Hence, by addition, $\left\{e^2\sqrt{\dfrac{1-x^2}{1-e^2x^2}}+\sqrt{\dfrac{1-e^2x^2}{1-x^2}}\right\}dx$

$$=\left\{\left(\frac{(1+e)^2-\dfrac{4ey^2}{(1+e)^2}}{(1+e)^2-y^2}\right)^{\frac{1}{2}}+\frac{(1-e^2)^2}{\{(1+e)^2-y^2\}^{\frac{1}{2}}\left\{(1+e)^2-\dfrac{4ey^2}{(1+e)^2}\right\}^{\frac{1}{2}}}\right\}dy.$$

On the Oscillations of a Viscous Spheroid. By H. Lamb, M.A.

[*Read Nov.* 10*th*, 1881.]

The chief problem discussed in this paper is the determination of the effect of viscosity on the gravitational oscillations of a spheroidal mass of liquid, but the solution can be readily adapted to cases where the tendency to the spherical form is due to other causes; for instance, it applies to the vibrations of a globule of water under the influence of the tension of the superficial film. I have put together, in Part I., some preliminary results of analysis; the application to the problem in hand then follows in Part II. The results of Part I. have an interest extending beyond the present application; they are useful, for instance, in the theory of the vibrations of an elastic sphere, and in the theory of the induction of electric currents in a spherical conductor.

I.

I propose to investigate the general solution of the following system of equations, $(\nabla^2+h^2)\,u=0,\ (\nabla^2+h^2)\,v=0,\ (\nabla^2+h^2)\,w=0$.........(1),

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0 \quad \text{.........................(2)},$$

subject to the condition of finiteness at the origin of coordinates. The symbol ∇^2 here stands for Laplace's operator $\dfrac{d^2}{dx^2}+\dfrac{d^2}{dy^2}+\dfrac{d^2}{dz^2}$, and h is a constant.

Let us begin with the case $h=0$. The functions u, v, w can then be expanded in series of solid harmonics, and it is plain that the terms of degree n in these expansions, say u_n, v_n, w_n, must separately satisfy (2). The equations $\nabla^2 u_n=0,\ \nabla^2 v_n=0,\ \nabla^2 w_n=0$ may therefore be put in the forms

$$\frac{d}{dy}\left(\frac{dv_n}{dx}-\frac{du_n}{dy}\right)=\frac{d}{dz}\left(\frac{du_n}{dz}-\frac{dw_n}{dx}\right),\ \text{\&c., \&c.}\ \text{............(3)}.$$

Hence the expression

$$\left(\frac{dw_n}{dy}-\frac{dv_n}{dz}\right)dx+(\ldots)\,dy+(\ldots)\,dz\;\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

is an exact differential, say it $=d\chi_n$. This gives

$$\frac{dw_n}{dy}-\frac{dv_n}{dz}=\frac{d\chi_n}{dx},\;\frac{du_n}{dz}-\frac{dw_n}{dx}=\frac{d\chi_n}{dy},\;\frac{dv_n}{dx}-\frac{du_n}{dy}=\frac{d\chi_n}{dz}\ldots\ldots\ldots(5).$$

We notice that $\nabla^2\chi_n=0$, so that χ_n must be a solid harmonic, of degree n. From (5) we also obtain

$$z\frac{d\chi_n}{dy}-y\frac{d\chi_n}{dz}=x\frac{du_n}{dx}+y\frac{du_n}{dy}+z\frac{du_n}{dz}+u_n-\frac{d}{dx}(xu_n+yv_n+zw_n)\ldots(6),$$

with two similar equations. Now it appears, from (1) and (2), with $h=0$, that

$$\nabla^2(xu_n+yv_n+zw_n)=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

so that we write

$$xu_n+yv_n+zw_n=\phi_{n+1}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

a solid harmonic of degree $n+1$. Hence (6) may be written

$$(n+1)\,u_n=\frac{d\phi_{n+1}}{dx}+z\frac{d\chi_n}{dy}-y\frac{d\chi_n}{dz},\text{ \&c., \&c}\ldots\ldots\ldots\ldots(9).$$

The factor $n+1$ may be dropped without loss of generality, so that we finally obtain, as the complete solution of the proposed system of equations, in the particular case of $h=0$,

$$u=\Sigma\left(\frac{d\phi_{n+1}}{dx}+z\frac{d\chi_n}{dy}-y\frac{d\chi_n}{dz}\right),\text{ \&c., \&c}\ldots\ldots\ldots\ldots(10),^*$$

where ϕ_n, χ_n stand for solid harmonics of degree n.

Before proceeding to the general case, it is convenient to exhibit the known solution of the equation

$$(\nabla^2+h^2)\,u=0\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

in the following form. Whatever the value of u, it can always be expanded in a series of terms of the form R_nS_n, where S_n is a surface harmonic of the n^{th} order, and R_n is a function of r or $(x^2+y^2+z^2)^{\frac{1}{2}}$, only. If we take any such term, and perform ∇^2+h^2 upon it, it is easily seen that we obtain S_n multiplied by a function of r. Hence these terms must satisfy (11) singly. If, then, we suppose R_nS_n expanded in rising powers of h^2, thus

$$R_nS_n=a_0-h^2a_1+h^4a_2-h^6a_3+\text{\&c.}\;\ldots\ldots\ldots\ldots\ldots(12),$$

we find, on substituting in (11), and equating separately to zero the

* Compare Borchardt's theorem, quoted by Mr. T. Craig, *Phil. Mag.*, Nov. 1880.

coefficients of the various powers of h^2,

$$\nabla^2 a_0 = 0,\ \nabla^2 a_1 = a_0,\ \nabla^2 a_2 = a_1,\ \&c. \quad \ldots\ldots(13).$$

The solution of the first of these, subject to the condition of finiteness at the origin, is

$$a_0 = \phi_n \quad \ldots\ldots(14),$$

where ϕ_n is any solid harmonic of positive degree n; and the solutions of the remaining equations of the system are obtained by repeated application of the known formula

$$\nabla^2 (r^m \phi_n) = m\,(2n+m+1)\, r^{m-2} \phi_n \quad \ldots\ldots(15).$$

We thus find $a_1 = \dfrac{r^2}{2\,.\,2n+3}\phi_n,\ a_2 = \dfrac{r^4}{2\,.\,4\,.\,2n+3\,.\,2n+5}\phi_n,\ \&c. \ldots(16).$

The complete solution of (13) would involve the addition, at each stage of this process, of an arbitrary solid harmonic of order n; but, as each term so introduced would form the starting point of a series of the form (16), it is plain that no loss of generality is incurred by the omission of these terms. We thus obtain, as the complete solution of (10),

$$u = \Sigma \psi_n \phi_n \quad \ldots\ldots(17),$$

where

$$\psi_n = 1 - \frac{h^2 r^2}{2\,.\,2n+3} + \frac{h^4 r^4}{2\,.\,4\,.\,2n+3\,.\,2n+5} - \&c. \ldots(18).$$

This is, of course, a well-known result.

The following properties of the functions ψ_n will be of frequent use:

$$r\frac{d\psi_{n-1}}{dr} = -\frac{h^2 r^2}{2n+1}\psi_n \quad \ldots\ldots(19),$$

$$\psi_n + \frac{1}{2n+1}\, r\frac{d\psi_n}{dr} = \psi_{n-1} \quad \ldots\ldots(20),$$

$$\psi_n - \psi_{n-1} = \frac{h^2 r^2}{2n+1\,.\,2n+3}\psi_{n+1} \quad \ldots\ldots(21).$$

It is evident that

$$\psi_0 = \frac{\sin hr}{hr} \quad \ldots\ldots(22),$$

and it follows, by repeated application of (19), that

$$\psi_n = (-)^n\,.\,1\,.\,3\,.\,5 \ldots 2n+1\,.\left(\frac{1}{z}\frac{d}{dz}\right)^n \frac{\sin z}{z} \quad \ldots\ldots(23),$$

where z is written for hr.

We can now proceed to the general solution of the proposed system of equations, (1) and (2). We will confine our attention, for the present, to the parts of u, v, w, say u_n, v_n, w_n, which involve surface harmonics of order n; and we will further suppose u_n, v_n, w_n to be expanded in rising powers of h^2, thus

$$\left.\begin{aligned} u_n &= \alpha_0 - h^2\alpha_1 + h^4\alpha_2 - \&c. \\ v_n &= \beta_0 - h^2\beta_1 + h^4\beta_2 - \&c. \\ w_n &= \gamma_0 - h^2\gamma_1 + h^4\gamma_2 - \&c. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(24).$$

On substituting these values in (1) and (2), we find

$$\nabla^2\alpha_0 = 0,\ \nabla^2\beta_0 = 0,\ \nabla^2\gamma_0 = 0 \quad \ldots\ldots\ldots\ldots(25),$$

$$\frac{d\alpha_0}{dx} + \frac{d\beta_0}{dy} + \frac{d\gamma_0}{dz} = 0 \quad \ldots\ldots\ldots\ldots(26),$$

$$\nabla^2\alpha_1 = \alpha_0,\ \nabla^2\beta_1 = \beta_0,\ \nabla^2\gamma_1 = \gamma_0 \quad \ldots\ldots\ldots\ldots(27),$$

$$\frac{d\alpha_1}{dx} + \frac{d\beta_1}{dy} + \frac{d\gamma_1}{dz} = 0 \quad \ldots\ldots\ldots\ldots(28),$$

and so on. The solution of the system (25) and (26) has already been proved to be

$$\alpha_0 = \frac{d\phi_{n+1}}{dx} + z\frac{d\chi_n}{dy} - y\frac{d\chi_n}{dz},\ \&c.,\ \&c. \quad \ldots\ldots\ldots\ldots(29).$$

The equations (27) are then satisfied, in virtue of (15), by

$$\alpha_1 = \frac{r^2}{2\,.\,2n+3}\alpha_0,\quad \beta_1 = \frac{r^2}{2\,.\,2n+3}\beta_0,\quad \gamma_1 = \frac{r^2}{2\,.\,2n+3}\gamma_0 \ldots\ldots(30);$$

but these values do not satisfy (28), for they make

$$\frac{d\alpha_1}{dx} + \frac{d\beta_1}{dy} + \frac{d\gamma_1}{dz} = \frac{n+1}{2n+3}\phi_{n+1} \quad \ldots\ldots\ldots\ldots(31).$$

But equations (27) are still satisfied if we add to the above values of $\alpha_1, \beta_1, \gamma_1$, any arbitrary solid harmonics of order n; and the form of (31) suggests, therefore, the addition of terms of the forms

$$Cr^{2n+5}\frac{d}{dx}\frac{\phi_{n+1}}{r^{2n+3}},\quad Cr^{2n+5}\frac{d}{dy}\frac{\phi_{n+1}}{r^{2n+3}},\quad Cr^{2n+5}\frac{d}{dz}\frac{\phi_{n+1}}{r^{2n+3}} \quad \ldots\ldots\ldots\ldots(32).$$

We find that (28) will now be satisfied, provided

$$C = \frac{n+1}{n+2}\cdot\frac{1}{2n+3\,.\,2n+5} \quad \ldots\ldots\ldots\ldots(33).$$

Hence a solution of the system (27) and (28) is given by

$$\alpha_1 = \frac{r^2}{2\,.\,2n+3}\alpha_0 + \frac{n+1}{n+2}\cdot\frac{r^{2n+5}}{2n+3\,.\,2n+5}\frac{d}{dx}\frac{\phi_{n+1}}{r^{2n+3}},\ \&c.,\ \&c. \ldots(34).$$

The *complete* solution of the system in question would involve the addition of terms of the same form as $\alpha_0, \beta_0, \gamma_0$; these are omitted for a reason already indicated.

The values of $\alpha_2, \beta_2, \gamma_2$ can now be written down by means of (15), viz., we have

$$\alpha_2 = \frac{r^4}{2.4.2n+3.2n+5}\alpha_0 + \frac{n+1}{n+2}\cdot\frac{1}{2n+3.2n+5}\cdot\frac{r^{2n+7}}{2.2n+7}\frac{d}{dx}\frac{\phi_{n+1}}{r^{2n+3}}$$

&c., &c....(35),

and it appears that these values make

$$\frac{d\alpha_2}{dx}+\frac{d\beta_2}{dy}+\frac{d\gamma_2}{dz}=0 \quad \text{..............................(36)},$$

so that no additional terms are necessary. The further course of the investigation is now apparent. Collecting our results, we find, as the complete solution of the equations (1) and (2), subject to the condition of finiteness at the origin,*

$$u = \Sigma\left\{\psi_n\left(\frac{d\phi_{n+1}}{dx}+y\frac{d\chi_n}{dz}-z\frac{d\chi_n}{dy}\right)\right.$$

$$\left.-\frac{n+1}{n+2}\cdot\frac{h^2}{2n+3.2n+5}\psi_{n+2}r^{2n+5}\frac{d}{dx}\frac{\phi_{n+1}}{r^{2n+3}}\right\}, \text{ \&c., \&c....(37)}.$$

It appears, then, that the solutions of the proposed system of equations are of two distinct types, which may be thus written—

First type, $$u = \psi_n\left(y\frac{d\chi_n}{dz}-z\frac{d\chi_n}{dy}\right), \text{ \&c., \&c. ....(38)};$$

Second type,

$$u = \psi_{n-1}\frac{d\phi_n}{dx}-\frac{n}{n+1}\cdot\frac{h^2r^2}{2n+1.2n+3}\psi_{n+1}.r^{2n+1}\frac{d}{dx}\frac{\phi_n}{r^{2n+1}}, \text{ \&c., \&c....(39)}.$$

The solutions of the second type may also be written in the form

$$u = \psi_{n-1}\frac{d\phi_n}{dx}-\frac{n}{n+1}(\psi_n-\psi_{n-1})\,r^{2n+1}\frac{d}{dx}\frac{\phi_n}{r^{2n+1}}, \text{ \&c., \&c.......(40)}.$$

We may remark that the solutions of the first type make

$$xu+yv+zw=0 \quad \text{...................................(41)},$$

and those of the second type

$$xu+yv+zw=n\psi_n\phi_n \quad \text{.................................(42)}.$$

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* This restriction may be removed if the summation Σ in (37) be supposed to include negative as well as positive values of n. The suffixes attached to the symbols ϕ_n, χ_n must then be taken to denote the *algebraical degree*, as distinguished from the *order*, of these harmonics.

II.

The equations of small motion of a viscous incompressible fluid are

$$\left.\begin{aligned}\rho\frac{du}{dt}&=\mu\nabla^2u+\rho\frac{dV}{dx}-\frac{dp}{dx}\\ \rho\frac{dv}{dt}&=\mu\nabla^2v+\rho\frac{dV}{dy}-\frac{dp}{dy}\\ \rho\frac{dw}{dt}&=\mu\nabla^2w+\rho\frac{dV}{dz}-\frac{dp}{dz}\end{aligned}\right\}\quad\text{.........................(43),}$$

and

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0\quad\text{.................................(44),}$$

where u, v, w are the component velocities at the point (x, y, z), ρ is the density, μ the coefficient of viscosity, V the gravitation potential, p the pressure. The components of stress at the point (x, y, z) of the fluid are given by the formulæ

$$\left.\begin{aligned}p_{xx}&=-p+2\mu\frac{du}{dx},\ \&c.,\ \&c.\\ p_{yz}&=\mu\left(\frac{dw}{dy}+\frac{dv}{dz}\right),\ \&c.,\ \&c.\end{aligned}\right\}\quad\text{........................(45),}$$

where p_{xx} denotes the tractional stress parallel to x across a plane perpendicular to x, p_{yz} the stress parallel to y across a plane perpendicular to z, &c. Let us apply these formulæ to the small oscillations of a liquid mass about the spherical form, and let the origin be taken at the centre of mass. The conditions to be satisfied at the boundary are that the stress there is to be wholly normal, and constant in amount. Hence we must have, at the boundary,

$$\left.\begin{aligned}xp_{xx}+yp_{xy}+zp_{xz}&=-Px\\ xp_{yx}+yp_{yy}+zp_{yz}&=-Py\\ xp_{zx}+yp_{zy}+zp_{zz}&=-Pz\end{aligned}\right\}\quad\text{...........................(46),}$$

where P denotes the uniform external pressure. If we substitute from (45), these become

$$2x\frac{du}{dx}+y\left(\frac{du}{dy}+\frac{dv}{dx}\right)+z\left(\frac{du}{dz}+\frac{dw}{dx}\right)=\frac{p-P}{\mu}\,.\,x,\ \&c.,\ \&c.,$$

or

$$\left(r\frac{d}{dr}-1\right)u+\frac{d}{dx}(xu+yv+zw)=\frac{p-P}{\mu}x,\ \&c.,\ \&c.\text{..........(47),}$$

where $\frac{d}{dr}$ denotes a differentiation along the radius vector r.

Let us now write $\nu=\mu/\rho$, and

$$\varpi=\frac{p}{\rho}-V\text{...(48).}$$

The equations (43) then become

$$\frac{du}{dt} = \nu\nabla^2 u - \frac{d\varpi}{dx}, \text{ \&c., \&c.} \quad \text{.........................(49).}$$

We now assume that u, v, w, ϖ all vary as $e^{-\alpha t}$; if we omit the exponential factor for shortness, the last written equations become

$$(\nabla^2 + h^2)\, u = \frac{1}{\nu}\frac{d\varpi}{dx}, \text{ \&c., \&c.} \quad \text{.........................(50),}$$

where

$$h^2 = \frac{\alpha}{\nu} \quad \text{...(51).}$$

It appears from (50) that we must have $\nabla^2\varpi = 0$, so that ϖ can be expanded in the form $\Sigma\varpi_n$, where ϖ_n is a solid harmonic of order n. The complete solution of (50) can now be written down at once; viz., we have $u = \dfrac{1}{\alpha}\dfrac{d\varpi}{dx}$ + terms of the types (38) and (39).

The terms of the form (38) represent motions which are everywhere perpendicular to the radius vector, and which are therefore unaffected by gravitation. It is clear, then, that the solutions of the two types are quite independent of one another.

The solutions of the first type may be very briefly dismissed. The equations (50) and (44) are satisfied by $\varpi = 0$ and

$$u = \psi_n\left(y\frac{d\chi_n}{dz} - z\frac{d\chi_n}{dy}\right), \text{ \&c., \&c.} \quad \text{.....................(38);}$$

and, substituting in (47), we find

$$\left[r\frac{d\psi_n}{dr} + (n-1)\,\psi_n\right] = 0 \quad \text{..........................(52),}$$

since, there being no radial motion, the surface value of V is constant. The square brackets are used to indicate that in the enclosed expression r is to be put equal to a, the radius vector of the surface. This equation determines the admissible values of h, which are all real, and the corresponding values of α are then given by (51).

We proceed now to the solutions of the second type, and assume

$$u = \frac{1}{\alpha}\frac{d\varpi_n}{dx} + \psi_{n-1}\frac{d}{dx}\frac{r^n}{a^n}T_n - \frac{n}{n+1}(\psi_n - \psi_{n-1})\frac{r^{2n+1}}{a^{2n+1}}\frac{d}{dx}\frac{a^{n+1}}{r^{n+1}}T_n, \text{ \&c., \&c.} \quad \text{......(53),}$$

where T_n is a surface harmonic of order n, and a is the mean radius of the spheroid. We shall find, in fact, that the solutions obtained by giving n different values in (53) are all independent. We have now to substitute from (53) in the surface condition (47). We find, at the surface,

$$\left(r\frac{d}{dr}-1\right)u = \frac{n-2}{a}\frac{d\varpi_n}{dx}+\left(r\frac{d}{dr}+n-2\right)\psi_{n-1}\cdot\frac{d}{dx}\frac{r^n}{a^n}T_n$$
$$-\frac{n}{n+1}\left(r\frac{d}{dr}+n-2\right)(\psi_n-\psi_{n-1})\cdot\frac{d}{dx}\frac{a^{n+1}}{r^{n+1}}T_n \quad\ldots(54),$$

$$\frac{d}{dx}(xu+yv+zw) = \frac{d}{dx}\left(\frac{n}{a}\varpi_n+n\psi_n\cdot\frac{r^n}{a^n}T_n\right)$$
$$= \frac{n}{a}\frac{d\varpi_n}{dx}+n\cdot\frac{1}{r}\frac{d\psi_n}{dr}\cdot x\frac{r^n}{a^n}T_n+n\psi_n\frac{d}{dx}\frac{r^n}{a^n}T_n$$
$$= \frac{n}{a}\frac{d\varpi_n}{dx}+n\psi_{n-1}\frac{d}{dx}\frac{r^n}{a^n}T_n+n(\psi_n-\psi_{n-1})\frac{d}{dx}\frac{a^{n+1}}{r^{n+1}}T_n \quad\ldots\ldots\ldots\ldots(55).$$

The reductions have been effected by means of (20), and of the known formula of Spherical Harmonics,

$$x\phi_n = \frac{r^2}{2n+1}\left(\frac{d\phi_n}{dx}-r^{2n+1}\frac{d}{dr}\frac{\phi_n}{r^{2n+1}}\right) \quad\ldots\ldots\ldots\ldots\ldots\ldots(56).$$

We have also made $r = a$ wherever possible.

Again, if we assume the equation to the surface of the spheroid to be

$$r = a+S_n \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(57),$$

where S_n is a surface harmonic of the n^{th} order, the surface value of V is known to be, if g denote the value of gravity at the surface,

$$V = \text{const.}-\frac{2(n-1)}{2n+1}gS_n$$
$$= \text{const.}-\frac{\beta^2}{n}aS_n \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(58),$$

if
$$\beta^2 = \frac{2n.n-1}{2n+1}\frac{g}{a} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(59).*$$

Hence we have, at the surface,

$$\frac{p-P}{\mu}.x = \frac{1}{\nu}(\varpi_n+V-P)x$$
$$= \frac{1}{\nu}\left(\varpi_n-\frac{\beta^2 a}{n}S_n\right)x$$
$$= \frac{a^2}{\nu}\cdot\frac{1}{2n+1}\left(\frac{d\varpi_n}{dx}-\frac{d}{dx}\frac{\varpi_n a^{2n+1}}{r^{2n+1}}\right)$$
$$-\frac{a^2}{\nu}\cdot\frac{1}{2n+1}\cdot\frac{\beta^2 a}{n}\left(\frac{d}{dx}\frac{r^n}{a^n}S_n-\frac{d}{dx}\frac{a^{n+1}}{r^{n+1}}S_n\right) \quad\ldots\ldots\ldots\ldots(60).$$

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* The introduction of this constant is suggested, of course, by Sir W. Thomson's result that the time of oscillation of a frictionless spheroid is $2\pi/\beta$, where β has the above value.

But, at the surface, we have also the kinematical condition

$$\frac{dS_n}{dt} = -aS_n = \frac{1}{a}(xu+yv+zw)$$

$$= \frac{1}{a}\left(n\frac{\varpi_n}{a}+n\psi_n T_n\right) \text{ ....(61).}$$

Hence (60) becomes

$$\frac{p-P}{\mu} = \frac{h^2a^2}{2n+1}\left\{\frac{1}{a}\left(1+\frac{\beta^2}{a^2}\right)\left(\frac{d\varpi_n}{dx}-\frac{d}{dx}\frac{\varpi_n a^{2n+1}}{r^{2n+1}}\right)\right.$$

$$\left.+\frac{\beta^2}{a^2}\psi_n\left(\frac{d}{dx}\frac{r^n}{a^n}T_n-\frac{d}{dx}\frac{a^{n+1}}{r^{n+1}}T_n\right)\right\} \text{ ....(62).}$$

Collecting our results, substituting in the three equations (47), and equating separately the surface harmonics of orders $n-1$ and $n+1$, we obtain, after some reductions,

$$\frac{2n-2}{a}\varpi_n+\left[r\frac{d\psi_{n-1}}{dr}+(2n-2)\psi_{n-1}\right]T_n$$

$$= \frac{h^2a^2}{2n+1}\left[\left(1+\frac{\beta^2}{a^2}\right)\frac{\varpi_n}{a}+\frac{\beta^2}{a^2}\psi_n T_n\right] \text{......................(63),}$$

and $$\frac{n}{n+1}\left[\left(r\frac{d}{dr}-3\right)(\psi_n-\psi_{n-1})T_n\right]$$

$$= \frac{h^2a^2}{2n+1}\left[\left(1+\frac{\beta^2}{a^2}\right)\frac{\varpi_n}{a}+\frac{\beta^2}{a^2}\psi_n T_n\right] \text{......................(64),}$$

where the square brackets indicate surface-values.* These may be

written $$\left\{2(n-1)-\frac{h^2a^2}{2n+1}\left(1+\frac{\beta^2}{a^2}\right)\right\}\frac{[\varpi_n]}{a}$$

$$+\left[2(n-1)\psi_{n-1}-\frac{h^2a^2}{2n+1}\left(1+\frac{\beta^2}{a^2}\right)\psi_n\right]T_n = 0 \text{ ....(65),}$$

$$-\frac{h^2a^2}{2n+1}\left(1+\frac{\beta^2}{a^2}\right)\frac{[\varpi_n]}{a}$$

$$+\left[-\frac{h^2a^2}{2n+1}\left(1+\frac{\beta^2}{a^2}\right)+\frac{n\,.\,2n+4}{n+1\,.\,2n+1}\frac{rd\psi_n}{dr}+\frac{h^2a^2}{n+1}\psi_n\right]T_n = 0 \text{...(66).}$$

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* Equations equivalent to (63) and (64) may be obtained in a different manner as follows:—first, by multiplying the three equations (47) in order by x, y, z, and adding; and secondly, by replacing the surface harmonics which arise in (47), by the solid harmonics of positive degree which agree with them at the surface, differentiating with respect to x, y, z in order, and adding. The method here indicated has the advantage of showing, analytically, that the solutions obtained by giving n different integral values in (38) and (53) are all independent.

Eliminating ϖ_n and T_n, we obtain, after several reductions,

$$2(n-1)\frac{h^2a^2}{2n+1}\left(1+\frac{\beta^2}{a^2}\right)\frac{[\psi_{n+1}]}{2n+1\,.\,2n+3}$$
$$=\left\{2(n-1)-\frac{h^2a^2}{2n+1}\left(1+\frac{\beta^2}{a^2}\right)\right\}\left[\frac{\psi_n}{n+1}-\frac{n\,.\,2n+4}{n+1\,.\,2n+1\,.\,2n+3}\,\psi_{n+1}\right] \quad\text{.............(67).}$$

This equation to determine the admissible values of ha is readily solved by approximation in the two extreme cases, where the viscosity is very large or very small.

When ν is very great, ha and a are infinitesimal, so that $[\psi_n]=1$, $[\psi_{n+1}]=1$, approximately. Hence, writing for shortness

$$\zeta=\frac{h^2a^2}{2n+1}\frac{\beta^2}{a^2}=\frac{1}{2n+1}\frac{\beta^2a^2}{\nu a} \quad\text{..............................(68),}$$

(67) becomes

$$\frac{2\,.\,n-1}{2n+1\,.\,2n+3}\,\zeta=\frac{2(n-1)-\zeta}{n+1}\left(1+\frac{n\,.\,2n+4}{2n+1\,.\,2n+3}\right) \quad\text{.........(69)}$$

approximately. Solving for ζ, and substituting in (68), we find

$$a=\frac{n}{2(n+1)^2+1}\cdot\frac{ga}{\nu} \quad\text{..................................(70).}$$

This is the result given by Mr. G. H. Darwin, at p. 10 of his paper "On the Bodily Tides of Viscous Spheroids, &c."*

When, on the other hand, ν is small,† it is evident from the investigation‡ of Sir W. Thomson, already alluded to, that $a=i\beta$, nearly, so that ha is large. The series (18) is then not a convenient expression for ψ_n; and we have recourse to the formula (23). From this we find that the most important part of $[\psi_n]$ is

$$(-)^n.1.3.5\,...\,(2n+1)\,\frac{\sin\left(ha+n\dfrac{\pi}{2}\right)}{(ha)^{n+1}} \quad\text{................(71).}$$

It appears that the ratio $[\psi_{n+1}]\,/\,[\psi_n]$ is of the order $1/ha$, so that (67) becomes, approximately,

$$2(n-1)-\frac{h^2a^2}{2n+1}\left(1+\frac{\beta^2}{a^2}\right)=0 \quad\text{.........................(72).}$$

This leads to

$$\frac{a}{i\beta}=1+\frac{n-1\,.\,2n+1}{h^2a^2}$$
$$=1+\frac{n-1\,.\,2n+1}{i\beta}\cdot\frac{\nu}{a^2},$$

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* *Phil. Trans.*, Part I., 1879.

† That is, relatively. The standard of comparison is βa^2.

‡ *Phil. Trans.*, 1863.

or
$$\alpha = i\beta + (n-1)(2n+1)\frac{\nu}{a^2} \quad \text{.....................(73).}$$

The most remarkable point about this result is the excessively minute extent to which the oscillations of a globe of moderate dimensions are affected by such a degree of viscosity as is ordinarily met with in nature. If we introduce a symbol τ to denote the "modulus of decay" of the oscillations, *i.e.*, the time which must elapse before their amplitude falls to $1/e$ of its original value, we have

$$\tau = \frac{1}{n-1\,.\,2n+1}\frac{a^2}{\nu} \quad \text{.......................(74).}$$

For a globe of the size of the earth, and of the same kinematic viscosity as water, we have, on the C. G. S. system, $a = 6{\cdot}37\times10^8$, $\nu = {\cdot}014$; and the corresponding value of τ for the oscillation of longest period ($n = 2$) is

$$\tau = 1{\cdot}84\times10^{11} \text{ years.}$$

Even with the value found by Mr. Darwin* for the viscosity of pitch near the freezing temperature, viz., $\mu = 1{\cdot}3\times10^8\times g$, we find, taking $g = 980$, the value
$$\tau = 150 \text{ hours}$$
as the modulus of decay of the slowest oscillation of a globe of the size of the earth, having the density of water and the viscosity of pitch.

The nature of the modification introduced by viscosity in the character of the motion will appear most clearly if we calculate the components ξ, η, ζ of the angular velocity of the fluid. Substituting the values of u, v, w from (53), we have

$$2\xi = \frac{dv}{dx} - \frac{du}{dy} = -\frac{h^2}{n+1}\psi_n\left(x\frac{d}{dy} - y\frac{d}{dx}\right)\frac{r^n}{a^n}T_n \quad \text{.........(75).}$$

Now it appears, from (23), that ψ_n is of the same order of magnitude as $e^{\pm ihr}/(hr)^{n+1}$. But we have, approximately,

$$h = \sqrt{\frac{\alpha}{\nu}} = \sqrt{\frac{i\beta}{\nu}} = q(1+i) \quad \text{..................(76),}$$

if $q^2 = \beta/2\nu$. Hence ψ_n is of the order $e^{qr}/(qr)^{n+1}$, where q is large. It appears, then, from (66) and (61), that we have, approximately,

$$\frac{h^2a^2}{n+1}[\psi_n]\,T_n = 2(n-1)\frac{[\varpi_n]}{a} \quad \text{.....................(77),}$$

and
$$\frac{[\varpi_n]}{a} = -\frac{\alpha a S_n}{n} \quad \text{............................(78).}$$

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* *Phil. Trans.*, 1879, p. 16.

Hence (75) becomes

$$2\xi = 2\frac{n-1}{n}\,\frac{a}{a}\cdot\frac{\psi_n}{[\psi_n]}\left(x\frac{d}{dy}-y\frac{d}{dx}\right)\frac{r^n}{a^n}S_n \quad\text{..............(79).}$$

Now, keeping only the most important terms,

$$\frac{\psi_n}{[\psi_n]} = \left(\frac{a}{r}\right)^{n+1}\frac{\sin\left(hr+n\frac{\pi}{2}\right)}{\sin\left(ha+n\frac{\pi}{2}\right)}$$

$$= \left(\frac{a}{r}\right)^{n+1}\{\sin q\,(r-a)+i\cos\,(qr-a)\}\,e^{q(r-a)} \quad\text{.........(80).}$$

We see that the vortices are arranged in spherical strata, and that the strength of the vortex motion rapidly diminishes, at the same time rapidly fluctuating in sign, as the depth below the surface increases. I have elsewhere pointed out the analogy between the diffusion of vortex motion in a viscous fluid and the conduction of heat. In the present case, owing to the oscillatory character of the motion, the sign of the vortex motion which is being diffused inwards from the surface is continually being reversed, so that the effect at a moderate depth becomes insensible, just as the fluctuations of temperature at the earth's surface cease to be felt at a depth of a few yards. In connection with this analogy, it is interesting to note that the kinematic viscosity of water ($\nu = \cdot 014$) does not much exceed the value found by Dr. Everett ($\cdot 01249$) for the thermometric conductivity of the Greenwich gravel.

By supposing S_n to be a sectorial harmonic, taking n a very large number, and fixing our attention on a small equatorial region of the spheroidal surface, we fall on the case of straight parallel waves on a plane sheet of water. The wave length λ must be taken $= 2\pi a/n$, so that the modulus of decay is

$$\tau = \frac{\lambda^2}{8\pi^2\nu} \quad\text{....................................(81).}$$

This differs somewhat from the result found by Stokes,* according to which the numerical factor in the denominator should be 16. I have verified the formula (81) by an independent calculation of the effect of viscosity on oscillatory waves.

The whole of the preceding investigations can readily be adapted to cases where the tendency to the spherical form is supplied by other forces than gravity. If, for instance, we consider the vibrations of a

* *Camb. Trans.*, t. 9, p. [61], 1850.

liquid globule under the surface-tension of the bounding film, the only modifications required in the foregoing process are, that we must make $V = 0$ throughout, and that in the surface condition (47) we must replace $p-P$ by

$$p-P-T\left(\frac{1}{r_1}+\frac{1}{r_2}\right) \quad\text{......................(82)},$$

where r_1, r_2 are the principal radii of curvature of the surface, and T is the surface-tension. It appears that

$$\frac{1}{r_1}+\frac{1}{r_2} = \frac{2}{a}+(n-1)(n+2)\frac{S_n}{a^2} \quad\text{..............(83);*}$$

so that (67) still holds, provided

$$\beta^2 = n(n-1)(n+2)\frac{T}{\rho a^3} \quad\text{......................(84)},$$

i.e., provided $2\pi/\beta$ is the time of a complete oscillation.† The rest of the solution, and the result (74), is then the same as before. This is only what we should expect from the fact that (74) does not involve g.

The modulus of decay of the slowest oscillation ($n = 2$) of a globule of water is, in seconds,

$$\tau = 14{\cdot}3a^2,$$

where the unit of a is the centimeter.

Addition to the preceding Paper.

A simple verification of the principal result given in the paper referred to can be obtained by means of the "Dissipation Function," whose value is given, in the case of an incompressible fluid, by the formula

$$\frac{F}{2\mu} = \left(\frac{du}{dx}\right)^2+\left(\frac{dv}{dy}\right)^2+\left(\frac{dw}{dz}\right)^2 + \tfrac{1}{2}\left(\frac{dw}{dy}+\frac{dv}{dz}\right)^2+\tfrac{1}{2}\left(\frac{du}{dz}+\frac{dw}{dx}\right)^2+\tfrac{1}{2}\left(\frac{dv}{dx}+\frac{du}{dy}\right)^2 \quad\text{......(1)},$$

the notation being the same as before. It is tolerably obvious *à priori*, and it can be verified by means of formulæ already given, that when the "kinematical viscosity" ν is small compared with βa^2,‡ we may in

* R. R. Webb, *Messenger of Mathematics*, t. 9, p. 177, 1880.

† Lord Rayleigh, *Proc. R. S.*, t. 29, 1879.

‡ $2\pi/\beta$ is the time of a complete oscillation, and a is the mean radius of the spheroid.

calculating the value of F neglect, for a first approximation, the modifications introduced into the values of u, v, w by viscosity. Now, in the absence of viscosity, the fundamental modes of vibration of a liquid spheroid would be irrotational, and the velocity-potential ϕ corresponding to the n^{th} mode would be of the form

$$\phi = A\frac{r^n}{a^n}S_n\cos\beta t,$$

where S_n is a surface harmonic of the n^{th} order, and A a constant. The corresponding equation to the external surface is

$$r = a+\frac{n}{\beta a}AS_n\sin\beta t \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

as we see from the consideration that at the surface we must have $dr/dt = d\phi/dr$. The value of the kinetic energy T is then given by

$$2T = \rho\iint\phi\frac{d\phi}{dr}\,.\,a^2d\omega = \rho naA^2\cos^2\beta t\,.\iint S_n^2\,d\omega,$$

where $d\omega$ denotes an elementary solid angle having its vertex at the centre of the spheroid. From the general theory of the fundamental modes of vibration of dynamical systems, it is plain that the potential energy V must be given by the formula

$$2V = \rho naA^2\sin^2\beta t\,.\iint S_n^2\,d\omega \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

and hence that the total energy is

$$W = \tfrac{1}{2}\rho naA^2\,.\iint S_n^2\,d\omega \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

If we write $u = d\phi/dx$, &c., the formula (1) becomes

$$\frac{F}{2\mu} = \left(\frac{d^2\phi}{dx^2}\right)^2+\left(\frac{d^2\phi}{dy^2}\right)^2+\left(\frac{d^2\phi}{dz^2}\right)^2+2\left(\frac{d^2\phi}{dy\,dz}\right)^2+2\left(\frac{d^2\phi}{dz\,dx}\right)^2+2\left(\frac{d^2\phi}{dx\,dy}\right)^2 \quad\ldots\ldots\ldots(5).$$

If we multiply the right-hand side of this by $dx\,dy\,dz$, and integrate over the volume of a sphere having its centre at the origin, and its radius $= r$, we obtain

$$\tfrac{1}{2}r^2\iint\frac{d\,.\,q^2}{dr}\,d\omega,\text{*}$$

where q denotes the value of the velocity at the surface of this sphere.

Now
$$r^2\iint q^2d\omega = \frac{d}{dr}\iint\phi\frac{d\phi}{dr}\,.\,r^2d\omega,$$

(each side, when multiplied by $\rho\,dr$, being double the kinetic energy of

* *Motion of Fluids*, p. 61.

the fluid contained between two spheres of radii r and $r+dr$,)

$$= \frac{d}{dr} \cdot n \frac{r^{2n+1}}{a^{2n}} A^2 \cos^2 \beta t \cdot \iint S_n^2 \, d\omega$$

$$= n \, . \, 2n+1 \, . \frac{r^{2n}}{a^{2n}} A^2 \cos^2 \beta t \cdot \iint S_n^2 \, d\omega.$$

From this we find

$$\tfrac{1}{2} r^3 \iint \frac{d \, . \, q^2}{dr} d\omega = n \, . \, n-1 \, . \, 2n+1 \, . \frac{r^{2n-1}}{a^{2n}} A^2 \cos^2 \beta t \cdot \iint S_n^2 \, d\omega.$$

Hence, making $r = a$, we find, from (5), that the total rate of dissipation of energy is

$$\frac{2\mu}{a} \, . \, n \, . \, n-1 \, . \, 2n+1 \, . \, A^2 \cos^2 \beta t \, . \iint S_n^2 \, d\omega.$$

The *average* rate of dissipation is obtained from this by omitting the factor $\cos^2 \beta t$, and dividing by 2. Hence

$$\frac{dW}{dt} = -\frac{\mu}{a} \, . \, n \, . \, n-1 \, . \, 2n+1 \, . \, A^2 \, . \iint S_n^2 \, d\omega.$$

The effect of the viscosity is to gradually diminish the amplitude of the oscillations, which is proportional to A. Hence, substituting the value of W from (4), we find

$$\frac{dA}{dt} + \frac{A}{\tau} = 0,$$

where $$\tau = \frac{1}{n-1 \, . \, 2n+1} \cdot \frac{a^2}{\nu}.$$

This agrees with the result obtained in the former paper.

The application of the method of the dissipation function to problems of this kind was first made by Stokes,* in investigating the effect of viscosity on ordinary water waves.

In the above investigation a knowledge of the value of β is not required, unless for the purpose of estimating how far the fundamental assumption that ν is small compared with βa^2 is (in any given case) legitimate. To find β we may calculate the potential energy corresponding to any prescribed form of the spheroidal surface. Thus, when the force governing the oscillations is the mutual gravitation of the parts of the spheroid, the potential energy when the surface has the form $r = a + T_n$ (T_n a surface harmonic of order n) may be found as follows. The gravitation potential at the surface of the spheroid

* *Camb. Trans.*, t. ix., p. [61]. There seems to be a slight oversight in Stokes's calculation. The rate of decay of the *kinetic* energy is equated to the dissipation calculated from (1) above. Since the average kinetic energy is half the total energy, the effect of this is to make the rate of decay of the oscillations come out double the true value.

$r = a + xT_n$ (x being a proper fraction) is known to be

$$= \text{const.} - g\left(1 - \frac{3}{2n+1}\right) xT_n,$$

where g is the value of gravity at the surface of the spheroid. Now let the value of x be increased to $x + dx$; this is equivalent to the addition of a superficial film of surface density $\rho\, dx \,.\, T_n$. Hence the increase of potential energy is

$$dV = 2g\rho \frac{n-1}{2n+1} x\, dx \iint T_n^2 \,.\, a^2 d\omega.$$

Integrating this with respect to x between the limits 0 and 1, we find for the total potential energy

$$V = g\rho a^3 \,.\, \frac{n-1}{2n+1} \iint T_n^2\, d\omega \quad \text{......................} (6).$$

In our case we have, by (2),

$$T_n = \frac{n}{\beta a} AS_n \,.\, \cos \beta t.$$

Substituting in (6), and comparing with (3), we find

$$\beta^2 = \frac{2n \,.\, n-1}{2n+1} \frac{g}{a}.$$

This agrees with the result obtained by Thomson, *Phil. Trans.*, 1863, p. 610.

On the Covariant Locus of the Vertex of a Pencil of Tangents to a Cubic in Involution. *By* J. J. Walker, M.A.

[*Read Dec.* 8*th*, 1881.]

In a recent communication to the Society by Mr. R. A. Roberts, "On Tangents to a Cubic forming a Pencil in Involution," the locus of the vertex has been very ingeniously found in the shape of the twelve cubics (besides the nine harmonic polars of the inflexion-points),

$$\left.\begin{array}{l} (1+8m^3)\, x^3 + 8m^3 U = 0 \\ (1+8m^3)\, y^3 + 8m^3 U = 0 \\ (1+8m^3)\, z^3 + 8m^3 U = 0 \end{array}\right\} \quad \text{......................} (1),$$

$$\left.\begin{array}{l} (1+8m^3)\,(x + y + z)^3 - 8(m-1)^3\, U = 0 \\ (1+8m^3)\,(x + \vartheta y + \vartheta^2 z)^3 - 8\,(m-1)^3\, U = 0 \\ (1+8m^3)\,(x + \vartheta^2 y + \vartheta z)^3 - 8\,(m-1)^3\, U = 0 \end{array}\right\} \quad \text{...............} (2),$$

$$\left.\begin{array}{l} (1+8m^3)\,(\vartheta x + y + z)^3 - 8\,(m-\vartheta)^3\, U = 0 \\ (1+8m^3)\,(x + \vartheta y + z)^3 - 8\,(m-\vartheta)^3\, U = 0 \\ (1+8m^3)\,(x + y + \vartheta z)^3 - 8\,(m-\vartheta)^3\, U = 0 \end{array}\right\} \quad \text{...............} (3),$$

$$\left.\begin{array}{l}(1+8m^3)(\vartheta^2x+y+z)^3-8(m-\vartheta^2)^3U=0\\(1+8m^3)(x+\vartheta^2y+z)^3-8(m-\vartheta^2)^3U=0\\(1+8m^3)(x+y+\vartheta^2z)^3-8(m-\vartheta^2)^3U=0\end{array}\right\}\quad\text{..............(4)},$$

the cubic being supposed to be written in one of its four canonical forms, viz.,

$$U\equiv x^3+y^3+z^3+6mxyz,$$

and ϑ, ϑ^2 being the imaginary cube roots of unity.

The equation of the nine harmonic polars has been exhibited as the square root of a covariant by Dr. Salmon (*Higher Plane Curves*, Art. 232).

It occurred to me, on becoming acquainted with the singular form in which Mr. R. A. Roberts found the locus above, to ascertain whether this part of the locus was expressible rationally in the covariant form, or whether, like the other part, its square only was so expressible;* but when I had gone so far as to satisfy myself on this head, I thought it might be as well to complete the work, and be able to exhibit the result in its entirety to those mathematicians who take an interest in such investigations.

As a preliminary it may be well to study briefly the character of these twelve cubics; the first for example is, in full,

$$(1+16m^3)x^3+8m^3(y^3+z^3+6mxyz)=0,$$

or, writing,

$$(1+16m^3)^{\frac{1}{3}}x=\xi,$$
$$2my=\eta,$$
$$2mz=\zeta,$$
$$\xi^3+\eta^3+\zeta^3+12\frac{m^2}{(1+16m^3)^{\frac{1}{3}}}\xi\eta\zeta=0,$$

the discriminant of which is the cube of

$$1+\frac{64m^6}{1+16m^3},$$

or of

$$\frac{(1+8m^3)^2}{1+16m^3};$$

so that the cubics will be proper curves of that order, without double points, if U has not one; and obviously each has three inflexions in common with U; viz., the first those lying on $x=0$; and so on.

Multiplying the triad (1), and observing that

$$\begin{aligned}(1+8m^3)(x^3+y^3+z^3) &= (1+2m^3)U-6mH,\\ (1+8m^3)^2(y^3z^3+z^3x^3+x^3y^3) &= -\Theta+m^3(2+m^3)U^2\\ &\quad -m(1+2m^3)UH+3m^2H^2,\\ (1+8m^3)x^3y^3z^3 &= m^3U+H,\end{aligned}$$

* My attention has been drawn to the *à priori* evidence of this in the fact that the product of the triads (1), (2), (3), (4) does not change sign when x and y are interchanged.

where H is the Hessian and Θ Dr. Salmon's covariant of the sixth degree in the variables, and eighth in the coefficients of U, the result is

$$-8m^3\,\Theta U+81m^6(1+8m^3)\,U^3-5m^4(1+80m^3)\,U^2H$$
$$+3m^2(1+8m^3)\,UH^2+H^3 \qquad \text{.......................(5).}$$

Similarly, the product of the triad (2) will be

$$-8m'^3\,\Theta'U'+81m'^6(1+8m'^3)\,U'^3-5m'^4(1+80m'^3)\,U'^2H'$$
$$+3m'^2(1+8m'^3)\,U'H'^2+H'^3 \qquad \text{.....................(6),}$$

the marked letters indicating the values of m, U, H for the coordinates $x'y'z'$ relative to a second canonical triangle, viz.,

$$x'=x+y+z,$$
$$y'=x+\vartheta y+\vartheta^2 z,$$
$$z'=x+\vartheta^2 y+\vartheta z.$$

These substitutions give—

$$m'=\frac{1-m}{1+2m},$$
$$1+8m'^3=9\,\frac{1+8m^3}{(1+2m)^3},$$
$$U'=\frac{9U}{1+2m},$$
$$H'=-\frac{27H}{(1+2m)^3},$$
$$\Theta'=-\frac{3^7\Theta}{(1+2m)^6};$$

by means of which (6) is expressed in terms of m, U, H, Θ without difficulty.

Also, the product of the triad (3) will be

$$-8m''^3\,\Theta''U''+81m''^6(1+8m''^3)\,U''^3-\ldots\ldots+H''^3 \qquad \text{.........(7),}$$

where

$$m''=\frac{\vartheta-m}{\vartheta^2+2\vartheta m}=\frac{1-\vartheta^2 m}{\vartheta+2m},$$
$$1+8m''^3=9\vartheta\,\frac{1+8m^3}{(\vartheta+2m)^3},$$
$$U''=\frac{9\vartheta U}{\vartheta+2m},$$
$$H''=-\frac{27\vartheta^2 H}{(\vartheta+2m)^3},$$
$$\Theta''=-\frac{3^7\vartheta^2\Theta}{(\vartheta+2m)^6}.$$

Lastly, the product of the triad (4) will, obviously, be derived from that of (3) by interchanging ϑ and ϑ^2.

Thus the products of the triads (2), (3), (4), cleared of extraneous factors, are respectively, writing Θ_1 for 8Θ,

$$\begin{aligned}
&(1+2m)(1-m)^3\Theta_1U+27(1-m)^6(1+8m^3)U^3\\
&\qquad+5(1-m)^4\{10(1+8m^3)-(1+2m)^4\}U^2H\\
&\qquad+9(1-m)^2(1+8m^3)UH^2-(1+2m)^4H^3,\\
&(\vartheta+2m)(\vartheta-m)^3\Theta_1U+27(\vartheta-m)^6\vartheta(1+8m^3)U^3\\
&\qquad+5(\vartheta-m)^4\{10\vartheta(1+8m^3)-(\vartheta+2m)^4\}U^2H\\
&\qquad+9(\vartheta-m)^2\vartheta(1+8m^3)UH^2-(\vartheta+2m)^4H^3,\\
&(\vartheta^2+2m)(\vartheta^2-m)^3\Theta_1U+27(\vartheta^2-m)\vartheta^2(1+8m^3)U\\
&\qquad+5(\vartheta^2-m)^4\{10\vartheta^2(1+8m^3)-(\vartheta^2+2m)^4\}U^2H\\
&\qquad+9(\vartheta^2-m)\vartheta^2(1+8m^3)UH^2-(\vartheta^2+2m)^4H^3;
\end{aligned}$$

and the continued product of these three, which is rational, is finally multiplied by (5)

$$-m^3\Theta_1U+81(1+8m^3)m^6U^3-5(1+80m^3)m^4U^2H+3(1+8m^3)m^2UH+H^3.$$

The result is, writing S, T, Δ for $m(1-m^3)$, $1-20m^3-8m^6$, $(1+8m^3)^3$, and rejecting the factor $1+8m^3$,

$$\begin{aligned}
&-S^3\Theta_1^4U^4+\{-81S^6TU^3+520S^4U^2H+3S^2TUH^2+(40S^3+T^2)H^3\}\Theta_1^3U^3\\
&+\{-3^7S^3\Delta U^6+120\times81S^4TU^5H\\
&\quad+S^2(-582\times64S^3+243T^2)U^4H^2+81T(40S^3+T^2)U^3H^3\\
&\quad-45S(5\times64S^3+7T^2)U^2H^4+72S^2TUH^5-3(128S^3+T^2)H^6\}\Theta_1^2U^2\\
&+\{-27^3S^6T\Delta U^9-165\times27^2S^4\Delta U^8H+243S^2T(128S^3+27T^2)U^7H^2\\
&\quad+(3875\times512S^6-138996S^3\Delta+2187\Delta^2)U^6H^3\\
&\quad+9ST(295\times64S^3-1125\Delta)U^5H^4\\
&\quad+3S^2(-525\times512S^3+14182\Delta)U^4H^5-9T(37\times64S^3+153\Delta)U^3H^6\\
&\quad+12S(110\times64S^3+209T^2)U^2H^7\\
&\quad+9\times64S^2TUH^8+(-512S^3+3\Delta)H^9\}\Theta_1U\\
&+3^{13}S^6\Delta U^{12}+95\times3^9S^4T\Delta U^{11}H+3^7S^2(722S^3+27\Delta)\Delta U^{10}H^2\\
&\quad+T(10125\times512S^6+45765S^3\Delta+19863\Delta^2)U^9H^3\\
&\quad+S(-6250\times4096S^6+289980S^3\Delta+10935\Delta^2)U^8H^4\\
&\quad+S^2T(-2025\times512S^3+268218\Delta)U^7H^5\\
&\quad+(3750\times4096S^6-348396S^3\Delta+86133\Delta^2)U^6H^6\\
&\quad+ST(1280\times64S^3+19404\Delta)U^5H^7\\
&\quad-3S^2(250\times4096S^3+128473\Delta)U^4H^8\\
&\quad+9T(-112\times64S^3+225\Delta)U^3H^9\\
&\quad+S(3200\times64S^3-4170\Delta)U^2H^{10}+1536S^2TUH^{11}-\Delta H^{12}=0.
\end{aligned}$$

Thursday, January 12th, 1882.

S. ROBERTS, Esq., F.R.S., President, in the Chair.

Dr. G. J. Allman, Queen's College, Galway, and Mrs. Bryant, B.Sc., F.C.P., were elected members, and Mr. G. H. Stuart was admitted into the Society. A vote of thanks was passed to the Norwegian Government for the present of a copy of the Second Edition of Abel's Works.

The following communications were made:—

"The Invariants of a certain Orthogonal Transformation, with special reference to their use in the Theory of the Strains and Stresses of an Elastic Solid:" Mr. W. J. C. Sharp.

"Certain Elliptic Function Formulæ:" Rev. M. M. U. Wilkinson.

"On the Calculation of Symmetric Functions:" Mr. J. Hammond.

"Complete Determination of the Real Foci, and of the Vector Equation, of the Pedal of a given Ellipse (and Parabola), with respect to any proposed Point:" Prof. Wolstenholme.

The following presents were received:—

Carte-de-Visite likeness from Prof. Genese.

"Educational Times," December, 1881, January, 1882.

"Atti della R. Academia dei Lincei—Transunti," Vol. vi., Fasc. 2°.

"Monatsbericht," November, 1881.

"Beiblätter zu den Annalen der Physik und Chemie," Band v., Stücke 11, 12.

"Œuvres complètes de Niels Henrik Abel," Nouvelle édition, Tomes i., ii.; Christiania, 1881.

"Archiv for Mathematik og Naturvidenskab," Femte Bind, Første, Andet, Tredie, Fjerde Hefte, & Sjette Bind, Første, Andet Hefte, 1881.

Complete Determination of the Real Foci, and of the Vector Equation, of the Pedal of a given Ellipse with respect to any proposed Point. *By* Prof. WOLSTENHOLME.

[*Read January 12th,* 1882.]

The equation of the pedal of the ellipse $a^2y^2+b^2x^2=a^2b^2$ with respect to a point O whose coordinates are X, Y, is

$$(x^2+y^2-xX-yY)^2=a^2(x-X)^2+b^2(y-Y)^2,$$

or, with the origin at O,

$$(x^2+y^2+xX+yY)^2=a^2x^2+b^2y^2.$$