and, adding vertical columns,

$$q_{aa} = 2h\left(\frac{e^{-\beta}}{1-e^{-\varpi}}+\frac{e^{-3\beta}}{1-e^{-3\varpi}}+\dots\right)$$

$$= 2h\sum_{n=0}^{n=\infty}\frac{e^{-(2n+1)\beta}}{1-e^{-(2n+1)\varpi}}$$

$$= h\sum_{n=0}^{n=\infty} e^{-(2n+1)\beta}\left\{1+\frac{1+e^{-(2n+1)\varpi}}{1-e^{-(2n+1)\varpi}}\right\}$$

$$= h\,\frac{e^{-\beta}}{1-e^{-2\beta}}+h\Sigma e^{-(2n+1)\beta}\coth\tfrac{1}{2}(2n+1)\varpi.$$

Now $$2\int_0^\infty \coth\pi x\sin mx\,dx = \coth\tfrac{1}{2}m,$$

therefore $$q_{aa} = \tfrac{1}{2}h\operatorname{cosech}\beta+2h\int_0^\infty \coth\pi x\sum_{n=0}^{n=\infty} e^{-(2n+1)\beta}\sin(2n+1)\,\varpi x\,dx$$

$$= \tfrac{1}{2}h\operatorname{cosech}\beta+2h\int_0^\infty \coth\pi x\,\frac{e^{-\beta}(1+e^{-2\beta})\sin\varpi x\,dx}{1-2e^{-2\beta}\cos2\varpi x+e^{-4\beta}}$$

$$= \tfrac{1}{2}h\operatorname{cosech}\beta+2h\cosh\beta\int_0^\infty\frac{\coth\pi x\sin\varpi x\,dx}{\cosh2\beta-\cos2\varpi x}.$$

Similarly $$q_{bb} = \tfrac{1}{2}h\operatorname{cosech}\alpha+2h\cosh\alpha\int_0^\infty\frac{\coth\pi x\sin\varpi x\,dx}{\cosh2\alpha-\cos2\varpi x},$$

and $$q_{ab} = \tfrac{1}{2}\operatorname{cosech}\varpi+2h\cosh\varpi\int_0^\infty\frac{\coth\pi x\sin\varpi x\,dx}{\cosh2\varpi-\cos2\varpi x};$$

equivalent to the expressions given by Poisson.

On Certain Systems of Partial Differential Equations of the First Order with several Dependent Variables. By H. W. Lloyd Tanner, M.A.

[*Read January 9th,* 1879.]

The simplest systems considered in this paper are composed of equations such as

$$\frac{dz_1}{dx_1}+\frac{dz_2}{dx_2}+\dots+\frac{dz_n}{dx_n}=0.$$

In the general case there are n independent variables, and $m(n-m)+1$ dependent variables. For the sake of symmetry, however, we assume a larger number of dependent variables, and between these relations are given sufficient to reduce them to the number stated above. The system comprises $\frac{\lfloor n}{\lfloor m-1 \, \lfloor n-m+1}$ equations, and each equation consists of $n-m+1$ terms. The general solution of such a system expresses each of the dependent variables as a Jacobian determinant of $n-m$ arbitrary functions of $x_1 \dots x_n$.

From the systems just described others are derived with their general solutions.

Some results* previously obtained by the author are in part simplified and in part generalized.

I.

1. The simplest case ($m = 1$) has been discussed by the author in the "Messenger of Mathematics."† There is a single equation

$$\frac{dz_1}{dx_1} + \frac{dz_2}{dx_2} + \dots + \frac{dz_n}{dx_n} = 0 \qquad (1),$$

and the general solution is

$$(-)^{i+1} z_i = \frac{d(y_1 \dots\dots\dots y_{n-1})}{d(x_1 \dots x_{i-1}, x_{i+1} \dots x_n)} \qquad (2),$$

where $y_1, \dots y_{n-1}$ are arbitrary functions of $x_1 \dots x_n$. That this is a solution of (1) is proved by substituting in (1). The expression on the left of (1) becomes

$$\begin{vmatrix} \frac{d}{dx_1}, & \frac{d}{dx_2}, & \dots\dots & \frac{d}{dx_n} \\ \frac{dy_1}{dx_1}, & \frac{dy_1}{dx_2}, & \dots\dots & \frac{dy_1}{dx_n} \\ \dots & \dots & & \dots \\ \frac{dy_{n-1}}{dx_1}, & \dots\dots\dots & & \frac{dy_{n-1}}{dx_n} \end{vmatrix}$$

which vanishes identically.

That the system (2) is the *general* solution of (1) is proved, by first showing that the suppositions

$$(-)^{i+1} z_i = \lambda \frac{d(y_1 \dots\dots\dots y_{n-1})}{d(x_1 \dots x_{i-1}, x_{i+1} \dots x_n)} \qquad (3)$$

leave $z_1 \dots z_n$ arbitrary. Introducing these values into (1), it is found

* Proc. London Math. Soc., Vol. IX., pp. 47—55.
† Vol. VII., pp. 107 *et sqq.*

that, to satisfy this equation, λ must be a function of $y_1 \dots y_{n-1}$ only; so that, when (1) is satisfied, (3) reduces to (2).

2. Before passing to the general investigation, it will perhaps be advantageous to deal with the case in which $m = 2$; as the discussion, otherwise unchanged, will be free from the complicated notation necessary when m is left undetermined.

In this case there are n equations:—

$$\left.\begin{array}{l}\dfrac{dz_{12}}{dx_2}+\dfrac{dz_{13}}{dx_3}+\dots+\dfrac{dz_{1n}}{dx_n}=0\\[2ex] \dfrac{dz_{21}}{dx_1}+\dfrac{dz_{23}}{dx_3}+\dots+\dfrac{dz_{2n}}{dx_n}=0\\ \dots\quad\dots\quad\dots\quad\dots\\ \dfrac{dz_{n1}}{dx_1}+\dfrac{dz_{n2}}{dx_2}+\dots+\dfrac{dz_{n,n-1}}{dx_{n-1}}=0\end{array}\right\}\quad\dots\dots\dots\dots(4).$$

The $n(n-1)$ dependent variables z_{ij} are subject to conditions of the form

$$\left.\begin{array}{c}z_{12}=-z_{21}\\ z_{12}\,.\,z_{34}-z_{13}\,.\,z_{24}+z_{14}\,.\,z_{23}=0\end{array}\right\}\quad\dots\dots\dots\dots(5).$$

If we suppose $$z_{ii}=0\quad\dots\dots\dots\dots(5'),$$

the second condition (5) includes the first. In virtue of (5), only $2n-3$ of the z's are mutually independent. For instance, from

$$z_{12},\ z_{13},\ \dots\dots\ z_{1n},$$
$$z_{23},\ \dots\dots\ z_{2n},$$

we can, by the second equation of (5), determine any other z.

The general solution of (4) is

$$z_{ij}=-z_{ji}=(-)^{i+j+1}\,.\,\frac{d(y_1\,\dots\dots\dots\dots\dots\dots\,y_{n-2})}{d(x_1\dots x_{i-1},\,x_{i+1}\dots x_{j-1},\,x_{j+1}\dots x_n)}\ \dots(6),$$
$$(i<j),$$

where $y_1, \dots y_{n-2}$ are $n-2$ arbitrary functions of $x_1 \dots x_n$. The solution may also be expressed by saying that z_{ij} is the coefficient of

$$\frac{d(\eta_1,\eta_2)}{d(x_i,x_j)},$$

in the expansion of $$\frac{d(\eta_1,\eta_2,y_1\dots y_{n-2})}{d(x_1,x_2,\dots\dots x_n)}$$

3. That the equations (6) do form a solution of (4) may be proved,

as in Art. 1, by substitution. If we replace z, in (4), by their values, the expressions on the left of (4) become the determinants of the matrix

$$\left\| \begin{matrix} \frac{d}{dx_1}, & \frac{d}{dx_2}, & \cdots\cdots & \frac{d}{dx_n} \\ \frac{dy_1}{dx_1}, & \frac{dy_1}{dx_2}, & \cdots\cdots & \frac{dy_1}{dx_n} \\ \cdots & \cdots & & \cdots \\ \frac{dy_{n-2}}{dx_1}, & \cdots\cdots\cdots & & \frac{dy_{n-2}}{dx_n} \end{matrix} \right\|,$$

each of which vanishes identically.

The values of z given by (6) also satisfy the algebraical relations (5). To prove this, replace $\frac{d}{dx_1}, \dots \frac{d}{dx_n}$, in the matrix last written, by a row of quantitative symbols $a_1, \dots a_n$. It is known that, if two (independent) determinants of the matrix thus formed vanish, then all the determinants of the matrix vanish. For example, if

$$\left. \begin{aligned} a_2 . z_{12} + a_3 . z_{13} + a_4 . z_{14} + \dots + a_n . z_{1n} &= 0 \\ a_1 . z_{21} + a_3 . z_{23} + a_4 . z_{24} + \dots + a_n . z_{2n} &= 0 \end{aligned} \right\} \quad \dots\dots\dots\dots\dots(7),$$

then also $\quad a_1 . z_{31} + a_2 . z_{32} + a_4 . z_{34} + \dots + a_n . z_{3n} = 0.$

But if from the first and second we eliminate a_3, we get

$$a_1 . z_{21} . z_{13} - a_2 . z_{12} . z_{23} + a_4 (z_{24} . z_{13} - z_{14} . z_{23}) + \&c. = 0,$$

which is consistent with the third equation only if we have

$$z_{13} . z_{24} - z_{14} . z_{23} = z_{12} . z_{34},$$

or

$$z_{12} . z_{34} - z_{13} . z_{24} + z_{14} . z_{23} = 0,$$

and the equations formed by replacing 4 herein by 5, ... n. But these are the equations (5).

It is to be noticed—and the remark will be useful in the sequel—that (5) are the sufficient conditions to ensure that, when the two equations (7) are satisfied, all the equations of the system are satisfied; upon the assumption that these two equations are independent, in other words that z_{12} does not vanish.

4. In proving that (6) is the general solution of (4), it is not possible to follow the course indicated in Art. 1. Instead of this, we shall discuss the equations of the preceding articles from a new point of view. We shall consider for the present that z_{ij} are known functions

of $x_1 \ldots x_n$, and shall regard the equations (6) as equations from which $y_1 \ldots y_{n-2}$ are to be determined. We assume that the given values of z_{ij} satisfy the conditions (4), (5); for, if they did not, no values of $y_1 \ldots y_{n-2}$ could satisfy the system (6).

If y represent any of the quantities $y_1 \ldots y_{n-2}$ or any function of them, then, identically,

$$\frac{d\,(y, y_1, \ldots y_{n-2})}{d\,(x_1 \ldots\ldots x_n)} = 0.$$

This is a system which may be written

$$\left.\begin{aligned} \Delta_1 y &= z_{12}\frac{dy}{dx_2} + z_{13}\frac{dy}{dx_3} + \ldots + z_{1n}\frac{dy}{dx_n} = 0 \\ \Delta_2 y &= z_{21}\frac{dy}{dx_1} + z_{23}\frac{dy}{dx_3} + \ldots + z_{2n}\frac{dy}{dx_n} = 0 \\ \Delta_3 y &= z_{31}\frac{dy}{dx_1} + z_{32}\frac{dy}{dx_2} + \ldots + z_{3n}\frac{dy}{dx_n} = 0 \\ \ldots & \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(8).$$

In virtue of the conditions (5), this system of n equations reduces to two only (Art. 3); and these two may be the first and second if only z_{12} does not vanish. But two equations such as (8) involving n independent variables cannot have more than $n-2$ common solutions; and, in order that they may have so many, certain conditions must be satisfied. These conditions—sufficient as well as necessary—may be written [using the Δ's as defined in (8)],

$$\frac{1}{z_{12}} \cdot \Delta_1 \frac{z_{23}}{z_{21}} = \frac{1}{z_{21}} \cdot \Delta_2 \frac{z_{13}}{z_{12}},$$

where the 3 is to be replaced by 4, ... n in succession. Observe now that the Δ's are lineal functions of $\frac{d}{dx_1} \ldots \frac{d}{dx_n}$; so that the equation just written may be reduced to the form

$$z_{12} \cdot \Delta_1 z_{23} - z_{12} \cdot \Delta_2 z_{13} - (z_{23} \cdot \Delta_1 - z_{13} \cdot \Delta_2)\, z_{12} = 0\,;$$

whence, since $$z_{23} \cdot \Delta_1 - z_{13} \cdot \Delta_2 = -\, z_{12} \cdot \Delta_3,$$

and z_{12} does not vanish,

$$\Delta_1 \cdot z_{23} - \Delta_2 \cdot z_{13} + \Delta_3 \cdot z_{12} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

But this equation, and others similar to it, are identically true in virtue of (4). For, written at length and rearranged, the left side of (9) is

$$
\begin{aligned}
&-z_{21}\frac{d}{dx_1}z_{13}+z_{31}\frac{d}{dx_1}z_{12}\\
&+z_{12}\frac{d}{dx_2}z_{23}+z_{32}\frac{d}{dx_2}z_{12}\\
&+z_{13}\frac{d}{dx_3}z_{23}-z_{23}\frac{d}{dx_3}z_{13}\\
&+z_{14}\frac{d}{dx_4}z_{23}-z_{24}\frac{d}{dx_4}z_{13}+z_{34}\frac{d}{dx_4}z_{12}\\
&+\&c.
\end{aligned}
$$

But, differentiating (5) with respect to x_p, we get

$$
\begin{aligned}
z_{14}\frac{d}{dx_p}z_{23}-z_{24}\frac{d}{dx_p}z_{13}+z_{34}\frac{d}{dx_p}z_{12}&\\
=-z_{23}\frac{d}{dx_p}z_{14}+z_{13}\frac{d}{dx_p}z_{24}-z_{12}\frac{d}{dx_p}z_{34}.&
\end{aligned}
$$

Making this substitution for $p=4, \dots n$ in succession in the expansion of the left side of (9) just written, and again rearranging, we find

$$
\left.
\begin{aligned}
&\Delta_1 z_{23}-\Delta_2 z_{13}+\Delta_3 z_{12}\\
=&-z_{12}\left\{\frac{dz_{31}}{dx_1}+\frac{dz_{32}}{dx_2}+\dots+\frac{dz_{3p}}{dx_p}+\dots\right\}\\
&+z_{13}\left\{\frac{dz_{21}}{dx_1}+\frac{dz_{23}}{dx_3}+\dots+\frac{dz_{2p}}{dx_p}+\dots\right\}\\
&-z_{23}\left\{\frac{dz_{12}}{dx_2}+\frac{dz_{13}}{dx_3}+\dots+\frac{dz_{1p}}{dx_p}+\dots\right\}
\end{aligned}
\right\}
\quad\dots\dots\dots\dots(10),
$$

and therefore vanishes identically by reason of (4).

Thus it appears that, because of (5), the system (8) reduces to two equations only; and because of (4) these two equations have $n-2$ common solutions, which indeed are found by integrating the integrable system

$$
\begin{aligned}
z_{23}dx_1-z_{13}dx_2+z_{12}dx_3&=0,\\
z_{24}dx_1-z_{14}dx_2+z_{12}dx_4&=0,\\
\&c.&
\end{aligned}
$$

Any set of $n-2$ solutions hence obtained will satisfy the equations (8) or (6).

5. The results of the last article have in effect proved that (6) is the general solution of (4). For it has been shown that, whatever $z_{ij}, \dots$ may be—subject only to the conditions (4), (5)—values of $y_1 \dots y_{n-2}$ can be found to satisfy the equations (6). But this is the same thing

as saying that every solution of (4), (5) is included in the form (6) when $y_1 \dots y_{n-2}$ are arbitrary, so that (6) is in fact the general solution of (4), (5).

6. At the commencement of Art. 4, it was remarked that the process employed to prove our solution to be general in the case of $m = 1$ was not applicable to the case in which $m = 2$. Indeed, if $z_{ij} \dots$ be taken as defined by (4), (5), the quantities $\lambda z_{ij}, \dots$ are not arbitrary; they are subject to certain conditions. If we write $z_{ij} \dots$ for $\lambda z_{ij}, \dots$, these may be written

$$\left.\begin{array}{l} z_{ij}\left\{\frac{dz_{k1}}{dx_1}+\frac{dz_{k2}}{dx_2}+\dots+\frac{dz_{kn}}{dx_n}\right\} \\ +z_{ki}\left\{\frac{dz_{j1}}{dx_1}+\frac{dz_{j2}}{dx_2}+\dots+\frac{dz_{jn}}{dx_n}\right\} \\ +z_{jk}\left\{\frac{dz_{i1}}{dx_1}+\frac{dz_{i2}}{dx_2}+\dots+\frac{dz_{in}}{dx_n}\right\}=0 \\ z_{ij}=-z_{ji},\ z_{ii}=0 \\ z_{ij}z_{kl}+z_{ik}z_{lj}+z_{il}z_{jk}=0 \\ i,j,k,l=1,2,\dots n \end{array}\right\} \dots\dots\dots\dots\dots(11);$$

and, as is suggested by the manner of obtaining this system, the general solution is

$$z_{ij} = -z_{ji} = \lambda \frac{d(y_1, y_2, \dots\dots\dots\dots\dots\dots y_{n-2})}{d(x_1 \dots x_{i-1}, x_{i+1}, \dots x_{j-1}, x_{j+1}, \dots x_n)},\ (i<j) \dots(12),$$

where $\lambda, y_1, \dots y_{n-2}$ are arbitrary functions of $x_1, \dots x_n$.

The results already obtained enable us to prove this without much difficulty.

First, to prove that (12) is a solution of (11), it is only necessary to observe that the differential equation in (11) is, by (10), merely a transformation of

$$\frac{1}{z_{12}}\Delta_1\frac{z_{23}}{z_{21}}=\frac{1}{z_{21}}\Delta_2\frac{z_{13}}{z_{12}},$$

or $$\left\{\frac{d}{dx_2}+\frac{z_{18}}{z_{12}}\cdot\frac{d}{dx_3}+\dots\right\}\frac{z_{23}}{z_{21}}=\left\{\frac{d}{dx_1}+\frac{z_{23}}{z_{21}}\cdot\frac{d}{dx_3}+\dots\right\}\frac{z_{13}}{z_{12}},$$

and therefore is not affected by a change of $z_{ij}, \dots$ into $\lambda z_{ij}, \dots$ for all values of i, j. The same is true of the other equations in (11), for they are homogeneous. But when the z's have the values given in (6), they satisfy the system (11). Hence also, when they have the values (12), namely when they are all multiplied by one and the same arbitrary function of $x_1 \dots x_n$, they still satisfy (11), though they no longer satisfy (4).

To prove the generality of the solution, it is only necessary to repeat the arguments of Art. 4. For the determination of $y_1, \ldots y_{n-2}$, we have the same equations (8) as before. We prove that (8) reduces to two equations by the algebraical relations in (11). The differential equation (11) shows that these two equations have $n-2$ common solutions, $y_1, \ldots y_{n-2}$, as we see from the identity (10). Hence, as in Art. 5, we infer that (12) is the general solution of (11).

II.

7. In the general case, every z has m subscripts. Should two of these subscripts be identical, the z vanishes; and a transposition of two subscripts changes the sign of the z. It will be convenient in most cases to represent $m-1$ of the subscripts by a single symbol. Thus $1, 2, \ldots m$, irrespective of order, might be represented by $\alpha 1$, or $\beta 2 \ldots$, or μm; where, for instance, α represents the subscripts $23 \ldots m$; $\beta = 13 \ldots m$; $\mu = 12 \ldots m-1$. To avoid ambiguity, we suppose that in any such group the subscripts are in order of magnitude.

By the use of this kind of abbreviation, it is easy to write down the algebraical relations connecting the z's. Let us select any m subscripts, say $1, 2, \ldots m$, and suppose these expressed as above. Choose also a second group πp, where p is a single subscript, π is a group of $m-1$ subscripts. Then the relations we seek may be written

$$z_{\pi 1} \cdot z_{\alpha p} - z_{\pi 2} \cdot z_{\beta p} + \ldots \pm z_{\pi m} \cdot z_{\mu p} \mp z_{\pi p} \cdot z_{\mu m} = 0 \quad \ldots\ldots\ldots(13),$$

the upper or lower sign being used according as m is odd or even.

By reason of these conditions, the number of mutually independent z's is reduced to $m(n-m)+1$. For instance, we can find the whole series if we know the z's which have the subscripts given in the following scheme, omitting those in brackets,

$$\left.\begin{array}{l} \alpha 1 \;\; ; \; \alpha, m+1 \; ; \; \ldots \; \alpha, p \; ; \; \ldots \; \alpha n \\ (\beta 2) \; ; \; \beta, m+1 \; ; \; \ldots \; \beta, p \; ; \; \ldots \; \beta n \\ \;\;\ldots \qquad \ldots \qquad \ldots \qquad \ldots \\ (\mu m) \; ; \; \mu, m+1 \; ; \; \ldots \; \mu, p \; ; \; \ldots \; \mu n \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots(14).$$

The groups $(\beta 2) \ldots (\mu m)$ are merely new arrangements of $\alpha 1$, and are therefore known when $\alpha 1$ is given. The other groups form m rows, and each row contains $n-m$ members; so that (14) includes $m(n-m)+1$ groups. All of these are different. For in the first row (we leave $\alpha 1$ out of sight for the present) the subscript 1 does not occur, in the second row no group comprises 2, and so on; but these subscripts occur in every other row. At the same time, every member of a column contains a subscript, $m+1, \ldots p, \ldots n$, which does not appear in any other column. Thus, for two groups to be identical, they must belong to the same column and to the same row.

The attempt to form any relation (13) between the z's indicated by (14) will fail. For, suppose that p is one of the numbers $1, 2, \dots m$; say $p = 1$. Then $z_{\beta p} = z_{\beta 1} = 0, \dots, \quad z_{\mu p} = z_{\mu 1} = 0$,
and (13) reduces to $\quad z_{\pi 1}\, z_{\alpha 1} \pm z_{\pi 1}\, z_{\mu m} = 0$
or $\quad z_{\alpha 1} \pm z_{\mu m} = 0.$
But the subscript μm is not included in the scheme (14).

If, on the other hand, p is not one of the numbers $1, 2, \dots m$, then π must be one of the groups $\alpha, \beta, \dots \mu$; for, otherwise, πp would be outside the scheme. But, if, for example, we put $p = \alpha$, we reproduce the equation $\quad z_{\alpha 1} \pm z_{\mu m} = 0.$

Thus it appears that the conditions (13) leave $m\,(n-m)+1$ of the z's mutually independent. It remains to show that, given the z's denoted in (14), all the other z's can be found by the help of (13). Now the groups in (14) comprise all arrangements of $m-1$ of the numbers $1, 2, \dots m$, with one of the numbers greater than m. Hence, if in (13) we take π as composed of $m-2$ of the first m numbers and one number greater than m, the groups $\pi 1, \pi 2, \dots \pi m$, are all included in (14), so that the first m terms o (13) are known. Also $z_{\mu m}$ is known; thus (13) becomes an equation to determine $z_{\pi p}$, a z which in its subscripts includes two of the numbers $m+1$, $m+2, \dots n$. By properly choosing p, π, we can thus determine all such z's. But now we may take π as a group involving two of the numbers above m. For $z_{\pi 1}, z_{\pi 2}, \dots z_{\pi m}$, whose subscripts involve only two numbers above m, are all known. In this case (13) becomes an equation for determining all the z's whose subscripts involves three numbers greater than m. By repetition of this process, it is evident that we can successively form all the z's if we know those given in (14). Hence only $m\,(n-m)+1$ are mutually independent.

8. The system of differential equations to be solved is

$$\frac{dz_{\pi 1}}{dx_1} + \frac{dz_{\pi 2}}{dx_2} + \dots + \frac{dz_{\pi n}}{dx_n} = 0 \quad \dots\dots\dots\dots\dots\dots (15),$$

where π is any combination of $1, 2, \dots n$ taken $m-1$ together. There are therefore $\dfrac{\lfloor n}{\lfloor m-1 \, \lfloor n-m+1}$ equations in the system. In each equation there are $n-m+1$ terms,—viz., one term for each subscript not included in π. There are $\dfrac{\lfloor n}{\lfloor m-1 \, \lfloor n-m}$ dependent variables. These being, however, subject to the conditions of Art. 7, $m\,(n-m)+1$, and no more of them, are mutually independent.

The general solution is

$$z_{ij\dots k} = (-)^{i+j+\dots k+\frac{m\,.\,m+1}{2}} \, \frac{d\,(y_1 \dots y_{n-m})}{d\,(x_1 \dots\dots x_n)}, \; i<j \dots <k \; \dots\dots (16),$$

where, in the range $x_1 \dots x_n$, $x_i, x_j, \dots x_k$ are omitted, and $y_1, \dots y_{n-m}$ are arbitrary functions of $x_1 \dots x_n$. The sign is $+$ or $-$ according as $i, j \dots k, 1, \dots n$, is reducible to the natural order $1, 2, \dots n$ by an even or an odd number of transpositions. The general solution may also be expressed by saying that $z_{ij\dots k}$ is the coefficient of $\frac{d(\eta_1, \dots \eta_m)}{d(x_i, x_j \dots x_k)}$ in the expansion of

$$\frac{d(\eta_1, \dots \eta_m, y_1, \dots y_{n-m})}{d(x_1, \dots\dots\dots\dots x_n)}.$$

Of course the conditions $i < j < \dots < l$ are not assumed here.*

---

* *Note by Prof. Cayley.*

The general result may be illustrated as follows. Take, for instance, $n = 7$; the problem is to find the most general values of $\left(\frac{7.6.5}{1.2.3} = \right)$ 35 quantities $y_{123}, \dots y_{567}$ (functions of the independent variables $x_1, x_2, x_3, x_4, x_5, x_6, x_7$) satisfying certain algebraical relations, and also certain differential relations.

It is convenient to assume that any one, for instance y_{123}, of the 35 y's, may be written under the different forms

$$y_{123}, y_{231}, y_{312}, -y_{132}, -y_{321}, -y_{213},$$

(and so in other cases, the sign being always + or − according to the number of interchanges,) that is, y_{231}, $-y_{132}$, &c. each mean y_{123}, and the number of the y's is thus = 35, as stated above.

The algebraical relations are the whole system of algebraical relations implied in the assumption that the y's are severally equal to the determinants formed out of the matrix

$$\left\| \begin{matrix} \frac{dP}{dx_1}, & \frac{dP}{dx_2} & \dots\dots & \frac{dP}{dx_7} \\ \frac{dQ}{dx_1}, & \frac{dQ}{dx_2} & \dots\dots & \frac{dQ}{dx_7} \\ \frac{dR}{x_1}, & \frac{dR}{dx_2} & \dots\dots & \frac{dR}{dx_7} \\ \frac{dS}{dx_1}, & \frac{dS}{dx_2} & \dots\dots & \frac{dS}{dx_7} \end{matrix} \right\|$$

where P, Q, R, S are arbitrary functions of $x_1, x_2 \dots x_7$, viz., $y_{123} = \pm$ the determinant formed with columns 4, 5, 6, 7 out of this matrix, or in a convenient notation

$$y_{123} = \pm \frac{\delta(P, Q, R, S)}{\delta(x_4, x_5, x_6, x_7)}, = \pm \frac{\delta(P, Q, R, S)}{\delta_{4567}(x, x, x, x)},$$

the $\pm$ sign being that answering to the arrangement 123 . 4567, considering as derived from 1234567 assumed to be positive; for the equation written down, the sign is of course +.

The differential equations are the *whole* system of differential equations such as

$$\frac{d}{dx_3} y_{123} + \frac{d}{dx_4} y_{124} + \frac{d}{dx_5} y_{125} + \frac{d}{dx_6} y_{126} + \frac{d}{dx_7} y_{127} = 0,$$

and the theorem is that the *general* values of y, satisfying the algebraical relations, and also these differential equations, are the above values

$$y_{123} = \pm \frac{\delta(P, Q, R, S)}{\delta_{4567}(x, x, x, x)},$$

where P, Q, R, S are arbitrary functions of $x_1, x_2, \dots x_7$.

9. As in Art. 3, it can be proved that the forms of z given by (16) satisfy the given equations (15); these reducing in fact to the identities—

$$\left\| \begin{matrix} \dfrac{d}{dx_1}, & \dfrac{d}{dx_2}, & \cdots\cdots & \dfrac{d}{dx_n} \\ \dfrac{dy_1}{dx_1}, & \dfrac{dy_1}{dx_2}, & \cdots\cdots & \dfrac{dy_1}{dx_n} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{dy_{n-m}}{dx_1}, & \cdots & \cdots & \dfrac{dy_{n-m}}{dx_n} \end{matrix} \right\| = 0.$$

That the conditions (13) are also fulfilled, may be proved in the same manner as in Art. 3. In the matrix just written we replace $\dfrac{d}{dx_1}, \dots \dfrac{d}{dx_n}$ by a row of quantities, $A_1, \dots A_n$. Suppose that

$$\frac{d\,(y_1, \dots\dots y_{n-m})}{d\,(x_{m+1}, \dots\dots x_n)}$$

does not vanish. Then it is known that, if all the determinants vanish which are formed by taking each of the first m columns in turn with the last $n-m$ columns, every determinant of the matrix anishes. That is, if

$$\left.\begin{aligned} A_1 . z_{\alpha 1} + A_{m+1} . z_{\alpha, m+1} + \dots + A_p . z_{\alpha p} + \dots + A_n . z_{\alpha n} &= 0 \\ A_2 . z_{\beta 2} + A_{m+1} . z_{\beta, m+1} + \dots + A_p . z_{\beta p} + \dots + A_n . z_{\beta n} &= 0 \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad & \dots \\ A_m . z_{\mu m} + A_{m+1} . z_{\mu, m+1} + \dots + A_p . z_{\mu p} + \dots + A_n . z_{\mu n} &= 0 \end{aligned}\right\},$$

then also $\quad A_1 . z_{\pi 1} + A_2 . z_{\pi 2} + \dots + A_p . z_{\pi p} + \dots + A_n . z_{\pi n} = 0.$

Here $\alpha, \beta, \dots \mu, \pi$ have the meanings attributed to them in Art. 7. But this implies that, multiplying the first m equations by $\lambda_1, \lambda_2, \dots \lambda_m$ respectively, and adding, we shall, upon suitable choice of $\lambda_1, \dots \lambda_m$, produce the last equation. Suppose this done, and compare coefficients of $A_1, A_2, \dots A_m$. Thus

$$\lambda_1 z_{\alpha 1} = z_{\pi 1},\ \lambda_2 z_{\beta 2} = z_{\pi 2},\ \dots\ \lambda_m z_{\mu m} = z_{\pi m}.$$

Also, comparing the coefficients of A_p,

$$\lambda_1 z_{\alpha p} + \lambda_2 z_{\beta p} + \dots + \lambda_m z_{\mu p} = z_{\pi p}.$$

Replace herein λ_1, λ_2, &c. by their values just found, remembering that $z_{\alpha 1} = -z_{\beta 2} = \dots = \pm z_{\mu m}$, and we get

$$z_{\pi 1} . z_{\alpha p} - z_{\pi 2} . z_{\beta p} + \dots \pm z_{\pi m} . z_{\mu p} \mp z_{\pi p} . z_{\mu m} = 0,$$

which is (13).

Thus the proof is complete that (16) is a solution of (15), (13).

10. We now regard (16) as a system of equations for the determination of $y_1, \dots y_{n-m}$. It is assumed that the z's satisfy the conditions

(15), (13), because these have just been proved essential to the existence of such forms for z as are given by (16). The functions $y_1, \dots y_{n-m}$ satisfy the equations, found as in Art. 4,

$$\left.\begin{array}{l}
\Delta_\alpha y = z_{\alpha 1}\dfrac{dy}{dx_1}+\dots+z_{\alpha p}\dfrac{dy}{dx_p}+\dots+z_{\alpha n}\dfrac{dy}{dx_n}=0\\
\Delta_\beta y = z_{\beta 2}\dfrac{dy}{dx_2}+\dots+z_{\beta p}\dfrac{dy}{dx_p}+\dots+z_{\beta n}\dfrac{dy}{dx_n}=0\\
\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\\
\Delta_\mu y = z_{\mu m}\dfrac{dy}{dx_m}+\dots+z_{\mu p}\dfrac{dy}{dx_p}+\dots+z_{\mu n}\dfrac{dy}{dx_n}=0\\
\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots
\end{array}\right\}\dots\dots\dots(17).$$

and
$$z_{\pi 1}\frac{dy}{dx_1}+\dots+z_{\pi p}\frac{dy}{dx_p}+\dots+z_{\pi n}\frac{dy}{dx_n}=0$$

represents any equation of the system.

We first show that, supposing $z_{\alpha 1}$ does not vanish, all the equations of the system are algebraical consequences of the first m equations, viz. (17). This follows at once from the fact that the z's satisfy (1). For, multiplying the first equation by $z_{\pi 1}$, the second by $-z_{\pi 2}$, the third by $z_{\pi 3}$, and so on, and adding the results, we get, by reason of (13),

$$z_{\alpha 1}\left\{z_{\pi 1}\frac{dy}{dx_1}+z_{\pi 2}\frac{dy}{dx_2}+\dots+z_{\pi p}\frac{dy}{dx_p}+\dots\right\}=0.$$

But, by hypothesis, $z_{\alpha 1}$ does not vanish. Hence the expression within brackets vanishes. But this is equivalent to saying, any equation of the system is satisfied.

11. The system (17), consisting of m mutually independent equations, and involving n independent variables, cannot have more than $n-m$ different solutions, nor can it have so many unless Boole's conditions are fulfilled. One of these conditions is

$$\frac{1}{z_{\alpha 1}}\cdot\Delta_\alpha\frac{z_{\beta q}}{z_{\beta 2}}=\frac{1}{z_{\beta 2}}\cdot\Delta_\beta\frac{z_{\alpha q}}{z_{\alpha 1}}\dots\dots\dots\dots\dots\dots\dots\dots(18),$$

and the rest are similar. But this condition becomes

$$z_{\alpha 1}\cdot\Delta_\alpha\cdot z_{\beta q}-z_{\alpha 1}\cdot\Delta_\beta\cdot z_{\alpha q}-\{z_{\beta q}\cdot\Delta_\alpha-z_{\alpha q}\cdot\Delta_\beta\}z_{\alpha 1}=0\ \dots\dots(18').$$

In this expression let us select the terms involving $\dfrac{d}{dx_p}$. They are

$$z_{\alpha 1}\cdot z_{\alpha p}\cdot\frac{d}{dx_p}\cdot z_{\beta q}-z_{\alpha 1}\cdot z_{\beta p}\cdot\frac{d}{dx_p}\cdot z_{\alpha q}-(z_{\beta q}\cdot z_{\alpha p}-z_{\alpha q}\cdot z_{\beta p})\cdot\frac{d}{dx_p}\cdot z_{\alpha 1}.$$

To simplify this expression we revert to (13), and assume therein

$\pi = 34 \ldots mq$. Then

$$z_{\pi 3} = z_{\pi 4} = \ldots = z_{\pi m} = 0,$$
$$z_{\pi 1} = z_{34 \ldots mq1} = \pm z_{134 \ldots mq} = \pm z_{\beta q},$$
$$z_{\pi 2} = z_{34 \ldots mq2} = \pm z_{234 \ldots mq} = \pm z_{\alpha q},$$

the upper or lower sign being taken in each case according as m is odd or even. Then (13) becomes

$$z_{\beta q} \cdot z_{\alpha p} - z_{\alpha q} \cdot z_{\beta p} \mp z_{\pi p} \cdot z_{\alpha 1} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (19),$$

the upper and lower signs still corresponding to m odd or even. Hence the expression we have to simplify becomes, on dividing by $z_{\alpha 1}$ which does not vanish,

$$z_{\alpha p} \cdot \frac{d}{dx_p} \cdot z_{\beta q} - z_{\beta p} \cdot \frac{d}{dx_p} \cdot z_{\alpha q} \mp z_{\pi p} \cdot \frac{d}{dx_p} \cdot z_{\alpha 1}.$$

But, differentiating (19) with respect to x_p, it appears that this last expression is identically equal to

$$-z_{\beta q} \cdot \frac{d}{dx_p} \cdot z_{\alpha p} + z_{\alpha q} \cdot \frac{d}{dx_p} \cdot z_{\beta p} \pm z_{\alpha 1} \cdot \frac{d}{dx_p} \cdot z_{\pi p}.$$

Summing this for all values of p from 1 to n inclusive, we get, for the value of the left side of (18′),

$$\left.\begin{array}{l} -z_{\beta q} \left\{ \dfrac{dz_{\alpha 1}}{dx_1} + \ldots + \dfrac{dz_{\alpha p}}{dx_p} + \ldots + \dfrac{dz_{\alpha n}}{dx_n} \right\} \\ +z_{\alpha q} \left\{ \dfrac{dz_{\beta 2}}{dx_2} + \ldots + \dfrac{dz_{\beta p}}{dx_p} + \ldots + \dfrac{dz_{\beta n}}{dx_n} \right\} \\ +z_{\rho m} \left\{ \dfrac{dz_{\pi 1}}{dx_1} + \ldots + \dfrac{dz_{\pi p}}{dx_p} + \ldots + \dfrac{dz_{\pi n}}{dx_n} \right\} \end{array} \right\} \ldots\ldots\ldots\ldots\ldots (20),$$

which identically vanishes by (15), proving that the conditions (18) are satisfied. Hence the system (17) for y admits of $n-m$ integrals found from the equations

$$z_{\alpha 1} dx_p - z_{\alpha p} dx_1 + z_{\beta p} dx_2 - \ldots \pm z_{\mu p} dx_m = 0.$$

In fine, then, it appears that, whatever the z's may be, we can always express them in the form (16), provided only the conditions (13), (15) are satisfied; in other words, no restriction upon the forms of z is imposed by (16) which is not also implied in (13), (15). In Art. 9, on the other hand, it has been shown that the equations (16) do not include any forms of z excluded by (13), (15). Thus the two definitions—the one given by (13), (15), the other by (16)—are precisely equivalent. We may express this by saying that (16) is the general solution of (13), (15); or, looking at the matter from another standpoint, that (13), (15) are the necessary and sufficient conditions that the transformations (16) should be possible.

12. No difficulty arises in extending the results of Art. 6 to the general case. Indeed, the reasoning of that article may be applied to the present case almost without change. Only it is desirable to modify the notation employed in (20) so as to put in evidence the relation between the symbols α, β, π, q. For this purpose we represent by ρ the group $34 \ldots m$, whence we get

$$\alpha = 2\rho,\ \beta = 1\rho,\ \pi = \rho q,$$

$$\beta q = 1\rho q,\ \alpha q = 2\rho q,\ \mu m = 12\rho = \rho 12.$$

Also observe that

$$z_{\beta q} \cdot \frac{dz_{\alpha p}}{dx_p} = z_{1\rho q} \cdot \frac{dz_{2\rho p}}{dx_p} = z_{\rho 1 q} \cdot \frac{dz_{\rho 2 p}}{dx_p},$$

since the same change of sign is effected in both factors. Thus the equation formed by expressing that (20) vanishes, takes the symmetrical form,

$$z_{\rho 2 q} \Sigma_p \frac{dz_{\rho 1 p}}{dx_p} - z_{\rho \cdot q} \Sigma_p \frac{dz_{\rho 2 p}}{dx_p} + z_{\rho 12} \Sigma_p \frac{dz_{\rho q p}}{dx_p} = 0 \qquad (21).$$

Supposing the z's to be still subject to (13), the general solution of the system of equations formed from (21) by making ρ, q take all values, is

$$z_{ij \ldots k} = (-)^{i+j+\ldots+k} \lambda \frac{d(y_1, \ldots y_{n-m})}{d(x_1, \ldots x_n)} \qquad (22),$$

where, in the range $x_1, \ldots x_n$, $x_i, x_j \ldots x_k$ are omitted, and λ, $y_1, \ldots y_{n-m}$ are arbitrary functions of $x_1, x_2, \ldots x_n$.

III.

13. When m becomes greater than $n-m$, and especially when the latter is small, the notation for the z's employed in the previous articles may be conveniently replaced by a complementary notation. Instead of writing m subscripts, we should write the $n-m$ subscripts that would have been omitted. To avoid confusion we shall mark the z's with an accent when this new notation is employed.

It is necessary to make some convention as to the order of subscripts to a z' which is equivalent to a given z; and it obviously suggests itself that in

$$z_{ij \ldots k} = \pm z'_{\iota \kappa \ldots \lambda} \qquad (23)$$

the upper or lower sign should be used according as the order

$i, j, \dots k, \iota, \kappa, \dots \lambda$ is reducible to $1, 2, \dots n$ in an even or an odd number of transpositions.

The propriety of this convention is confirmed by its effect upon the transformation of (16). Suppose that $ij \dots k\, \iota\kappa \dots \lambda$ is reducible to the order $12 \dots n$ by an even number of transpositions, so that in (23) the upper sign is used. Then, from the way in which the sign of (16) was determined, it follows that

$$z_{ij\dots k} = + \frac{d\,(y_1 \dots y_{n-m})}{d\,(x_\iota, x_\kappa, \dots x_\lambda)}.$$

If, on the other hand, the reduction requires an odd number of transpositions, so that the lower sign is used in (23), we should have

$$z_{ij\dots k} = - \frac{d\,(y_1 \dots y_{n-m})}{d\,(x_\iota, x_\kappa, \dots x_\lambda)}.$$

In each case $$z'_{\iota\kappa\dots\lambda} = \frac{d\,(y_1 \dots y_{n-m})}{d\,(x_\iota, x_\kappa, \dots x_\lambda)} \quad\dots\dots\dots\dots(24).$$

14. The transformation of equations such as (13) or (21), where the subscripts of the z's include group symbols, depends upon the following considerations. Let $$z_\pi = z'_\rho,$$
so that $$\pi\rho = 1, 2, \dots n.$$

(The sign of equality is here used to imply that each of the two groups is reducible to the other in an even number of transpositions.) Also, a being another symbol representing a group of subscripts, let

$$z_{\pi a} = z'_\sigma.$$

Then $$\pi a \sigma = 1, 2, \dots n = \pi\rho,$$
$$\therefore\ a\sigma = \rho.$$

This means that the subscript σ is formed from the subscript ρ by removing from the left of the latter the group a.

For instance, let $n = 6$, $\pi = 5$; then

$$z_\pi = z_5 = z'_{12346}.$$

Hence $$z_{\pi 12} = z'_{346}.$$

Again, to transform $z_{\pi 36}$, observe that

$$z'_{12346} = z'_{36214},$$

so that $$z_{536} = z_{\pi 36} = z'_{214}.$$

If any numbers are common to π, a,

$$z_{\pi a} = 0.$$

Hence, if we have to remove from the subscript of z' a number not in it, that z' vanishes. Say, for example, we wish to transform $z_{\pi 5}$, π itself containing 5. In the subscript of z' corresponding to z_π there is no 5 to be removed; so that we infer the z' vanishes.

15. Consider now the case in which $n-m=1$. In the equation (15) the group symbol π comprises $m-1$, that is $n-2$, numbers.

Suppose $$z_\pi = z'_{ij};$$

then $$z_{\pi i} = z'_j; \quad z_{\pi j} = -z'_i,$$

and all the other z''s vanish. Thus (15) becomes

$$\frac{dz'_j}{dx_i} - \frac{dz'_i}{dx_j} = 0,$$

and the system consists of all the equations that can be formed by giving i, j all values from 1 to n inclusive. From (24) the general solution of the system is (since $n-m=1$)

$$z'_i = \frac{dy_1}{dx_i},$$

where y_1 is an arbitrary function of $x_1 \ldots x_n$.

Take now the equation (21). The group symbol ρ comprises $m-2$, that is $n-3$, numbers. Let

$$z_\rho = z_{12q}.$$

Then $$z_{\rho 2q} = z'_1, \; z_{\rho 12} = z'_q, \; z_{\rho 13} = 0, \; \ldots \; z_{\rho 1q} = -z'_2 \ldots,$$

&c. &c.,

so that (21) becomes

$$z'_1\left(\frac{dz'_q}{dx_2} - \frac{dz'_2}{dx_q}\right) + z'_2\left(-\frac{dz'_q}{dx_1} + \frac{dz'_1}{dx_q}\right) + z'_q\left(\frac{dz'_2}{dx_1} - \frac{dz'_1}{dx_2}\right) = 0,$$

and the general solution of the whole system is, by (22),

$$z'_i = \lambda \frac{dy_1}{dx},$$

where λ, y_1 are arbitrary functions of $x_1 \ldots x_n$.

16. Next suppose $n-m=2$.

In this case it is necessary to take the equations (13) into account.

Let us put $z_\pi = z'_{123}$, so that $z_{\pi 1} = z'_{23}, \; z_{\pi 2} = -z'_{13}, \; z_{\pi 3} = z'_{12}$,

and all the others vanish. Then put $p=5$, and let

$$z_{\alpha 1} = -z_{\beta 2} = z_{\gamma 3} = \text{\&c.} = z'_{45},$$

$$\therefore \; z_\alpha = z'_{145}, \text{ and } z_{\alpha 5} = z'_{14},$$

$$z_\beta = -z'_{215}, \; \therefore \; z_{\beta 5} = -z'_{24},$$

$$z_\gamma = z'_{345}, \; \therefore \; z_{\gamma 5} = z'_{34}.$$

Substituting these values in (13), it becomes

$$z'_{23} \,.\, z'_{14} - z'_{13} \,.\, z'_{24} + z'_{12} \,.\, z'_{34} = 0.$$

The other conditions may be got in the same way, or may be inferred by changing the suffixes. It will be observed that the relations are the same as were obtained in the case of $m = 2$; and there is no difficulty in proving that the relations for z' may be obtained by the process described in Art. 7.

The differential equations (15) become

$$\frac{dz'_{jk}}{dx_i} - \frac{dz'_{ik}}{dx_j} + \frac{dz'_{ij}}{dx_k} = 0 \;\; (i, j, k = 1 \dots n),$$

and the general solution of this system is

$$z'_{ij} = \frac{d\,(y_1, y_2)}{d\,(x_i, x_j)}.$$

The system (21) is replaced by

$$\begin{aligned} & z'_{il} \left\{ \frac{dz'_{kl}}{dx_i} - \frac{dz'_{jl}}{dx_k} + \frac{dz'_{jk}}{dx_l} \right\} \\ - & z'_{jl} \left\{ \frac{dz'_{kl}}{dx_i} - \frac{dz'_{il}}{dx_k} + \frac{dz'_{ik}}{dx_l} \right\} \\ + & z'_{kl} \left\{ \frac{dz'_{jl}}{dx_i} - \frac{dz'_{il}}{dx_k} + \frac{dz'_{ij}}{dx_l} \right\} = 0, \\ & (i, j, k, l = 1, \dots n,) \end{aligned}$$

and the general solution of the system is

$$z'_{ij} = \lambda \frac{d\,(y_1, y_2)}{d\,(x_i, x_j)},$$

where λ, y_1, y_2, are arbitrary functions of $x_1 \dots x_n$.

IV.

17. Some further notice seems to be required of the fact that a process for proving the generality of the solution (16), which was successfully used in the case of $m = 1$, failed when $m = 2$. In the case of $m = 1$, we had

$$z_i = (-)^{i-1} \frac{d\,(y_1 \dots y_{n-1})}{d\,(x_1 \dots x_{i-1} x_{i+1} \dots x_n)},$$

so that z_i involves $n-1$ arbitrary functions of $x_1 \dots x_n$. If we introduce a new arbitrary function λ, as a coefficient to the Jacobian, the z's would involve n arbitrary functions of n variables $x_1 \dots x_n$, and, as we have seen, become themselves arbitrary.

In the case of $m = 2$ on the other hand, the z's involve only $n-2$ arbitrary functions. Hence, by the introduction of one new arbitrary function, they become dependent upon $n-1$ arbitrary functions, so that they cannot be themselves arbitrary functions of the n variables $x_1 \dots x_n$. But the question arises, whether we cannot, by introducing *two* (or, generally, m) new arbitrary functions into the values of z, make these arbitrary. If so, how are they to be distributed? The question is not without importance. For it will be remembered that the equations (10), (21), and the general solutions thereof, were suggested by the introduction of a supplementary arbitrary function such as had, in the case of $m = 1$, made z_i arbitrary. Judging from analogy, one would be led to suppose that, by introducing r new functions in a suitable manner, then we should get forms of z which for $m = r$ would be unconditioned, but which when $m > r$ would satisfy certain systems of differential equations; these systems would moreover be of the same form for all values of m greater than r [compare (21) for instance].

Nor are there wanting indications of the answers to be given to these two questions:—How are the supplementary functions to be distributed? and to what forms of differential equations are we thus led? In fact, in the case in which $n-m = 1$, a complete answer to each question has been given; and, as far as it goes, it confirms the remarks made above. We have already deduced, as particular cases of our general results, that the system

$$\frac{dz_i}{dx_j} - \frac{dz_j}{dx_i} = 0 \ (i, j = 1, 2 \dots n) \dots\dots\dots\dots\dots\dots (25)$$

implies
$$z_i = \frac{dy_1}{dx_i};$$

and that

$$z_i \left\{ \frac{dz_j}{dx_k} - \frac{dz_k}{dx_j} \right\} - z_j \left\{ \frac{dz_i}{dx_k} - \frac{dz_k}{dx_i} \right\} + z_k \left\{ \frac{dz_i}{dx_j} - \frac{dz_j}{dx_i} \right\} = 0 \dots (26)$$
$$(i, j, k = 1, 2 \dots n)$$

implies
$$z_i = \lambda \frac{dy_1}{dx_i};$$

the two statements in each case being precisely equivalent. But, as is well known, these results can be extended. For instance, the system

$$\left\{ \frac{dz_i}{dx_j} - \frac{dz_j}{dx_i} \right\} \left\{ \frac{dz_k}{dx_l} - \frac{dz_l}{dx_k} \right\}$$
$$- \left\{ \frac{dz_i}{dx_k} - \frac{dz_k}{dx_i} \right\} \left\{ \frac{dz_j}{dx_l} - \frac{dz_l}{dx_j} \right\}$$
$$+ \left\{ \frac{dz_i}{dx_l} - \frac{dz_l}{dx_i} \right\} \left\{ \frac{dz_j}{dx_k} - \frac{dz_k}{dx_j} \right\} = 0 \dots\dots\dots\dots (27)$$

is precisely equivalent to this other system

$$z_i = \lambda \frac{dy_1}{dx_i} + \frac{dy_2}{dx_i}.$$

If now from (27) we form a new system in the same way as (26) was derived from (25), this will be equivalent to

$$z_i = \lambda_1 \frac{dy_1}{dx_i} + \lambda_2 \frac{dy_2}{dx_i}.$$

This process may be repeated until z_i involves n arbitrary functions $\lambda_1, y_1, \lambda_2, y_2, \ldots,$ when it becomes itself arbitrary.

These results suggest what appears to be a very probable answer to the questions proposed. We infer that, when $m = 2$,

$$\pm z = \lambda \frac{d(y_1, y_{m+2} \ldots y_n)}{d(x_1 \ldots\ldots\ldots x_n)} + \frac{d(y_2, y_{m+2} \ldots y_n)}{d(x_1 \ldots\ldots\ldots x_n)}$$

is arbitrary, involving as it does n arbitrary functions $\lambda, y_1, y_2, y_4 \ldots y_n$. When, however, m is greater than 2, z ceases to be arbitrary, and satisfies a system of equations analogous to (27). If we represent the equation (15) by the symbol

$$(\pi) = 0,$$

one of the equations of the system would be

$$(\rho ij)(\rho kl) - (\rho ik)(\rho jl) + (\rho il)(\rho jk) = 0 \quad \ldots\ldots\ldots\ldots (28),$$

where, of course, ρ is a symbol for a group containing $m-3$ letters.

Again, the supposition

$$z = \lambda_1 \frac{d(y_1, y_{m+2}, \ldots y_n)}{d(x_1 \ldots\ldots\ldots x_n)} + \lambda_2 \frac{d(y_2, y_{m+2}, \ldots y_n)}{d(x_1 \ldots\ldots\ldots x_n)}$$

leaves z arbitrary unless $m > 3$. If this inequality holds, then z would satisfy equations derived from (28), just as (26) is obtained from (25).

And so on.

I have ventured to make these remarks without a formal proof (a deficiency I hope soon to amend), because I cannot help thinking that the evidence in favour of them is of the strongest. In the first place, when z is a Jacobian, we have the equations (15) for the whole range of m. When z is proportional to a Jacobian, we have the equations (21) from $m = 2$ onwards; and the persistence of its form is very remarkable. Then, there is the *à priori* probability that, z being composed of $n-m$ arbitrary functions, the proper introduction of m more arbitrary functions should make z an arbitrary function of the n independent variables—a probability which becomes a certainty in the extreme cases $m = 1$, $m = n-1$. At one end of the series $(m = n-1)$ we have the complete set of systems of differential equations formed in succession by an alternating rule, and a set of corresponding general solutions which can also be formed in succession by an alternating process. The first step in each of these processes has been proved to hold good for all values of m. Moreover, it is difficult to believe that the persistence of form noted above should not hold good of the higher systems; and there is a confirmation of the opposite hypothesis in the fact that the differential equations formed by analogy break off at the right place. For example, in (28) we have a group symbol, ρ, involving $m-3$ numbers; so that m in this case cannot be less than 3, which is the limit otherwise arrived at. Add to these the close analogy between Jacobians and differential coefficients, and a body of evidence is completed which is scarcely less convincing than actual proof.

18. In a paper,—"On Partial Differential Equations of the first order with several Dependent Variables,"—published in the Society's "Proceedings,"* I have discussed the conditions that

$$\Sigma z_{ij\ldots k} \frac{d\,(u_1, \ldots u_m)}{d\,(x_i x_j \ldots x_k)} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (29)$$

should be expressible in the form

$$\lambda \frac{d\,(y_1, \ldots y_{n-m}, u_1, \ldots u_m)}{d\,(x_1 \ldots\ldots\ldots\ldots\ldots\ldots x_n)} = 0.$$

The conditions there obtained were left in a form equivalent to (18); and it seems desirable to draw attention to the simpler form (21), (13) in which they can be expressed. The conditions that the left side of (29) should itself be a Jacobian, *i.e.*, not merely a product of a Jacobian into an arbitrary function, are given by the systems (13), (15). The same sets of conditions are necessary and sufficient to ensure that a system of simultaneous equations such as (29) should reduce to the form

$$\lambda \frac{d\,(y_1 \ldots y_r, u_1, \ldots u_m)}{d\,(x_1 \ldots\ldots\ldots\ldots\ldots\ldots x_n)} = 0,$$

where $m+r < n$.

Should the remarks of Art. 17 be confirmed, the theory of such equations as (29) will be considerably simplified: for these results would enable us to effect transformations of expressions such as that on the left of (29) in manner not unlike the transformations of

$$\Sigma y_1 dx_1,$$

first effected by Pfaff.

* Vol. IX., pp. 41—61.

February 13th, 1879.

C. W. MERRIFIELD, Esq., F.R.S., President, in the Chair.

Sir James Cockle, F.R.S., was admitted into the Society, and Mr. R. Hargreaves, M.A., Fellow of St. John's College, Cambridge, and Prof. W. E. Story, Ph.D. Leipsic, were proposed for election.

The following communications were made :—

"On the number of Conics which satisfy Five Independent Conditions," by M. Halphen (communicated by Dr. Hirst).

"Construction of Magic Squares," by Sir J. Cockle, F.R.S. (Messrs. Cayley, Henrici, Hart, Harley, and others took part in a discussion on the communication.)

"Notes on Frames," by Prof. Henrici, F.R.S.

"On a Modular Equation," and "On the Formula for four Abelian Functions answering to the Formula for four Theta Functions," by Prof. H. J. S. Smith, F.R.S.

"Quaternion Proof of Minding's Theorem," by Mr. J. J. Walker.

The following presents were made to the Society :—

"Proceedings of the Royal Society," Vol. xxviii., No. 192.

"Educational Times," February, 1879.

"Monatsbericht," Nov. 1878; Berlin, 1879.

"Mémoires de la Société des Sciences Physiques et Naturelles de Bordeaux," 2e Série, Tome iii, 1er cahier; Paris, 1878.

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