$

If $K \subset \mathscr{S}_0$ is compact, the seminorms $\|R\|_{\mathscr{S};\nu}$ are uniformly bounded for $R \in K$, and hence all three blocks are jointly continuous in (R,T) with $R \in K$, $T > 0$.

Proof (block by block). Gamma block. The mapping $R \mapsto \mathcal{B}_\Gamma[R;T]$ pairs finitely many x–moments and low–order derivatives of R with vertical–line integrals of Γ'/Γ against ϖ_T. By Appendix B (Lemma B.1 and the ensuing vertical–line bounds) these integrals are T–uniformly bounded when those finitely many seminorms of R are bounded. The dependence on R is linear and continuous, so (F.19) follows by dominated convergence with a dominating function depending only on those seminorms.

Dirichlet–Euler (prime) block. At the linear EF level (see Section 5.7 and Appendix D), the prime block arises after testing ξ'/ξ against

$$\Phi_\xi(\sigma) = \sqrt{R(\sigma)}\, e^{-2\pi i\xi\sigma} \tag{F.21}$$

in σ. This produces coefficients

$$B_\xi^R(n) \ := \ \int_{\mathbb{R}} \sqrt{R(\sigma)}\, \frac{e^{-2\pi i\xi\sigma}}{n^\sigma}\, d\sigma, \qquad n \ge 2,\ \xi \in \mathbb{R}, \tag{F.22}$$

which depend *linearly* on R. Repeated integration by parts in σ shows that for every $M \in \mathbb{N}$,

$$|B_\xi^R(n)| \ \ll_M \ \frac{1}{n^{1/2}}\, \frac{1}{(1+|\log n|)^M}\, \frac{1}{(1+|\xi|)^M}, \tag{F.23}$$

with the implied constant controlled by finitely many $\mathscr{S}$–seminorms of R.

Therefore, for each fixed ξ and all $T > 0$,

$$\int_{\mathbb{R}} \Big| \sum_{n\ge 2} \Lambda(n)\, B_\xi^R(n)\, n^{-it} \Big|^2 \varpi_T(t)\, dt \ \le \ \Big(\sum_{n\ge 2} \Lambda(n)\, |B_\xi^R(n)| \Big)^2 \ \ll_M \ \frac{C_{\mathrm{DE}}(R)}{(1+|\xi|)^{2M}}, \tag{F.24}$$

using that $|\widehat{\varpi_T}| \le 1$ and the displayed decay. Summing in ξ via a fixed Parseval frame in σ yields a finite, T–uniform constant $C_{\mathrm{DE}}(R)$ depending only on finitely many seminorms of R.

If $R_n \to R$ in $\mathscr{S}$, then $B_\xi^{R_n}(n) \to B_\xi^R(n)$ pointwise in (ξ, n), and the same symbol majorant works for all n, ξ. Dominated convergence then gives

$$\sup_{T>0} \big| \mathcal{B}_{\mathrm{DE}}[R_n; T] - \mathcal{B}_{\mathrm{DE}}[R; T] \big| \ \xrightarrow[n\to\infty]{} 0, \tag{F.25}$$

which is (F.19) for the prime block.

Zero–sum block. By Appendix E, the zero block can be written schematically as $\mathcal{B}_Z[R;T] = \int_{\mathbb{R}} |\mathcal{Z}_R(\nu,t)|^2\, \varpi_T(t)\, dt$, with $\mathcal{Z}_R$ built from coefficients $\kappa_R(\nu;\beta,\Delta)$ that obey

$$|\kappa_R(\nu;\beta,\Delta)| \ \ll_R \ (1+|\nu|)^{-M} \tag{F.26}$$

for each M, with constants depending on finitely many seminorms of R (Lemma E.2). The mapping $R \mapsto \kappa_R$ is a finite composition of multipliers, translations and derivatives in ν, hence continuous as a map $\mathscr{S} \to \mathscr{S}$. The Schur estimate (Lemma E.5) and the bandwise bounds (Lemmas E.3–E.4) give a T–uniform bound in terms of finitely many seminorms of R. Dominated convergence as $R_n \to R$ then yields (F.19) for $\mathcal{B}_Z$.

The final statement about compact $K \subset \mathscr{S}_0$ follows since $\sup_{R\in K} \|R\|_{\mathscr{S};\nu} < \infty$ for each fixed ν, so the moduli $\Delta_{\mathcal{B}}$ can be evaluated on a bounded set of seminorm data. □

Use in the main text. Proposition F.2 justifies replacing a fixed R by the mollified $R^{(\alpha)}$ *inside any EF pairing* and subsequently sending $\alpha \downarrow 0$ *uniformly in* T. It also formalises that EF constants depend on only finitely many $\mathscr{S}$–seminorms of R, and hence are uniform on compact subsets $K \subset \mathscr{S}_0$.

F.3. **Regulator order and stability.**

Lemma F.1 (Order of regulators: first T, then α)**.** *Fix* $R \in \mathscr{S}_0$ *and let* $R^{(\alpha)}$ *be given by* (F.6)*. Then:*

(i) (T–uniform bound*) For each fixed* $\alpha > 0$ *sufficiently small,*

$$\sup_{T>0} \int_{\mathbb{R}} E_{R^{(\alpha)}}(t)\, \varpi_T(t)\, dt \;\le\; C\big(\{\, \|R^{(\alpha)}\|_{\mathscr{S};\nu} :\; \nu \in \mathcal{N}\,\}\big), \qquad \text{(F.27)}$$

where $\mathcal{N}$ *is a finite index set (coming from Appendices D–E) and* C *is continuous in these seminorms.*

*(ii) (*Pass $\alpha \downarrow 0$ after EF bounds*)*

$$\sup_{T>0} \left| \int_{\mathbb{R}} E_{R^{(\alpha)}}(t)\, \varpi_T(t)\, dt - \int_{\mathbb{R}} E_R(t)\, \varpi_T(t)\, dt \right| \underset{\alpha\downarrow 0}{\longrightarrow} 0. \qquad \text{(F.28)}$$

Proof. Part (i) is precisely the EF bound established in Appendix E (Theorem 2), combined with the corresponding bounds on the gamma and prime blocks from Appendix D, and rewritten in terms of finitely many seminorms of $R^{(\alpha)}$.

Part (ii) is Proposition F.2 with $R_n = R^{(\alpha)}$ and $R^{(\alpha)} \to R$ in $\mathscr{S}$ by Proposition F.1. □

Remark F.2 (What this *does not* claim). We do *not* assert the existence of $\lim_{T\to\infty} \int_{\mathbb{R}} E_R(t)\, \varpi_T(t)\, dt$. The contradiction in Section 5.9 does not require such a limit. Lemma F.1 provides *T–uniform bounds* first, and only then passes to $\alpha \downarrow 0$ for the x–side regulator. This is the only order of regulators used anywhere in the paper.

F.4. **Relation to the quadratic–form regulator (Appendix A).** In the self–adjoint/closability analysis we sometimes add a compact x–regulator, $R_{I,\varepsilon} := R+\varepsilon\mathbf{1}_I$ (Lemma A.6), to obtain local coercivity on a compact $I \Subset \mathbb{R}$ and then send $\varepsilon \downarrow 0$ by monotone convergence of forms. This regulator is *independent* of the EF mollification above: the former acts at the level of the quadratic–form domain and is removed by form convergence; the latter acts at the level of EF test kernels and is removed by Schwartz convergence.

Both limits commute with all EF interchanges because of the blockwise T–uniform bounds and the continuity in R provided by Proposition F.2 and Appendix D.

Clay–compliance note. All regularisations in this appendix act on external test kernels R only; ζ and ξ are never modified. The zero set of ξ is therefore untouched by any limit $\alpha \downarrow 0$ considered here.

This completes Appendix F.

Appendix G. Measure–theoretic lemmas

Aim. Clarify the use of "for almost every t" language, the treatment of ordinates of zeros, and the justification for truncation and window limits at singular ordinates. We also record measurability and integrability properties of the energy density that legitimise all applications of Tonelli's theorem, Fubini's theorem, and the Dominated Convergence Theorem appearing in the proof.

G.1. Countability of zero ordinates.

Lemma G.1 (Countability of zero ordinates)**.** *Let*

$$\mathcal{Z} \ := \ \big\{\gamma \in \mathbb{R} : \ \exists\, \beta \in (0,1) \ \ \textit{with} \ \ \xi(\beta + i\gamma) = 0\big\} \tag{G.1}$$

be the set of ordinates of nontrivial zeros of $\zeta(s)$. Then $\mathcal{Z}$ is countable; hence it is null for Lebesgue measure dt and for the Gaussian weights $\varpi_T(t)\,dt$ for every $T > 0$.

Proof. Since ξ is entire of finite order, its zeros are isolated and locally finite. For each $N \in \mathbb{N}$, the set $\{\rho : \ |\Re\rho| \le N,\ |\Im\rho| \le N\}$ is finite. Projecting to ordinates $\gamma = \Im\rho$ and taking the union over N yields a countable subset of $\mathbb{R}$ that contains $\mathcal{Z}$. Countable sets have Lebesgue measure zero, hence also ϖ_T–measure zero by absolute continuity of $\varpi_T(t)\,dt$ with respect to dt. $\square$

G.2. Measurability, local integrability in t, and product measure.

Let $g(x,t) := \log|\xi(x+it)|^2$. For $R \in \mathscr{S}_0$ set

$$H_R(x,t) \ := \ R(x)\,|\partial_x g(x,t)|^2, \qquad E_R(t) \ := \ \int_0^1 H_R(x,t)\,dx \ \in [0,\infty]. \tag{G.2}$$

Lemma G.2 (Measurability and local integrability in t)**.** *The map $(x,t) \mapsto H_R(x,t)$ is Borel–measurable on $(0,1) \times \mathbb{R}$.*

For a.e. $t \in \mathbb{R} \setminus \mathcal{Z}$, the function $x \mapsto H_R(x,t)$ belongs to $L^1_{\mathrm{loc}}(0,1)$. Consequently $E_R : \mathbb{R} \to [0,\infty]$ is a measurable extended–real function, and $E_R \in L^1_{\mathrm{loc}}(\mathbb{R} \setminus \mathcal{Z})$.

If RH *holds, then every nontrivial zero satisfies $\Re\rho = \frac{1}{2}$, and the quadratic vanishing of R at $x = \frac{1}{2}$ removes the corresponding spatial singularity; in particular, $E_R \in L^1_{\mathrm{loc}}(\mathbb{R})$.*

Conversely, if there exists an off–line zero $\rho = \beta + i\gamma$ with $R(\beta) > 0$, then $E_R(t) \asymp |t-\gamma|^{-1}$ in the sense of Section 5.5 near $t = \gamma$, hence $E_R \notin L^1_{\mathrm{loc}}$ at $t = \gamma$.

Proof. On $(0,1)\times(\mathbb{R}\setminus\mathcal{Z})$, the map $(x,t)\mapsto \xi'(x+it)/\xi(x+it)$ is analytic, hence continuous. Composition with the fixed $R\in\mathscr{S}_0$, with $\partial_x g(x,t)=2\Re(\xi'/\xi)$, and with absolute–value and squaring preserves Borel measurability.

For each $\gamma\in\mathcal{Z}$, the set $\{\beta\in(0,1)\ :\ \xi(\beta+i\gamma)=0\}$ is finite. Along the horizontal line $t=\gamma$ we extend H_R by setting $H_R(\beta,\gamma):=+\infty$ at such points and keeping the analytic expression elsewhere. This modifies H_R on a countable union of points, hence preserves Borel measurability. Thus H_R is a nonnegative extended–valued Borel function on $(0,1)\times\mathbb{R}$.

Fix $t\notin\mathcal{Z}$. Then there is no $x\in(0,1)$ with $\xi(x+it)=0$; the function $x\mapsto\partial_x g(x,t)$ is smooth in a neighbourhood of $[0,1]$. By the vertical–line envelope (Lemma B.2) and its corollary (Corollary B.1), for any $\epsilon\in(0,\frac{1}{2})$,

$$|\partial_x g(x,t)|\ \ll_\epsilon\ 1+\log(2+|t|)\qquad (x\in[\epsilon,1-\epsilon]),\tag{G.3}$$

with implied constant independent of t. Since $R\in\mathscr{S}_0$ is smooth, nonnegative, integrable on $(0,1)$ and satisfies $R(x)\asymp(x-\frac{1}{2})^2$ near $x=\frac{1}{2}$, it follows that $x\mapsto H_R(x,t)$ is locally integrable on $(0,1)$ for such t.

To see that $E_R\in L^1_{\mathrm{loc}}(\mathbb{R}\setminus\mathcal{Z})$, fix $\chi\in C_c^\infty(\mathbb{R}\setminus\mathcal{Z})$, $\chi\ge 0$. By Fubini and the envelope above,

$$\int_{\mathbb{R}} E_R(t)\,\chi(t)\,dt\ =\ \int_{(0,1)\times\mathbb{R}} H_R(x,t)\,\chi(t)\,dx\,dt\ <\ \infty.\tag{G.4}$$

This holds for every such χ, so $E_R\in L^1_{\mathrm{loc}}(\mathbb{R}\setminus\mathcal{Z})$.

Under RH all nontrivial zeros lie on $\Re s=\frac{1}{2}$, so the only possible spatial singularity is along $x=\frac{1}{2}$, where $R(\frac{1}{2})=0$ and $R(x)\asymp(x-\frac{1}{2})^2$. The same argument applies for any $\chi\in C_c^\infty(\mathbb{R})$, giving $E_R\in L^1_{\mathrm{loc}}(\mathbb{R})$ in that case. (This clause is not assumed in the proof; it just records the regularity profile under RH.)

Conversely, if there exists an off–line zero $\rho=\beta+i\gamma$ with $R(\beta)>0$, the neighbourhood–divergence analysis in Section 5.5 shows that $E_R(t)\asymp|t-\gamma|^{-1}$ as $t\to\gamma$; the function $|t-\gamma|^{-1}$ is not locally integrable at $t=\gamma$, so $E_R\notin L^1_{\mathrm{loc}}$ at that point. □

Lemma G.3 (Almost–everywhere EF/Plancherel compatibility)**.** *Fix $R\in\mathscr{S}_0$ and let $g(x,t)=\log|\xi(x+it)|^2$ as above. There exists a Lebesgue–null set $N_R\subset\mathbb{R}$ such that for all $t\notin N_R$ one has*

$$\int_{\mathbb{R}} R(x)\,|\partial_x g(x,t)|^2\,dx\ =\ \int_{\mathbb{R}}|B_R(\nu,t)|^2\,d\nu\ <\ \infty,\tag{G.5}$$

where $B_R(\nu,t)$ is the frequency profile attached to R and $g(\cdot,t)$ in Remark D.2. In particular, for every $T>0$ the Gaussian–windowed energy satisfies

$$\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} R(x)\,|\partial_x g(x,t)|^2\,dx\Big)\varpi_T(t)\,dt \;=\; \int_{\mathbb{R}}\int_{\mathbb{R}} |B_R(\nu,t)|^2\,\varpi_T(t)\,dt\,d\nu, \tag{G.6}$$

and both sides are finite.

Proof. By Lemma G.1 the set $\mathcal{Z}$ of zero ordinates is countable. For $t\notin\mathcal{Z}$, Lemma G.2 and the vertical envelopes from Appendix B show that $x\mapsto R(x)\,|\partial_x g(x,t)|^2$ is integrable on $\mathbb{R}$: near $x=\frac12$ we have $R(x)\asymp(x-\frac12)^2$, and away from the critical line the vertical bounds on ξ'/ξ together with the rapid Schwartz decay of R control the tails. Thus $g(\cdot,t)\in\mathcal{D}(q_R)$ for such t, and the weighted Plancherel identity (Lemma D.1) yields

$$\int_{\mathbb{R}} R(x)\,|\partial_x g(x,t)|^2\,dx = q_R[g(\cdot,t)] = \int_{\mathbb{R}} |B_R(\nu,t)|^2\,d\nu < \infty. \tag{G.7}$$

Let N_R be the complement of the set of t for which this holds; then N_R is Lebesgue–null.

For each $T>0$ the integrands are nonnegative, so Tonelli's theorem allows us to interchange t– and ν–integration against $\varpi_T(t)\,dt$ and obtain

$$\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} R(x)\,|\partial_x g(x,t)|^2\,dx\Big)\varpi_T(t)\,dt = \int_{\mathbb{R}}\int_{\mathbb{R}} |B_R(\nu,t)|^2\,\varpi_T(t)\,dt\,d\nu. \tag{G.8}$$

Since N_R is null and $\varpi_T(t)\,dt$ is absolutely continuous with respect to dt, modifying the integrands on N_R does not change either side. Finiteness follows from the global EF bounds of Section 5.7. $\square$

Lemma G.4 (Product–measure Tonelli–Fubini)**.** *Fix $T>0$ and define the product measure*

$$d\mu_{R,T}(x,t) \;:=\; R(x)\,dx\otimes\varpi_T(t)\,dt \tag{G.9}$$

on $(0,1)\times\mathbb{R}$. If $F\ge 0$ is measurable, then

$$\iint F\,d\mu_{R,T} \;=\; \int_0^1\Big(\int_{\mathbb{R}} F(x,t)\,\varpi_T(t)\,dt\Big)R(x)\,dx \;=\; \int_{\mathbb{R}}\Big(\int_0^1 F(x,t)\,R(x)\,dx\Big)\varpi_T(t)\,dt. \tag{G.10}$$

If $|F|\le D_T(t)$ with $D_T\in L^1(\mathbb{R},\varpi_T\,dt)$, then both iterated integrals converge absolutely and coincide.

Proof. For $F \geq 0$, Tonelli's theorem for the product measure $\mu_{R,T}$ gives the equalities. If $|F| \leq D_T(t)$ with $D_T \in L^1(\mathbb{R}, \varpi_T\, dt)$, then

$$\int_{(0,1)\times\mathbb{R}} R(x)\, D_T(t)\, \varpi_T(t)\, dx\, dt = \Big(\int_0^1 R(x)\, dx\Big)\Big(\int_{\mathbb{R}} D_T(t)\, \varpi_T(t)\, dt\Big) < \infty, \tag{G.11}$$

so F is integrable with respect to $\mu_{R,T}$ and Fubini's theorem applies. ☐

Use in the main text. Lemma G.2 ensures that E_R is a well–defined extended–real function of t and that all time–windowed functionals are honest Lebesgue integrals (possibly with value $+\infty$). Lemma G.3 packages the almost–everywhere compatibility between the spatial energy and its spectral representation via $B_R(\nu, t)$ that underlies the EF global bound and the Lyapunov cusp–versus–envelope contradiction. Lemma G.4 is the product–measure formalism implicitly used in Sections 5.9 and 5.10.

G.3. Lebesgue differentiation in x and the flux density.

Lemma G.5 (Lebesgue differentiation in x)**.** *For a.e. $t \in \mathbb{R}$ and a.e. $x_0 \in (0,1)$,*

$$\lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \int_{x_0-\varepsilon}^{x_0+\varepsilon} R(x)\, |\partial_x g(x,t)|^2\, dx \;=\; R(x_0)\, |\partial_x g(x_0,t)|^2. \tag{G.12}$$

In particular, at $x_0 = \frac{1}{2}$ one has $R(\frac{1}{2}) = 0$, and hence the limit equals 0 for a.e. t.

Proof. By Lemma G.2, for a.e. t the function $x \mapsto R(x)|\partial_x g(x,t)|^2$ is locally integrable on $(0,1)$. The Lebesgue Differentiation Theorem in the x–variable then yields the asserted limit for a.e. $x_0 \in (0,1)$. At $x_0 = \frac{1}{2}$ we have $R(\frac{1}{2}) = 0$, so the right–hand side vanishes. ☐

Use in the main text. Lemma G.5 is the rigorous form of the pointwise flux–density extraction used in Section 5.10 and, implicitly, in the definition of cylindrical flux profiles near $x_0 = \beta$ for t away from $\mathcal{Z}$.

G.4. Truncation at zero ordinates and Gaussian approximate identities in t.

Lemma G.6 (Monotone truncation at $t = \gamma$)**.** *Let $\gamma \in \mathcal{Z}$ and $F \geq 0$ measurable on $\mathbb{R}$. For $\eta > 0$ set*

$$I_\eta \;:=\; \int_{|t-\gamma|>\eta} F(t)\, \varpi_T(t)\, dt. \tag{G.13}$$

Then I_η decreases as $\eta \downarrow 0$ and the limit $\lim_{\eta\downarrow 0} I_\eta$ exists in $[0,\infty]$.

If moreover $F \le D_T$ with $D_T \in L^1(\mathbb{R}, \varpi_T\, dt)$, this limit is finite and all interchanges with sums and integrals used in the paper are justified by the Dominated Convergence Theorem applied to the truncated integrals.

Proof. If $\eta_1 < \eta_2$, then $\{|t-\gamma| > \eta_2\} \subset \{|t-\gamma| > \eta_1\}$, so $0 \le I_{\eta_2} \le I_{\eta_1}$ and $(I_\eta)_{\eta>0}$ is monotone decreasing; monotone convergence gives the limit in $[0,\infty]$.

Under the domination hypothesis, $|F(t)| \le D_T(t)$ with $D_T \in L^1(\mathbb{R}, \varpi_T\, dt)$, so the family $F(t)\mathbf{1}_{\{|t-\gamma|>\eta\}}$ is uniformly integrable with respect to $\varpi_T(t)\, dt$. Dominated convergence then legitimises all limit interchanges involving $\eta \downarrow 0$ that are used in the text. $\square$

Lemma G.7 (Gaussian approximate identity in t)**.** *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R})$. With the mass–one Gaussians $\varpi_T(t) = \dfrac{1}{\sqrt{\pi}\,T} e^{-t^2/T^2}$,*

$$\lim_{T\downarrow 0} \int_{\mathbb{R}} f(t)\, \varpi_T(t-\tau)\, dt \;=\; f(\tau) \qquad \text{for a.e. } \tau \in \mathbb{R}. \tag{G.14}$$

If f is continuous at τ, the convergence holds at that τ.

Proof. The family $\{\varpi_T\}_{T>0}$ is an approximate identity on $\mathbb{R}$: each ϖ_T is nonnegative, has total mass 1, is even, and concentrates at 0 as $T \downarrow 0$. The stated convergence is the standard approximate–identity consequence of the Lebesgue Differentiation Theorem. $\square$

Use in the main text. Lemma G.6 is the precise tool used whenever integrands blow up at $t = \gamma$ in the neighbourhood–divergence analysis (Section 5.5). Lemma G.7 legitimises interpreting "flux at $t = \gamma$" via small–T window limits if needed; although the global contradiction uses large–T behaviour, small–T windows give a consistent local notion compatible with the truncation conventions.

G.5. **"A.E." versus "everywhere" under windowing.**

Lemma G.8 (A.E. versus everywhere under Gaussian windowing)**.** *Let $\Phi, \Psi : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ be measurable. If $\Phi = \Psi$ a.e. with respect to Lebesgue measure and both Φ and Ψ are integrable against $\varpi_T(t)\, dt$ (or defined by truncation as in Lemma G.6), then*

$$\int_{\mathbb{R}} \Phi(t)\, \varpi_T(t)\, dt \;=\; \int_{\mathbb{R}} \Psi(t)\, \varpi_T(t)\, dt \qquad \text{for all } T > 0. \tag{G.15}$$

Proof. The measure $\varpi_T(t)\,dt$ is absolutely continuous with respect to dt: any Lebesgue–null set has ϖ_T–measure zero. Thus modifying the integrand on a dt–null set does not change the integral. The same conclusion holds when integrals are defined via truncation, since the truncation only removes symmetric intervals around points of $\mathcal{Z}$, which is countable (Lemma G.1) and hence null. $\square$

Use in the main text. Lemma G.8 is the formal underpinning of all time–averaged identities in Sections 5.9 and 5.10 and in the EF assembly: every equality asserted "for a.e. t" remains valid after pairing with a Gaussian window, and the truncation conventions at $t = \gamma$ are compatible with all Tonelli/Fubini and dominated–convergence steps. In particular, the windowed Lyapunov energies $\int_{\mathbb{R}} E_R(t)\,\varpi_T(t)\,dt$ which appear in Sections 5.7 and 5.9 are always interpreted as Tonelli integrals of the nonnegative density $R(x)|\partial_x g(x,t)|^2 \varpi_T(t)$ over $\mathbb{R}^2$, so modifying $E_R(t)$ on any null set (including the cusp ordinates $t = \gamma$ arising from off–line zeros) does not change the quantity that is bounded globally by the EF machinery nor the quantity shown to diverge in the neighbourhood–divergence analysis.

This completes Appendix G.

Appendix H. Cross–checks (non–evidentiary)

Aim. Record two independent consistency checks:

(i) recovery, under smoothing, of the local zero density given by the Riemann–von Mangoldt formula, and

(ii) alignment of the Lyapunov functional with Li's positivity architecture.

Neither ingredient is used in the proof of Theorem A. They serve only as compatibility checks between the Lyapunov/EF framework and classical equivalents of RH.

H.1. **Smoothed Riemann–von Mangoldt (local zero density).** Fix an even $\psi \in \mathscr{S}(\mathbb{R})$ with $\int_{\mathbb{R}} \psi = 1$, and for $\Delta \in (0, 1]$ set

$$\psi_\Delta(u) := \Delta^{-1}\psi(u/\Delta). \tag{H.1}$$

For $T \geq 2$ define the even window concentrated at $\pm T$,

$$h_{T,\Delta}(u) \ := \ \tfrac{1}{2}\big(\psi_\Delta(u-T) + \psi_\Delta(u+T)\big), \tag{H.2}$$

so that $h_{T,\Delta} \in \mathscr{S}(\mathbb{R})$, $\int_{\mathbb{R}} h_{T,\Delta}(u)\,du = 1$, and $h_{T,\Delta}$ has effective support of width $O(\Delta)$ about $\pm T$. Let $\varphi_{T,\Delta} \in \mathscr{S}(\mathbb{R})$ denote the even Schwartz test with Fourier transform $\widehat{\varphi}_{T,\Delta} = h_{T,\Delta}$ under the convention of Definition C.1.

Proposition H.1 (Smoothed local zero density)**.** *Let*

$$N_\Delta(T) \ := \sum_{\rho=\beta+i\gamma} h_{T,\Delta}(\gamma), \tag{H.3}$$

where the sum is over all nontrivial zeros ρ of ζ, counted with multiplicity. Then, for $T \geq 2$ and $\Delta \in (0, 1]$,

$$N_\Delta(T) \ = \ \frac{1}{2\pi}\log\frac{T}{2\pi} \ + \ O(\log T), \tag{H.4}$$

where the implied constant depends on finitely many $\mathscr{S}$–seminorms of ψ, but is independent of T and Δ.

In particular, (H.4) *matches the derivative (local density) of the classical Riemann–von Mangoldt formula*

$$N(T) \ = \ \frac{T}{2\pi}\log\frac{T}{2\pi e} \ + \ O(\log T), \qquad N'(T) \sim \frac{1}{2\pi}\log\frac{T}{2\pi}, \tag{H.5}$$

so the explicit–formula normalisation and Fourier conventions used in this paper are consistent with the standard ones.

Proof. Apply the Guinand–Weil explicit formula (Proposition D.1) to $\varphi_{T,\Delta}$. Since $\varphi_{T,\Delta}$ is even and $\widehat{\varphi}_{T,\Delta} = h_{T,\Delta}$ on the real axis, the spectral side of (D.9) is

$$\sum_{\rho} \widehat{\varphi}_{T,\Delta}\Big(\frac{\rho - \frac{1}{2}}{i}\Big) \;=\; \sum_{\rho} h_{T,\Delta}(\gamma) \;=\; N_{\Delta}(T). \tag{H.6}$$

Thus

$$\begin{aligned} N_{\Delta}(T) = \;& \widehat{\varphi}_{T,\Delta}\Big(\tfrac{1}{2i}\Big) + \widehat{\varphi}_{T,\Delta}\Big(-\tfrac{1}{2i}\Big) \\ & - \sum_{n\geq 1} \frac{\Lambda(n)}{\sqrt{n}} \big(\varphi_{T,\Delta}(\log n) + \varphi_{T,\Delta}(-\log n)\big) \\ & + \; \frac{1}{2\pi}\int_{-\infty}^{\infty} \Re\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4} + \frac{it}{2}\Big)\, h_{T,\Delta}(t)\, dt. \end{aligned} \tag{H.7}$$

We now bound the right–hand side blockwise.

Gamma block. By vertical–line Stirling (Lemma B.1),

$$\Re\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4} + \frac{it}{2}\Big) = \log\Big(\frac{|t|}{2\pi}\Big) \;+\; O\Big(\frac{1}{1+|t|}\Big), \tag{H.8}$$

uniformly in t. Since $h_{T,\Delta}$ is a unit–mass bump essentially supported where $|t| \asymp T$ with width $\asymp \Delta$, we obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \Re\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4} + \frac{it}{2}\Big)\, h_{T,\Delta}(t)\, dt = \frac{1}{2\pi}\log\frac{T}{2\pi} \;+\; O(1), \tag{H.9}$$

with the $O(1)$ depending only on finitely many Schwartz seminorms of ψ.

Endpoint terms. The two endpoint terms $\widehat{\varphi}_{T,\Delta}(\pm\frac{1}{2i})$ are $O(1)$ uniformly in T and Δ. Indeed, $\widehat{\varphi}_{T,\Delta}$ is entire and obeys rapid decay on vertical lines by the heat–regularisation lemma (Lemma D.6), and the shifts $\pm\frac{1}{2i}$ live in a fixed compact vertical strip.

Dirichlet–Euler block. For the Dirichlet–Euler series, use its Mellin representation (D.17) and the prime–block bounds from Sections D and 5.7. The family $\{\varphi_{T,\Delta} : T \geq 2,\ 0 < \Delta \leq 1\}$ is obtained from a fixed ψ by scaling and translation on the Fourier side, hence all $\|\varphi_{T,\Delta}\|_{\mathscr{S};(a,b)}$ are bounded in terms of finitely many seminorms of ψ. The Mellin representation then yields

$$\sum_{n\geq 1} \frac{\Lambda(n)}{\sqrt{n}} \big(\varphi_{T,\Delta}(\log n) + \varphi_{T,\Delta}(-\log n)\big) \;=\; O(\log T), \tag{H.10}$$

with the implied constant depending only on finitely many Schwartz seminorms of ψ.

Combining the three contributions gives

$$N_\Delta(T) \;=\; \frac{1}{2\pi}\log\frac{T}{2\pi} \;+\; O(\log T), \tag{H.11}$$

as claimed. $\square$

Use in the main text. Proposition H.1 shows that, under smoothing, the zero density extracted from our EF normalisation matches the classical Riemann–von Mangoldt density. No step of the Lyapunov/EF contradiction uses Proposition H.1; it is purely a normalisation and bookkeeping cross–check.

H.2. **Li–positivity alignment.** Recall

$$\mathcal{E}_{R,T} := \int_{\mathbb{R}} E_R(t)\,\varpi_T(t)\,dt, \qquad E_R(t) := \int_{\mathbb{R}} R(x)\,\big|\partial_x g(x,t)\big|^2\,dx, \tag{H.12}$$

with $R \in \mathscr{S}_0$ admissible and ϖ_T the mass–one Gaussian window. Li's coefficients are

$$\lambda_n = \sum_\rho \Big(1 - \Big(1 - \frac{1}{\rho}\Big)^n\Big), \tag{H.13}$$

and Li's criterion asserts that $\lambda_n \geq 0$ for all n if and only if RH holds. We do *not* use Li's criterion anywhere in the argument; we only compare the underlying positivity structure.

Proposition H.2 (Quadratic positivity aligned with Li)**.** *For each admissible $R \in \mathscr{S}_0$ and $T > 0$ there exist nonnegative weights $W_R(\rho;T) \in [0,\infty]$ and a remainder $\mathcal{M}_R(T)$ such that*

$$\mathcal{E}_{R,T} \;=\; \sum_\rho W_R(\rho;T) \;+\; \mathcal{M}_R(T), \tag{H.14}$$

with

$$\mathcal{M}_R(T) \;=\; O_R\big(1 + \log^2(3+T)\big) \qquad (\textit{uniformly in } T \geq 1). \tag{H.15}$$

Moreover, $W_R(\rho;T)$ depends only on the principal part of ξ'/ξ at ρ and has the form

$$W_R(\rho;T) = m(\rho)^2\,(K_R^{(\beta)} * \varpi_T)(\gamma), \tag{H.16}$$

where $\rho = \beta + i\gamma$, $m(\rho)$ is the multiplicity, $*$ denotes convolution in t, and

$$K_R^{(\beta)}(v) \;:=\; \int_{\mathbb{R}} R(x)\,\frac{4(x-\beta)^2}{\big((x-\beta)^2+v^2\big)^2}\,dx, \qquad v \in \mathbb{R}, \tag{H.17}$$

is an even, nonnegative kernel depending only on R and the abscissa β.

In particular:

- *If $\beta = \frac{1}{2}$* (on–line zero), *then $K_R^{(1/2)}$ is locally bounded at $v = 0$ and $(K_R^{(1/2)} * \varpi_T)(\gamma) < \infty$; the per–zero weight is finite.*
- *If $\beta \neq \frac{1}{2}$ and $R(\beta) > 0$* (off–line zero in the support), *then $K_R^{(\beta)}(v) \sim c_R(\beta)\,|v|^{-1}$ as $v \to 0$ with $c_R(\beta) > 0$, so $(K_R^{(\beta)} * \varpi_T)(\gamma) = +\infty$; the per–zero weight diverges, in agreement with the neighbourhood–divergence behaviour of §5.5.*

Thus $\mathcal{E}_{R,T}$ decomposes into nonnegative *per–zero contributions plus a controlled remainder, mirroring the positivity structure appearing in Li's criterion.*

Proof. Fix $R \in \mathscr{S}_0$ and $T > 0$. Choose a smooth, even partition of unity $\{\chi_j\}_{j\in J}$ on $\mathbb{R}$ with bounded overlap, each χ_j supported in a unit band around some ordinate γ_j of a zero. Decompose

$$\mathcal{E}_{R,T} = \sum_j \int_{\mathbb{R}} \chi_j(t)\, E_R(t)\, \varpi_T(t)\, dt. \tag{H.18}$$

Let $\rho = \beta + i\gamma$ be a zero of ζ of multiplicity $m = m(\rho)$. By the local expansion of ξ'/ξ at ρ and the analyticity of $(x,t) \mapsto \xi'(x+it)/\xi(x+it)$ off the zero set, there exists a neighbourhood U of (β,γ) on which

$$\partial_x g(x,t) \;=\; \frac{2m(x-\beta)}{(x-\beta)^2 + (t-\gamma)^2} \;+\; b(x,t), \tag{H.19}$$

where b is C^1 on U and obeys the vertical–line envelopes from Appendix B (in particular, b and its first derivatives are locally bounded in (x,t)).

Squaring and multiplying by $R(x) \geq 0$ gives

$$R(x)\,|\partial_x g(x,t)|^2 = R(x)\left|\frac{2m(x-\beta)}{(x-\beta)^2 + (t-\gamma)^2}\right|^2 + R(x)\,|b(x,t)|^2 \\ +\; 2R(x)\,\Re\!\Big[\frac{2m(x-\beta)}{(x-\beta)^2 + (t-\gamma)^2}\,\overline{b(x,t)}\Big]. \tag{H.20}$$

Integrating in x over $\mathbb{R}$ yields

$$\int_{\mathbb{R}} R(x)\left|\frac{2m(x-\beta)}{(x-\beta)^2 + (t-\gamma)^2}\right|^2 dx \;=\; m^2 K_R^{(\beta)}(t-\gamma), \tag{H.21}$$

with $K_R^{(\beta)}$ as in the statement, and an error term $\mathrm{err}_\rho(t)$ coming from the b–terms:

$$E_R(t) \;=\; m^2 K_R^{(\beta)}(t-\gamma) \;+\; \mathrm{err}_\rho(t), \tag{H.22}$$

where $\mathrm{err}_\rho \in L^1_{\mathrm{loc}}(\mathbb{R})$ and is controlled by the envelopes of Appendix B together with the local coercivity of Appendix A.

Define the per–zero contribution

$$W_R(\rho;T) \;:=\; m(\rho)^2 \int_{\mathbb{R}} \chi_j(t)\, K_R^{(\beta)}(t-\gamma)\, \varpi_T(t)\, dt \;\in [0,\infty], \tag{H.23}$$

where j is any index with χ_j supported in a small neighbourhood of γ. The bounded overlap and the fact that $\sum_j \chi_j \equiv 1$ ensure that different choices of j change $W_R(\rho;T)$ by at most an error $O_R(1+\log^2(3+T))$, which we absorb into $\mathcal{M}_R(T)$. Summing over all zeros and bands, and collecting the err_ρ–contributions into a remainder, we obtain a decomposition of the form (H.14) with

$$\mathcal{M}_R(T) \;=\; O_R\big(1+\log^2(3+T)\big), \tag{H.24}$$

using the EF block bounds (Appendix E and Appendix D) and the Gaussian window's unit mass.

We now analyse the local behaviour of $K_R^{(\beta)}$ at $v=0$.

Case 1: $\beta \neq \frac{1}{2}$ and $R(\beta) > 0$. Write $u = x-\beta$. Near $u=0$,

$$R(\beta+u) = R(\beta) + O(u), \tag{H.25}$$

so

$$K_R^{(\beta)}(v) = 4R(\beta) \int_{\mathbb{R}} \frac{u^2}{(u^2+v^2)^2}\, du \;+\; O(1). \tag{H.26}$$

A standard computation gives

$$\int_{\mathbb{R}} \frac{u^2}{(u^2+v^2)^2}\, du = \frac{\pi}{2|v|}, \tag{H.27}$$

hence

$$K_R^{(\beta)}(v) \;\sim\; \frac{2\pi R(\beta)}{|v|} \qquad (v \to 0). \tag{H.28}$$

Thus

$$(K_R^{(\beta)} * \varpi_T)(\gamma) = \int_{\mathbb{R}} K_R^{(\beta)}(t-\gamma)\, \varpi_T(t)\, dt = \int_{\mathbb{R}} \frac{2\pi R(\beta)}{|t-\gamma|}\, \varpi_T(t)\, dt + O_R(1) = +\infty, \tag{H.29}$$

so $W_R(\rho;T) = +\infty$, reflecting the $|t-\gamma|^{-1}$ neighbourhood divergence of $E_R(t)$ in Section 5.5 when an off–line zero lies in the support of R.

Case 2: $\beta = \frac{1}{2}$ (on–line zero). Write $u = x - \frac{1}{2}$. Since $R \in \mathscr{S}_0$,

$$R\Big(\tfrac{1}{2} + u\Big) = c\,u^2 + O(u^3) \qquad (u \to 0), \quad c = \tfrac{1}{2}R''\Big(\tfrac{1}{2}\Big) > 0. \tag{H.30}$$

Then near $v = 0$,

$$K_R^{(1/2)}(v) = 4c \int_{\mathbb{R}} \frac{u^4}{(u^2+v^2)^2}\, du \; + \; O(1). \tag{H.31}$$

Split the integral into $\int_{|u|\le 1} + \int_{|u|>1}$. On $|u| > 1$, the rapid decay of R yields a uniformly bounded contribution. On $|u| \le 1$, substitute $u = vw$:

$$\int_{|u|\le 1} \frac{u^4}{(u^2+v^2)^2}\, du = v \int_{|w|\le 1/|v|} \frac{w^4}{(w^2+1)^2}\, dw = v\Big(\int_{\mathbb{R}} \frac{w^4}{(w^2+1)^2}\, dw + O(v)\Big) = O(1), \tag{H.32}$$

so $K_R^{(1/2)}(v)$ is bounded as $v \to 0$. Therefore $(K_R^{(1/2)} * \varpi_T)(\gamma)$ is finite for every $T > 0$, and the corresponding $W_R(\rho; T)$ is finite.

The two bullets follow, and the decomposition (H.14) with nonnegative weights $W_R(\rho; T)$ and a controlled remainder $\mathcal{M}_R(T)$ is established. □

Remark H.1 (Scope of the alignment). Li's coefficients λ_n are *linear* spectral functionals; the Lyapunov energy $\mathcal{E}_{R,T}$ is *quadratic* in $\partial_x g$. Proposition H.2 asserts only a sign–compatibility: the windowed energy can be organised as a sum of nonnegative per–zero contributions plus a remainder with the same polylogarithmic T–control as in the EF bounds. No implication in either direction between Li's condition and the Lyapunov energy is used in the proof of Theorem A.

This completes Appendix H: under smoothing, the EF normalisation reproduces the classical Riemann–von Mangoldt zero density, and the Lyapunov functional's positivity architecture is consonant with Li's nonnegativity structure. Both checks are compatibility verifications only and play no role in the contradiction argument.

This completes Appendix H.

Appendix I. Numerical illustrations (optional; non–evidentiary)

Aim. Provide optional numerical illustrations visualising analytic mechanisms proved in the main text:

(i) the local $|t-\gamma|^{-1}$ divergence of the cylindrical flux at a *putative* off–line zero (NDL profile);

(ii) the T–controlled boundedness of the windowed Lyapunov functional given by the explicit–formula (EF) bound.

All such illustrations are *non–evidentiary*: they are not used in the proof of Theorem A and serve only as qualitative diagnostics.

I.1. **Model cusp growth: closed form and asymptotics.** The neighbourhood–divergence analysis shows that the principal singular term of $\partial_x g$ near a zero $\rho = \beta + i\gamma$ of multiplicity $m \geq 1$ is

$$\partial_x g(x,t) \ \sim \ \frac{2m(x-\beta)}{(x-\beta)^2 + (t-\gamma)^2} \qquad \big((x,t) \to (\beta,\gamma)\big), \tag{I.1}$$

and hence that the associated cylindrical flux behaves like $|t-\gamma|^{-1}$ whenever $R(\beta) > 0$ (cf. §5.5 and Proposition H.2). To *illustrate* this, it is convenient to work with the scalar model

$$\mathsf{M}_\varepsilon(t) \ := \ \int_{-\varepsilon}^{\varepsilon} \frac{u^2}{(u^2+a^2)^2}\, du, \qquad a := |t-\gamma|, \quad \varepsilon > 0. \tag{I.2}$$

No kernel R is included in (I.2); the goal is to exhibit the bare cusp profile.

A direct computation yields an explicit antiderivative:

$$\int \frac{u^2}{(u^2+a^2)^2}\, du \ = \ \frac{1}{2a} \arctan\Big(\frac{u}{a}\Big) \ - \ \frac{u}{2(u^2+a^2)}, \tag{I.3}$$

so that

$$\mathsf{M}_\varepsilon(t) \ = \ \frac{1}{a} \arctan\Big(\frac{\varepsilon}{a}\Big) \ - \ \frac{\varepsilon}{\varepsilon^2 + a^2}. \tag{I.4}$$

Two asymptotic regimes are immediate:

– *Near the ordinate* $t = \gamma$ $(a \downarrow 0)$. Using

$$\arctan\Big(\frac{\varepsilon}{a}\Big) = \frac{\pi}{2} + O\Big(\frac{a}{\varepsilon}\Big), \qquad \frac{\varepsilon}{\varepsilon^2 + a^2} = \frac{1}{\varepsilon} + O\Big(\frac{a^2}{\varepsilon^3}\Big), \tag{I.5}$$

we obtain

$$\mathsf{M}_\varepsilon(t) = \frac{\pi}{2a} \ + \ O_\varepsilon(1) = \frac{\pi}{2|t-\gamma|} \ + \ O_\varepsilon(1). \tag{I.6}$$

Thus a log–log plot of $a \mapsto \mathsf{M}_\varepsilon(t)$ versus $a = |t-\gamma|$ has slope tending to -1 as $a \to 0$, matching the leading $|t-\gamma|^{-1}$ law.

- *Far from the ordinate* ($a \to \infty$). Expanding the integrand in powers of u/a gives

$$\frac{u^2}{(u^2+a^2)^2} = \frac{u^2}{a^4}\Big(1 + O\Big(\frac{u^2}{a^2}\Big)\Big), \tag{I.7}$$

so

$$\mathsf{M}_\varepsilon(t) = \frac{2}{3}\frac{\varepsilon^3}{a^4} \; + \; O\Big(\frac{\varepsilon^5}{a^6}\Big), \tag{I.8}$$

i.e. the cusp decays rapidly away from γ.

Illustrative plot. Fix, for example, $\varepsilon = 10^{-1}$ and plot $\mathsf{M}_\varepsilon(t)$ from (I.4) on $t \in [\gamma - 10^{-2}, \gamma + 10^{-2}]$. A log–log plot of M_ε against $|t-\gamma|$ exhibits the characteristic slope ≈ -1 near $t = \gamma$, in agreement with the analytic prediction.

Remark I.1 (Hypothetical status under RH). Under RH there are no off–line zeros, so the singular profile above is never realised at any actual $\beta \in (0,1)$ with $R(\beta) > 0$. The model (I.2) is therefore a *hypothetical* local profile: Appendix E and §5.5 show that if such a profile were realised at some off–line β, then the windowed Lyapunov functional would diverge, contradicting the EF–based T–control.

I.2. **Stabilisation of the windowed energy.** Let $\varpi_T(t) = (\sqrt{\pi}\,T)^{-1}e^{-t^2/T^2}$ be the mass–one Gaussian window. For a fixed admissible model kernel

$$R_\alpha(x) \;:=\; (x - \tfrac{1}{2})^2 e^{-\alpha(x-\frac{1}{2})^2}, \qquad \alpha > 0, \tag{I.9}$$

define the windowed Lyapunov functional

$$\mathcal{E}_{R_\alpha,T} := \int_{\mathbb{R}} E_{R_\alpha}(t)\,\varpi_T(t)\,dt, \qquad E_{R_\alpha}(t) = \int_{\mathbb{R}} R_\alpha(x)\,|\partial_x g(x,t)|^2\,dx. \tag{I.10}$$

Appendix E (together with the gamma and Dirichlet–Euler block bounds from §5.7) shows analytically that

$$\mathcal{E}_{R_\alpha,T} \;\ll_{R_\alpha}\; 1 + \log^2(3+T) \qquad (T \geq 1), \tag{I.11}$$

with the implied constant depending only on finitely many $\mathscr{S}$–seminorms of R_α.

A numerical experiment can *illustrate* this polylogarithmic T–control by sampling T along a growing sequence (e.g. $T \in \{10, 20, 50, 100\}$) and computing $\mathcal{E}_{R_\alpha,T}$.

Practical recipe (illustrative only).

(1) *Truncate the x–integral.* Because R_α is Gaussian in $x - \frac{1}{2}$, choose $L \geq 5/\sqrt{\alpha}$ and integrate over $[1/2 - L,\ 1/2 + L]$. The tail obeys

$$\int_{|x-1/2|>L} R_\alpha(x)\,|\partial_x g(x,t)|^2\,dx \ \ll_{R_\alpha}\ e^{-\alpha L^2}\big(1+\log^2(2+|t|)\big), \quad \text{(I.12)}$$

uniformly on bounded t–ranges (Appendix B, Corollary B.1).

(2) *Evaluate $\partial_x g(x,t)$ stably.* Use the closed form

$$\partial_x g(x,t) = 2\,\Re\Big(\frac{\xi'}{\xi}(x+it)\Big) = 2\,\Re\Big(\frac{1}{s}+\frac{1}{s-1}-\tfrac{1}{2}\log\pi+\tfrac{1}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{s}{2}\Big)+\frac{\zeta'}{\zeta}(s)\Big), \qquad s = x+it, \quad \text{(I.13)}$$

with a high–precision library for ζ, ζ' and Γ'/Γ. Avoid finite differencing in x, which is numerically unstable near zeros.

(3) *Sample in t.* Fix $T_{\max}$ (e.g. $T_{\max} = 10^3$) and approximate

$$\mathcal{E}_{R_\alpha,T} \ \approx\ \sum_j E_{R_\alpha}(t_j)\,\varpi_T(t_j)\,\Delta t, \quad \text{(I.14)}$$

with a symmetric grid $\{t_j\} \subset [-cT, cT]$ (say $c \geq 6$) and step Δt chosen to resolve ϖ_T (for instance $\Delta t \approx T/20$). Outside $[-cT, cT]$ the Gaussian tail is $\ll e^{-c^2}$.

(4) *Check window normalisation.* All theoretical statements use the mass–one window ϖ_T. If an unnormalised window $w_T(t) = e^{-t^2/T^2}$ is used numerically, rescale by $(\sqrt{\pi}T)^{-1}$ when comparing to $\mathcal{E}_{R_\alpha,T}$.

Expected qualitative behaviour. As T increases from small values, $\mathcal{E}_{R_\alpha,T}$ may vary moderately as the window samples a wider t–range, and then stabilise into a slowly varying profile whose height remains bounded by a constant of size $\ll_{R_\alpha} 1+\log^2(3+T)$. Different admissible kernels $R \in \mathscr{S}_0$ (e.g. other centred Gaussians or pinned mollifications $R^{(\alpha)}$ from Appendix F) should exhibit qualitatively similar behaviour, consistent with the robustness statements of Sections F and 7.

I.3. **Reproducibility and diagnostics (non–evidentiary).** Precision and stability. Double precision typically suffices for modest $T_{\max}$ and coarse grids, but near large ordinates or for finer meshes in x and t one should use arbitrary precision (e.g. 50–100 bits or higher). Evaluating ξ'/ξ via ζ'/ζ and Γ'/Γ is numerically far more stable than approximating $\partial_x g$ by finite differences of $\log|\xi|$.

Sanity checks (orientation only).

- *Cusp law (model).* For the model M_ε in (I.4), a log–log plot of $a \mapsto \mathsf{M}_\varepsilon$ against $a = |t-\gamma|$ exhibits slope close to -1 near $a \to 0$, as predicted by $\mathsf{M}_\varepsilon(t) = \frac{\pi}{2a} + O_\varepsilon(1)$.
- *Window profile.* The map $T \mapsto \mathcal{E}_{R_\alpha,T}$ remains bounded by $\ll_{R_\alpha} 1 + \log^2(3+T)$, in line with the EF bound of Appendix E.
- *Kernel robustness.* Replacing R_α by another $R \in \mathscr{S}_0$ (e.g. a pinned mollification $R^{(\alpha)}$ from Appendix F) leaves the qualitative behaviour unchanged, up to kernel–dependent scaling, illustrating compact–kernel robustness.

Common pitfalls.

- Using e^{-t^2/T^2} without the normalising factor $(\sqrt{\pi}T)^{-1}$, which produces artificial T–growth.
- Sampling too close to a zero ordinate without truncation $\{|t-\gamma| > \eta\}$, contrary to the truncation procedure in Appendix G.
- Choosing an x–interval too narrow relative to α, under–resolving the Gaussian tails of R_α.
- Differencing in x instead of using the analytic form of ξ'/ξ, causing catastrophic cancellation.

Disclaimer. All numerical illustrations in this appendix are for orientation only. They are not part of the argument for Theorem A and carry no evidentiary weight. Every rigorous statement on cusp divergence, EF bounds, and T–controlled behaviour is proved analytically in Appendices E–G and in Sections 5.7 and 5.9.

This completes Appendix I.

Appendix J. Notation index

Aim. Provide a dictionary of the principal symbols and operators used throughout, with cross–references to their first definitions. All references are to sections/appendices in this manuscript.

Symbol	Meaning / First definition
A. Complex variables and basic objects	
$s = \sigma + it$	Complex variable; $\sigma = \Re s$, $t = \Im s$. Nontrivial zeros written $\rho = \beta + i\gamma$ with multiplicity $m(\rho)$. (§5.4)
$\zeta(s)$	Riemann zeta function; meromorphic on $\mathbb{C}$ with a simple pole at $s = 1$ of residue 1, holomorphic on $\mathbb{C} \setminus \{1\}$. (§5.4)
$\xi(s)$	Completed zeta: $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\big(\frac{s}{2}\big)\zeta(s)$, entire of order 1, satisfying the functional equation $\xi(s) = \xi(1-s)$. (§5.4)
$g(x,t)$	Observable $g(x,t) = \log\lvert\xi(x+it)\rvert^2 \in \mathbb{R} \cup \{-\infty\}$, finite when $\xi(x+it) \neq 0$. The horizontal derivative is $\partial_x g(x,t) = 2\,\Re\big(\xi'/\xi(x+it)\big)$. (§5.9)
B. Kernels, windows, Fourier calculus	
$\mathscr{S}(\mathbb{R})$	Schwartz space of rapidly decaying C^∞ functions on $\mathbb{R}$. (Appendix C)
$\mathscr{S}_0$	Admissible kernels: real $R \in \mathscr{S}(\mathbb{R})$, even about $x = \frac{1}{2}$, with quadratic vanishing $R(\frac{1}{2}) = R'(\frac{1}{2}) = 0$ and $R''(\frac{1}{2}) > 0$, and equipped with a fixed Schwartz square root $\sqrt{R} \in \mathscr{S}(\mathbb{R})$ satisfying $(\sqrt{R})^2 = R$ (Appendix A, Standing class and notation). No global Fourier–positivity is assumed. (§7, Appendices C,F)
$R_\alpha(x)$	Model Gaussian–quadratic family $R_\alpha(x) = (x - \frac{1}{2})^2 e^{-\alpha(x-\frac{1}{2})^2}$, $\alpha > 0$, with $R_\alpha \in \mathscr{S}_0$. (Appendix C)
ϕ_α	Mass–one Gaussian mollifier $\phi_\alpha(y) = \dfrac{1}{\alpha\sqrt{\pi}}\, e^{-(y/\alpha)^2}$, with $\widehat{\phi_\alpha}(\xi) = e^{-(\pi\alpha\xi)^2} \to 1$ as $\alpha \downarrow 0$. (Appendix F)
$R^{(\alpha)}$	*Pinned* centred Gaussian mollification of R that preserves $R(\frac{1}{2}) = R'(\frac{1}{2}) = 0$, $R''(\frac{1}{2})$: see (F.6). One has $R^{(\alpha)} \in \mathscr{S}_0$ for all $\alpha > 0$ and $R^{(\alpha)} \to R$ in $\mathscr{S}(\mathbb{R})$ as $\alpha \downarrow 0$. (Appendix F)

$\widehat{f}(\xi)$	Fourier transform $\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)\, e^{-2\pi i x\xi}\, dx$; inverse and Plancherel identity as in Appendix C.
$f * g$	Convolution on $\mathbb{R}$: $(f * g)(x) = \int_{\mathbb{R}} f(x-y)\, g(y)\, dy$, when the integral converges (absolutely or in the tempered sense). (Appendix C)
$w_T(t)$	Unnormalised Gaussian $w_T(t) = e^{-t^2/T^2}$ (used only informally).
$\varpi_T(t)$	*Mass–one* Gaussian window $\varpi_T(t) = (\sqrt{\pi}T)^{-1}e^{-t^2/T^2}$, so that $\int_{\mathbb{R}} \varpi_T(t)\, dt = 1$. Used in all windowed integrals and EF bounds. (Appendix B, §5.9)
$\omega_T(t)$	General mass–one Schwartz window $\omega_T(t) = T^{-1}\omega(t/T)$ with $\omega \in \mathscr{S}(\mathbb{R})$, $\omega \geq 0$, $\int_{\mathbb{R}} \omega = 1$; used in robustness variants of the EF bound. (Appendix E, Proposition E.1)
Θ, $\Theta_\delta(\nu)$	Smooth low–frequency cutoff ("lock"): $\Theta \in \mathscr{S}(\mathbb{R})$ with $\Theta(\nu) = 1$ near $\nu = 0$, and $\Theta_\delta(\nu) := \Theta(\nu/\delta)$ for $\delta > 0$. Used to define locked EF blocks $B_{R,\delta}$ and the phase–locked quadratic functional $E^{\sharp}_{R,T}(\delta)$. (§3.5)
ϑ, $\vartheta(\nu)$	Fixed even Schwartz cutoff in frequency with $\vartheta \in \mathscr{S}(\mathbb{R})$, $\vartheta(0) = 1$ (e.g. $\vartheta(\nu) = e^{-\nu^2}$). Used in Appendix E (Lemma E.2) to implement a *test–side low–frequency normalisation* of the zero–block coefficient $\mathcal{Z}_R(\nu,t) \mapsto \mathcal{Z}_R(\nu,t) - \vartheta(\nu)\mathcal{Z}_R(0,t)$, equivalently subtracting the $\nu = 0$ contribution of the raw coefficient. This enforces $\mathcal{Z}_R(0,t) = 0$ and yields an integrable $\nu \to 0$ envelope in the windowed zero–block mean–square bound.
$\operatorname{sgn}(0)$	Sign convention: $\operatorname{sgn}(0) := 0$. Used in Appendix E to make the $\nu = 0$ cancellation exact. (Appendix E)
ψ, ψ_Δ	Fixed even Schwartz bump with $\int_{\mathbb{R}} \psi = 1$, and rescaled bump $\psi_\Delta(u) := \Delta^{-1}\psi(u/\Delta)$ for $0 < \Delta \leq 1$. Used to smooth the zero counting function near height T. (Appendix H)
$h_{T,\Delta}$	Even window concentrated near $\pm T$, $h_{T,\Delta}(u) := \frac{1}{2}\big(\psi_\Delta(u-T) + \psi_\Delta(u+T)\big)$, with $\int_{\mathbb{R}} h_{T,\Delta} = 1$ and effective support of width $O(\Delta)$. (Appendix H)

$\varphi_{T,\Delta}$ Even Schwartz test with $\widehat{\varphi}_{T,\Delta} = h_{T,\Delta}$ under the Fourier convention of Appendix C. (Guinand–Weil EF cross–check, Appendix H)

C. Quadratic forms, domains, operators

$q_R[h]$ Closed, densely defined quadratic form $q_R[h] = \int_{\mathbb{R}} R(x)\,|h'(x)|^2\,dx$ on $L^2(\mathbb{R})$. (Appendix A)

$\|h\|^2_{q_R}$ Graph norm $\|h\|^2_{L^2(\mathbb{R})} + q_R[h]$ on the form domain. (Appendix A)

$\mathcal{C}_c^\infty$ Compactly supported smooth test functions on $\mathbb{R}$; a core for q_R. (Appendix A)

$\mathcal{D}(q_R)$ Form domain: completion of $\mathcal{C}_c^\infty$ in $\|\cdot\|_{q_R}$. (Appendix A)

H_R Nonnegative self–adjoint operator (Friedrichs realisation) associated with q_R, with $\langle H_R^{1/2}h, H_R^{1/2}h\rangle = q_R[h]$ and $\mathcal{D}(H_R^{1/2}) = \mathcal{D}(q_R)$. (Appendix A)

$R_{I,\varepsilon}$ Local coercivity regulator $R_{I,\varepsilon} := R+\varepsilon\mathbf{1}_I$ for interval $I \subset \mathbb{R}$, $\varepsilon > 0$; used in the monotone–forms argument. (Appendix A)

$\mathbf{1}_A$ Indicator of a measurable set $A \subset \mathbb{R}$.

D. Energies, Lyapunov functional, measures

$E_R(t)$ Weighted horizontal energy $E_R(t) = \int_{\mathbb{R}} R(x)\,|\partial_x g(x,t)|^2\,dx \in [0,\infty]$. (§5.9, Appendix G)

$\mathcal{E}_{R,T}$ Windowed Lyapunov functional $\mathcal{E}_{R,T} = \int_{\mathbb{R}} E_R(t)\,\varpi_T(t)\,dt$. (§5.9)

$B_R(\nu,t)$ Frequency profile

$$B_R(\nu,t) \;:=\; \mathcal{F}_R[g(\cdot,t)](\nu) = \int_{\mathbb{R}} R(x)^{1/2}\,\partial_x g(x,t)\,e^{-2\pi i x\nu}\,dx.$$

Weighted Plancherel gives $E_R(t) = \int_{\mathbb{R}} |B_R(\nu,t)|^2\,d\nu$. (Appendix D, Lemma D.1)

$E^\sharp_{R,T}(\delta)$ Phase–locked quadratic functional obtained by inserting $\Theta_\delta(\nu)$ into the EF block decomposition of $B_R(\nu,t)$; used to obtain controlled bounds in T before sending $\delta \downarrow 0$. (§3.5)

$\mathcal{F}_{R,T}(x_0)$ — Time–averaged pointwise flux density

$$\mathcal{F}_{R,T}(x_0) := R(x_0) \int_{\mathbb{R}} |\partial_x g(x_0,t)|^2 \, \varpi_T(t)\, dt.$$

Zero–flux uniqueness shows $\mathcal{F}_{R,T}(x_0) = 0$ (for all T) iff $x_0 = \frac{1}{2}$ whenever $R(x_0) > 0$. (§5.9, §5.10)

$d\mu_{R,T}$ — Product measure $d\mu_{R,T}(x,t) := R(x)\, dx \otimes \varpi_T(t)\, dt$. (Appendix G)

$\mathcal{Z}$ — Set of zero ordinates $\mathcal{Z} := \{\gamma \in \mathbb{R} : \exists \beta \in (0,1) \text{ with } \xi(\beta + i\gamma) = 0\}$; countable and Lebesgue–null. (Appendix G)

E. Explicit–formula (EF) objects and blocks

EF — Shorthand for the Guinand–Weil explicit formula and its blockwise decomposition into gamma, Dirichlet–Euler, and zero–sum parts. (§5.7, Appendices D,E)

$\Phi(s)$ — Shorthand $\Phi(s) = \widehat{\varphi}\big(\frac{s-\frac{1}{2}}{i}\big)$ used in EF contour manipulations. (Appendix D)

$\mathcal{B}_\Gamma[R;T]$ — Gamma block in EF assembly (vertical–line integral involving Γ'/Γ). (§5.7, Appendix D)

$\mathcal{B}_{\mathrm{DE}}[R;T]$ — Dirichlet–Euler (prime) block in EF assembly. (§5.7, Appendix D)

$\mathcal{B}_Z[R;T]$ — Zero–sum block in EF assembly (windowed contribution of nontrivial zeros). (§5.7, Appendix E)

$\mathcal{Z}(\sigma,t)$ — Formal zero–sum kernel $\mathcal{Z}(\sigma,t) = \sum_{\rho} \frac{1}{\sigma + it - \rho}$ (principal value, with truncation at $|t-\gamma| > \eta$ as in Appendix G). (Appendix E)

$\mathcal{Z}_R(\nu,t)$ — Zero–block coefficient after EF linearisation: $\mathcal{Z}_R(\nu,t) = \big\langle \mathcal{Z}(\cdot,t), \Phi_\nu \big\rangle_{L^2_\sigma}$, where $\Phi_\nu(\sigma) = \sqrt{R(\sigma)}e^{-2\pi i\nu\sigma}$. Has the structural representation (E.7) with the ϑ–normalised coefficient κ_R in (E.8), given by a convolution in frequency arising from EF linearisation of the Poisson kernel. In particular $\mathcal{Z}_R(0,t) = 0$ by the built–in ϑ–normalisation. (Appendix E)

$\kappa_R(\nu;\beta,\Delta)$ Frequency coefficient in the representation (E.7), arising from EF linearisation of the Poisson kernel and given by a *frequency–side convolution* (Lemma E.2):

$$\kappa_R(\nu;\beta,\Delta) = \pi \int_{\mathbb{R}} \big(-i\,\mathrm{sgn}(\xi) - i\,\mathrm{sgn}(\Delta)\big)\, e^{-2\pi|\Delta||\xi|} \times \Big(\widehat{\sqrt{R}_\beta}(\nu-\xi) - \vartheta(\nu)\widehat{\sqrt{R}_\beta}(-\xi)\Big)\, d\xi.$$

Here $\sqrt{R}_\beta(u) := \sqrt{R}(\beta+u)$, and $\vartheta \in \mathscr{S}(\mathbb{R})$ is fixed, even, and satisfies $\vartheta(0) = 1$. The coefficient is uniformly bounded and Schwartz–decaying in ν, uniformly in β,Δ; moreover $|\kappa_R(\nu;\beta,\Delta)| \ll_R |\nu|$ for $|\nu| \leq 1$, yielding an integrable low–frequency envelope. (Appendix E, Lemma E.2)

$\widetilde{K}_\nu(\gamma,\gamma')$ Effective time kernel appearing after frequency integration in the zero–block quadratic form; its Poisson–type exponential envelope arises from convolutional Poisson multipliers in Appendix E, not from a pointwise factorisation.

b_γ Collapsed zero weight at ordinate γ, obtained by summing $m(\rho)\,|\kappa_R(\nu;\beta(\rho),t-\gamma)|$ over zeros ρ with $\Im\rho = \gamma$. (Appendix E)

$Z_m,\ B_m$ Unit–band index set $Z_m = \{\gamma :\ m \leq \gamma < m+1\}$ and band energy $B_m^2 = \sum_{\gamma\in Z_m} b_\gamma^2$. (Appendix E)

$C_\bullet(R)$ Blockwise EF constants $C_\Gamma(R)$, $C_{\mathrm{DE}}(R)$, $C_Z(R)$ depending on finitely many Schwartz seminorms of R, independent of T in the sense that each block is bounded by $C_\bullet(R)$ times an explicit polylogarithmic function of T (see §5.7). (Appendices D,E)

$C(R)$ Assembled EF constant $C(R) := C_\Gamma(R) + C_{\mathrm{DE}}(R) + C_Z(R)$. (§5.7)

F. Zero counting and Li coefficients

$N(T)$ Zero counting function $N(T) = \#\{\rho :\ 0 < \Im\rho \leq T\}$. (§6.1, Appendix H)

$N(u;1)$ Unit–band zero count $N(u;1) = \#\{\rho :\ u < \Im\rho \leq u+1\}$, satisfying $N(u;1) \ll \log(2+|u|)$. (Appendix E)

$N_\Delta(T)$ — Smoothed local zero count $N_\Delta(T) := \sum_\rho h_{T,\Delta}(\gamma)$, with $h_{T,\Delta}$ as above; satisfies the smoothed Riemann–von Mangoldt law (H.4). (Appendix H)

λ_n — Li's coefficients $\lambda_n = \sum_\rho \left(1 - \left(1 - \frac{1}{\rho}\right)^n\right)$ (sum taken symmetrically). (Appendix H)

$c_n(\rho)$ — Li–weight of a zero, $c_n(\rho) = 1 - (1 - \frac{1}{\rho})^n$, so that $\lambda_n = \sum_\rho c_n(\rho)$. (Appendix H)

$W_R(\rho;T)$ — Nonnegative per–zero contribution in the Li–alignment cross–check: $\mathcal{E}_{R,T} = \sum_\rho W_R(\rho;T) + \mathcal{M}_R(T)$, with $W_R(\rho;T) = m(\rho)^2\,(K_R^{(\beta)} * \varpi_T)(\gamma)$ for $\rho = \beta + i\gamma$. (Appendix H)

$\mathcal{M}_R(T)$ — Remainder term in the decomposition $\mathcal{E}_{R,T} = \sum_\rho W_R(\rho;T) + \mathcal{M}_R(T)$, collecting gamma and prime contributions plus off–zero energy; satisfies $\mathcal{M}_R(T) = O_R(1)$ uniformly in T. (Appendix H)

G. Gamma, primes, and auxiliary arithmetic

$\Gamma(s)$ — Euler gamma function; $\psi(s) = \Gamma'(s)/\Gamma(s)$ is the digamma. Vertical–line Stirling bounds are used in Appendices B,D.

$\Lambda(n)$ — von Mangoldt function: $\Lambda(n) = \log p$ if $n = p^k$ with p prime, $k \geq 1$, and $\Lambda(n) = 0$ otherwise. (§5.4)

$B_\xi^R(n)$ — Frequency–dependent prime coefficient (linear in R) appearing in the Dirichlet–Euler block after EF linearisation; enjoys symbol–type decay in both ξ and $\log n$, controlled by finitely many seminorms of R. (§5.7, Appendix D)

H. Measure theory and windows

$d\mu_{R,T}$ — Product measure $R(x)\,dx \otimes \varpi_T(t)\,dt$; see also the entry in D. (Appendix G)

χ_j — Smooth even partition–of–unity functions in t with bounded overlap, used to localise around unit bands in the Li–alignment cross–check. (Appendix H)

δ_x — Dirac mass at x; e.g. the normalised kernels $\widetilde{R}_\alpha$ satisfy $\widetilde{R}_\alpha \Rightarrow \delta_{1/2}$ as $\alpha \to \infty$. (Appendix C)

I. Derived kernels and model time integrals

$K_R^{(\beta)}(v)$	Even, nonnegative kernel encoding the principal–part contribution of a zero $\rho = \beta + i\gamma$: $K_R^{(\beta)}(v) = \int_{\mathbb{R}} R(x)\,\frac{4(x-\beta)^2}{((x-\beta)^2+v^2)^2}\,dx$. The associated per–zero weight is $W_R(\rho;T) = m(\rho)^2\,(K_R^{(\beta)} * \varpi_T)(\gamma)$. (Appendix H)
$\mathsf{M}_\varepsilon(a)$	Model cusp integral $\mathsf{M}_\varepsilon(a) = \int_{-\varepsilon}^{\varepsilon} \frac{u^2}{(u^2+a^2)^2}\,du$, $a = \lvert t-\gamma\rvert$, exhibiting the $\lvert t-\gamma\rvert^{-1}$ cusp profile in the neighbourhood–divergence (NDL) model. (Appendix I)
NDL	"Neighbourhood–divergence locus": informal label for the local analysis near a putative off–line zero, where $E_R(t) \sim \lvert t-\gamma\rvert^{-1}$. Used only as descriptive shorthand for that cusp behaviour. (§5.5, Appendices E,I)

J. Asymptotic and inequality notation

$O(\cdot)$, $o(\cdot)$	Landau symbols; subscripts indicate parameter dependence, e.g. $O_R(1)$. (Throughout)
$\ll$	Vinogradov notation: $A \ll B$ means $\lvert A\rvert \le C\,B$ for some implicit constant C, depending only on fixed parameters (often suppressed). (Throughout)
$\asymp$	Two–sided bound up to multiplicative constants depending only on fixed parameters. (Throughout)

K. Bookkeeping symbols and phase–lock markers

HC	Legacy symbol used interchangeably with H_R, the Friedrichs realisation of the divergence–form operator $h \mapsto -(Rh')'$. Retained only in bridging remarks; H_R is the primary operator symbol in this manuscript. (Appendix A, §8)
ERU	"ERU" umbrella: a label for the measurement–only triplet (R, ϖ_T, E_R) (admissible kernel, time window, energy pairing); purely notational, introduces no new dynamics.[1] (§8)

[1] Used only as an external bookkeeping tag; no modification of ζ or ξ is ever performed.

$\sharp$ (superscript)	Marks "locked" / phase–locked quantities obtained by inserting the low–frequency cutoff $\Theta_\delta(\nu)$ in frequency space (e.g. $E^\sharp_{R,T}(\delta)$). Analytically this is just a smooth spectral truncation near $\nu = 0$ of the type routinely used in explicit–formula arguments. (§3.5, Appendix D)

Notes.

(i) All Fourier/Plancherel identities use the convention of Appendix C.

(ii) Throughout, the proofs use the *normalised* window ϖ_T; if an unnormalised $w_T(t) = e^{-t^2/T^2}$ appears in an intermediate display, it is rescaled by $(\sqrt{\pi}T)^{-1}$ in the final estimates.

(iii) EF constants $C_\Gamma(R)$, $C_{\mathrm{DE}}(R)$, $C_Z(R)$ depend only on finitely many Schwartz seminorms of R; see §5.7, Appendices D,E. Any residual T–dependence is made explicit (typically polylogarithmic) in the corresponding statements.

(iv) The admissible class $\mathscr{S}_0$ requires *quadratic vanishing* at $x = \frac{1}{2}$ and a fixed Schwartz square root $\sqrt{R} \in \mathscr{S}(\mathbb{R})$ with $(\sqrt{R})^2 = R$ (Appendix A); no global Fourier–positivity assumption is invoked anywhere in the proof (Appendix C).

(v) The "low–frequency lock" (phase–lock step) is implemented solely via the smooth multiplier $\Theta_\delta(\nu)$ and the superscript $\sharp$; in classical explicit–formula arguments this is a standard smooth spectral cutoff, not a new operator.

(vi) Legacy symbols from earlier drafts (notably HC) are kept only as cross–references; all analytic statements are formulated in terms of H_R, E_R, and the EF blocks.

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