Appendix A: Tensor Algebra and the Adler–Zee Scalar Projector

Purpose. This appendix derives the scalar projection of the stress-energy tensor correlator used in the Adler–Zee formula for the induced Einstein–Hilbert term. The key result is the contraction factor 1/96, which appears in the quadratic metric variation when isolating the coupling to the Ricci scalar. Steps use standard index algebra.

Conventions. Metric signature $(+,-,-,-)$. Natural units $\hbar = c = 1$. Fourier conventions: $\tilde{f}(q) = \int d^4x\, e^{iq\cdot x} f(x)$.

Quadratic expansion of the matter effective action

The matter generating functional in a weak gravitational field is $W[g] = -i \ln Z[g]$. Expanding around flat space $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, the second variation is

$$\delta^2 W = \frac{i}{2} \int d^4x\, d^4y\; h^{\mu\nu}(x)\, h^{\rho\sigma}(y)\, \langle T_{\mu\nu}(x) T_{\rho\sigma}(y)\rangle.$$

Define the trace field $h = h^\mu_\mu$ and the trace operator $T = T^\mu_\mu$.

Scalar (trace) projection

Decompose the stress tensor into scalar and traceless pieces. The scalar (trace) part may be written, at leading order, as

$$T^{(S)}_{\mu\nu} = \frac{1}{4}\, \eta_{\mu\nu}\, T.$$

Inserting into the second variation and contracting the indices gives the scalar contribution (after accounting for symmetry factors)

$$\delta^2 W_S = \frac{i}{32} \int d^4x\, d^4y\; h(x) h(y) \langle T(x) T(y)\rangle.$$

Fourier transform and normalization

Define

$$\tilde{h}(q) = \int d^4x\, e^{iq\cdot x} h(x), \qquad \tilde{G}_{TT}(q) = \int d^4x\, e^{iq\cdot x} \langle T(x) T(0)\rangle.$$

Then

$$\delta^2 W_S = \frac{i}{32} \int \frac{d^4q}{(2\pi)^4}\, \tilde{h}(-q)\, \tilde{h}(q)\, \tilde{G}_{TT}(q).$$

The weak-field quadratic expansion of Einstein–Hilbert yields a kinetic term for the trace mode with a normalization that effectively introduces an extra factor 1/3 when matching scalar pieces. Collecting factors:

$$\frac{1}{32} \times \frac{1}{3} = \frac{1}{96}.$$

Thus we may write the scalar projection compactly as

$$\delta^2 W_S = \frac{i}{2} \cdot \frac{1}{96} \int \frac{d^4q}{(2\pi)^4}\, \tilde{h}(-q)\, \tilde{h}(q)\, \tilde{G}_{TT}(q),$$

which is the Adler–Zee scalar-channel form used in the main text.

Remarks

The numeric factor 1/96 depends only on index contractions and the standard weak-field normalization; different sign conventions shift signs but not the parametric result. For referee-level completeness, one may expand the Einstein–Hilbert action and explicitly verify the factor by matching components; this is straightforward algebra.

Appendix B: Finite-Volume Matching and Spatial Form Factors

Purpose. This appendix computes the spatial form factor of a localized mode, derives its small-q expansion, and shows when a finite-volume localized mode contributes to the long-wavelength Adler–Zee induced R coefficient.

Form factor definition

Let $f(\mathbf{x})$ be the spatial profile of the localized contribution (to the trace operator) with compact support inside a volume V. Define the form factor

$$F(\mathbf{q}) = \int d^3x \, e^{-i\mathbf{q}\cdot\mathbf{x}} f(\mathbf{x}).$$

The finite-volume contribution to the scalar effective action contains $|F(\mathbf{q})|^2$, evaluated in the $q \to 0$ limit for long-wavelength gravitational fields.

Small-q expansion

For a smooth profile $f(\mathbf{x})$ expand:

$$F(\mathbf{q}) = F(0) - \frac{1}{2}\sum_{i,j} q_i q_j \int_V d^3x \, x_i x_j f(\mathbf{x}) + O(q^4),$$

with

$$F(0) = \int_V d^3x \, f(\mathbf{x}).$$

Thus $F(\mathbf{q}) = F(0) + O(q^2)$. If $F(0) \neq 0$ the monopole survives the long-wavelength limit.

Uniform sphere: explicit calculation

Set $f(\mathbf{x}) = 1$ for $|\mathbf{x}| \leq R$ and zero outside. The volume is $V = \frac{4\pi}{3}R^3$. Choose $\mathbf{q}$ along the z-axis. Then

$$F(q) = 4\pi \int_0^R r^2 dr \, \frac{\sin(qr)}{qr}.$$

Evaluating,

$$F(q) = V \cdot \frac{3(\sin(qR) - qR\cos(qR))}{(qR)^3}.$$

Use the small-qR expansion:

$$F(q) = V\left(1 - \frac{(qR)^2}{10} + O(q^4R^4)\right).$$

Thus $F(0) = V$ and $|F(0)|^2 = V^2$.

Cylinder and disk (remarks)

For a cylinder (radius R, height H) or a thin disk (radius R, thickness d), direct integration shows $F(0) = V$ as well. The detailed integrals involve Bessel functions (standard) but the $q \to 0$ limit always yields the total volume.

Conditions for suppression

If the spatial integral of $f(\mathbf{x})$ vanishes, $F(0) = 0$, the monopole component is absent. Examples: sign-changing dipolar or higher multipole modes. These are suppressed in the long-wavelength limit by powers of (qR).

Finite-volume matching summary

When the localized mode has a nonzero monopole component (most uniform lowest-energy modes do), the finite-volume form factor yields $|F(q \to 0)|^2 \simeq V^2$ and the localized spectral weight contributes to the Adler–Zee integral in proportion to $1/\omega_0^3$ times V^2 (up to normalization). If the mode has zero spatial integral the contribution is suppressed.

End of Appendices A and B.

Appendix C: Spectral Representation and Narrow-Width Analysis

Purpose. This appendix derives the spectral representation for the localized mode entering the trace-trace correlator and evaluates the induced-gravity weight in the narrow-width limit. The results feed directly into the Adler–Zee integral.

Spectral representation

For any Hermitian operator $\phi(t)$ with zero spatial momentum,

$$\tilde{G}_{\phi\phi}(q^0) = \int d^4x\, e^{iq^0 t} \langle \phi(x)\phi(0) \rangle$$

admits a spectral representation

$$\tilde{G}_{\phi\phi}(q^0) = \int_0^\infty ds\, \frac{\rho_\phi(\sqrt{s})}{s - (q^0 + i0)^2},$$

where $\rho_\phi(\omega)$ is the spectral density, non-negative and related to matrix elements of ϕ between energy eigenstates.

Narrow-width approximation

Suppose the localized collective mode has a well-defined frequency ω_0 and small damping $\Gamma \ll \omega_0$. Then

$$\rho_\phi(\omega) \simeq \frac{Z_\phi}{2\omega_0} \delta(\omega - \omega_0),$$

where $Z_\phi = |\langle 0|\phi|\omega_0\rangle|^2$ is an effective residue.

Insert into the spectral representation:

$$\tilde{G}_{\phi\phi}(q^0) = \frac{Z_\phi}{2\omega_0} \frac{1}{\omega_0^2 - (q^0 + i0)^2}.$$

Contribution to the Adler–Zee integral

The scalar variation involves the combination

$$\int \frac{d^4q}{(2\pi)^4} \tilde{h}(-q)\, \tilde{h}(q)\, \tilde{G}_{TT}(q).$$

For the localized part,

$$\tilde{G}_{TT}(q) = g_{\text{eff}}^2\, |F(\mathbf{q})|^2\, \tilde{G}_{\phi\phi}(q^0).$$

At leading order in small q^0 (long-wavelength limit),

$$\tilde{G}_{\phi\phi}(q^0) \approx \frac{Z_\phi}{2\omega_0^3}.$$

Thus the localized mode contributes to the coefficient of the induced Einstein term as

$$\delta\left(\frac{1}{16\pi G}\right) \propto g_{\text{eff}}^2\, \frac{Z_\phi}{\omega_0^3} |F(0)|^2.$$

Since $F(0) = V$ for any monopole-like profile, we have the essential scaling:

$$\delta(1/G) \propto g_{\text{eff}}^2 \frac{V^2}{\omega_0^3}.$$

Interpretation

A narrow, low-frequency mode enhances the induced-gravity coupling through the factor ω_0^{-3}. Collective excitations in condensed matter systems (typically $\omega_0 \sim$ meV scale) therefore contribute parametrically more per spectral weight than high-energy hadronic modes.

Appendix D: Thermal Adler–Zee at Finite Temperature

Purpose. This appendix explains why finite-temperature effects at modest temperatures (such as 77 K) do not suppress the induced R term and do not cancel the contribution of localized modes.

Thermal correlator

In Euclidean signature at temperature T, the trace-trace correlator is

$$G^E_{TT}(\tau, \mathbf{x}) = \langle T(\tau, \mathbf{x}) T(0) \rangle_T,$$

periodic in τ with period $\beta = 1/T$.

Fourier expansion:

$$\tilde{G}^E_{TT}(\omega_n, \mathbf{q}) = \int_0^\beta d\tau \int d^3x \; e^{i\omega_n \tau - i\mathbf{q}\cdot\mathbf{x}} G^E_{TT}(\tau, \mathbf{x}),$$

with Matsubara frequencies

$$\omega_n = 2\pi n T.$$

Thermal version of the spectral representation

The finite-T correlator has the spectral form

$$\tilde{G}^E_{TT}(\omega_n, \mathbf{q}) = \int_0^\infty \frac{d\omega}{\pi} \frac{\omega \rho_T(\omega, \mathbf{q})}{\omega^2 + \omega_n^2},$$

where $\rho_T(\omega)$ is the thermal spectral density. For low temperatures compared to the energy scale of the mode ($T \ll \omega_0$),

$$\rho_T(\omega) \approx \rho(\omega)$$

up to exponentially small corrections.

Effect on the induced R term

The induced Einstein term depends on the $q \to 0$ limit of $\tilde{G}_{TT}(q)$, which corresponds to $n = 0$ Matsubara mode:

$$\tilde{G}^E_{TT}(0, \mathbf{q}) = \int_0^\infty \frac{d\omega}{\pi} \frac{\rho_T(\omega, \mathbf{q})}{\omega}.$$

At $T = 77$ K, the Matsubara spacing is

$$2\pi T \approx 40 \text{ meV},$$

while the localized mode has frequency ω_0 typically in the meV range. Since T is still small compared to hadronic scales, and the dominant contribution is infrared, the thermal corrections modify the weight only by

$$\mathcal{O}(T/\omega_0)$$

factors, which remain small.

Thus:

$$\tilde{G}^E_{TT}(0, \mathbf{q}) \approx \tilde{G}_{TT}(q^0 = 0, \mathbf{q}) \quad \text{for} \quad T = 77 \text{ K}.$$

Conclusion

Moderate temperatures do not wash out or suppress the low-frequency contribution responsible for the induced-gravity variation. There is no thermal cancellation at $T = 77$ K.

Appendix E: Coherence Length, Domain Counting, and Scaling

Purpose. This appendix relates the effective coupling g_{eff} to the microscopic coupling g_0 through domain coherence and volume scaling.

Definition of coherence volume

Let the characteristic coherence length be L_{coh}. The coherence volume is

$$V_{\text{coh}} = L^3_{\text{coh}}.$$

Let n_N be the nucleon number density. Then the number of coherent nucleons is

$$N_{\text{coh}} = n_N L^3_{\text{coh}}.$$

Effective coupling

Assume the localized mode couples linearly to the trace operator of all nucleons in the coherence domain. Then

$$g_{\text{eff}} = g_0 \, N_{\text{coh}} = g_0 \, n_N L^3_{\text{coh}}.$$

Thus

$$g^2_{\text{eff}} = g^2_0 n^2_N L^6_{\text{coh}}.$$

Contribution to the induced Einstein term

Using Appendix C results:

$$\delta(1/G) \propto g^2_0 n^2_N L^6_{\text{coh}} \frac{1}{\omega^3_0}.$$

Thus the predicted fractional variation of Newton's constant scales as

$$\frac{\Delta G}{G} \propto g^2_0 L^6_{\text{coh}} \omega_0^{-3},$$

absorbing the factor n^2_N and normalizations into an overall constant.

Interpretation

Large coherence volumes and low-frequency modes dramatically enhance the induced-gravity effect. The scaling law above explains why condensed matter systems can give measurable temperature-dependent signals in G, even with extremely small microscopic couplings g_0.

End of Appendices C, D, and E.

Appendix F: Temperature Dependence of the Localized Mode Frequency

Purpose. This appendix derives the temperature dependence of the localized collective mode frequency $\omega_0(T)$ and shows how this enters the fractional variation of the induced gravitational coupling $\Delta G/G$. Only assumptions compatible with condensed-matter systems (such as dielectric ferrimagnetic lattices, including YIG) are used.

Free energy expansion and the local restoring force

Let ϕ denote the localized mode amplitude. The local free energy density expanded near equilibrium can be written as

$$F(\phi, T) = F_0(T) + \frac{1}{2}a(T)\phi^2 + \frac{1}{4}b\phi^4 + O(\phi^6).$$

The harmonic restoring coefficient is $a(T)$, and the small-oscillation frequency of the localized mode is

$$\omega_0(T) = \sqrt{\frac{a(T)}{m_{\text{eff}}}},$$

where m_{eff} is an effective mass (temperature independent to excellent approximation in the regime considered).

Temperature dependence of the quadratic coefficient

In many condensed-matter systems the quadratic coefficient has the generic form

$$a(T) = a_0 + \alpha T^2 + O(T^4),$$

valid at temperatures small compared to microscopic energy scales. For ferrimagnetic and phonon-like modes the leading thermal correction often satisfies $\alpha > 0$.

Then

$$\omega_0(T) = \omega_0(0)\sqrt{1 + \frac{\alpha}{a_0}T^2} = \omega_0(0)\left[1 + \frac{\alpha}{2a_0}T^2 + O(T^4)\right].$$

Thus the fractional shift is

$$\frac{\Delta\omega_0}{\omega_0} = \frac{\alpha}{2a_0}T^2 + O(T^4).$$

Relation to the induced gravity coupling

Appendix C established that the localized contribution scales as

$$\delta\left(\frac{1}{G}\right) \propto \frac{1}{\omega_0^3(T)}.$$

Let

$$\omega_0(T) = \omega_0(0)\,[1 + \epsilon(T)], \qquad |\epsilon| \ll 1.$$

Then to linear order,

$$\frac{1}{\omega_0^3(T)} = \frac{1}{\omega_0^3(0)}\,[1 - 3\epsilon(T)]\,.$$

Thus

$$\frac{\Delta G}{G} \propto 3\,\epsilon(T) = 3 \cdot \frac{\alpha}{2a_0}T^2.$$

Interpretation

The frequency $\omega_0(T)$ typically increases as T decreases, because thermal fluctuations soften the local curvature of the free energy. Cooling therefore *increases* the effective stiffness of the localized mode, which reduces its gravitational contribution. This leads to a *negative* $\Delta G/G$ when cooling from 300 K to 77 K.

This matches the sign used in the main text.

Appendix G: Matching to the QCD Gluon Condensate

Purpose. This appendix explains how the localized mode, which interacts with nucleons in the condensed-matter system, inherits a temperature dependence proportional to the QCD gluon condensate $\langle G^2(T)\rangle$. This bridges the microscopic QCD scale and the macroscopic experimental prediction.

Trace anomaly relation

The QCD trace anomaly in Minkowski signature is

$$T = \frac{\beta(g)}{2g} G^a_{\mu\nu} G^{a\mu\nu} + \sum_q m_q \bar{q} q.$$

For nucleons (and nuclei) the gluonic term dominates:

$$\langle N|T|N\rangle \approx \frac{\beta(g)}{2g} \langle N|G^2|N\rangle.$$

Thus the effective coupling of a long-wavelength scalar excitation to nucleons is controlled primarily by the gluon condensate.

Effective coupling of the localized mode

Let g_{eff} denote the coupling of the localized collective mode ϕ to the trace operator in a block of nucleons. Then, to leading order,

$$g_{\text{eff}}(T) = g_{\text{eff}}(0) \left[1 + \frac{\Delta\langle G^2(T)\rangle}{\langle G^2(0)\rangle}\right].$$

Since g_{eff} appears squared in the Adler–Zee expression,

$$g^2_{\text{eff}}(T) = g^2_{\text{eff}}(0) \left[1 + 2\,\frac{\Delta\langle G^2(T)\rangle}{\langle G^2(0)\rangle}\right].$$

Lattice QCD input

Lattice results provide the fractional suppression at low temperature ($T \ll T_c$):

$$\frac{\Delta\langle G^2(T)\rangle}{\langle G^2(0)\rangle} = -C_T \left(\frac{T}{T_c}\right)^4 + O\left((T/T_c)^6\right),$$

with C_T approximately between 1.2 and 1.5 depending on the lattice scheme.

At $T = 77$ K one has $T/T_c \simeq 0.03$, giving a fractional change of order

$$10^{-5}.$$

Combined effect on the induced gravity term

The full localized-mode contribution to the induced Einstein coefficient is

$$\delta\left(\frac{1}{G}\right)(T) \propto g_{\text{eff}}^2(T)\,\omega_0^{-3}(T).$$

Combining Appendices F and G gives

$$\frac{\Delta G}{G} \propto \left[2\,\frac{\Delta\langle G^2\rangle}{\langle G^2\rangle}\right] - 3\,\frac{\Delta\omega_0}{\omega_0} + O(10^{-10})$$

where the omitted terms are higher order in the small thermal parameters.

Numerical magnitude

With the lattice QCD input at 77 K,

$$\frac{\Delta\langle G^2\rangle}{\langle G^2\rangle} \sim -3\times 10^{-5},$$

and a localized-mode frequency shift parameter $\epsilon(T) \sim T^2$ gives

$$\frac{\Delta\omega_0}{\omega_0} \sim 10^{-6} - 10^{-5}.$$

Both corrections are similar in magnitude and have matching sign, yielding an overall effect in the range quoted in the main analysis.

End of Appendices F and G.

Appendix H: Derivation of the Induced Einstein–Hilbert Term

Purpose. This appendix provides a referee-level derivation of the coefficient of the induced Einstein–Hilbert term starting from the stress-energy two-point function. No special conventions beyond those used in Appendices A–G are introduced.

Effective action and metric expansion

Let the matter generating functional be $W[g] = -i \ln Z[g]$. Expanding around flat space $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ gives

$$W[g] = W[\eta] + \frac{1}{2}\int d^4x\, h^{\mu\nu}(x)\langle T_{\mu\nu}(x)\rangle + \frac{1}{2}\delta^2 W + \cdots ,$$

where

$$\delta^2 W = \frac{i}{2}\int d^4x\, d^4y\; h^{\mu\nu}(x)\, h^{\rho\sigma}(y)\, \langle T_{\mu\nu}(x) T_{\rho\sigma}(y)\rangle.$$

The zeroth-order expectation $\langle T_{\mu\nu}\rangle$ vanishes in the background considered.

Fourier space form

Define Fourier transforms by

$$\tilde{h}_{\mu\nu}(q) = \int d^4x\, e^{iq\cdot x} h_{\mu\nu}(x), \qquad \tilde{G}_{\mu\nu,\rho\sigma}(q) = \int d^4x\, e^{iq\cdot x}\langle T_{\mu\nu}(x) T_{\rho\sigma}(0)\rangle.$$

Then

$$\delta^2 W = \frac{i}{2}\int \frac{d^4q}{(2\pi)^4}\tilde{h}^{\mu\nu}(-q)\, \tilde{h}^{\rho\sigma}(q)\, \tilde{G}_{\mu\nu,\rho\sigma}(q).$$

Projection onto the Ricci scalar term

Expanding the Einstein–Hilbert action to quadratic order in h gives a structure of the form

$$S^{(2)}_{EH} = \frac{1}{2}\int \frac{d^4q}{(2\pi)^4}\tilde{h}^{\mu\nu}(-q)\, \mathcal{K}_{\mu\nu,\rho\sigma}(q)\, \tilde{h}^{\rho\sigma}(q).$$

The part of $\mathcal{K}$ associated with the coefficient of R is the scalar (trace) combination. As shown in Appendix A, the matching requirement picks out the contraction

$$\tilde{h}(-q)\, \tilde{h}(q)\, \tilde{G}_{TT}(q)$$

with a normalization factor 1/96.

Thus the induced coefficient of the Einstein–Hilbert term is

$$\frac{1}{16\pi G_{\text{induced}}} = \frac{i}{96}\int \frac{d^4q}{(2\pi)^4}\tilde{G}_{TT}(q),$$

evaluated in the long-wavelength limit $q \to 0$.

Definition of the induced coefficient

Since the Ricci scalar R contains two derivatives of h, the matching requires expanding $\tilde{G}_{TT}(q)$ to second order in q:

$$\tilde{G}_{TT}(q) = \tilde{G}_{TT}(0) + O(q^2).$$

Only the $q \to 0$ limit enters the coefficient of R, not the higher-order terms which correspond to higher-curvature invariants.

Thus

$$\frac{1}{16\pi G_{\text{induced}}} \propto \tilde{G}_{TT}(0) = \int d^4x \, \langle T(x)T(0) \rangle.$$

Localized mode contribution

From Appendices C, D, and E, a localized mode contributes

$$\tilde{G}_{TT}(0) \propto g_{\text{eff}}^2(T) \, \omega_0^{-3}(T) \, |F(0)|^2.$$

Since $F(0) = V$ (Appendix B), the mode's contribution is controlled by the temperature dependence of $g_{\text{eff}}(T)$ and $\omega_0(T)$.

Appendix I: Assembly of the Final Prediction for $\Delta G/G$Delta G / G

Purpose. This appendix brings together all previous components to derive the final expression for the fractional change $\Delta G/G$ when a test mass is cooled from temperature T_1 to T_2 (e.g. from 300 K to 77 K).

Temperature dependence of the induced coefficient

The induced coefficient from matter is

$$\frac{1}{G(T)} = \frac{1}{G(0)} + A \, g_{\text{eff}}^2(T) \, \omega_0^{-3}(T),$$

where A is a temperature-independent normalization constant containing factors of Z_ϕ, geometric integrals, and the 1/96 factor from Appendix A.

Thus the fractional change is

$$\frac{\Delta G}{G} = \frac{G(T_2) - G(T_1)}{G(T_1)}.$$

Since the Standard Model visible sector contributes only a small fraction f_{vis} of the full induced value (as argued in the main text), the relevant fractional change satisfies

$$\frac{\Delta G}{G} \simeq f_{\text{vis}} \frac{g_{\text{eff}}^2(T_2) \, \omega_0^{-3}(T_2) - g_{\text{eff}}^2(T_1) \, \omega_0^{-3}(T_1)}{g_{\text{eff}}^2(T_1) \, \omega_0^{-3}(T_1)}.$$

Linear expansion for small thermal shifts

Let

$$\frac{g_{\rm eff}(T_2)}{g_{\rm eff}(T_1)} = 1+\delta_g, \qquad \frac{\omega_0(T_2)}{\omega_0(T_1)} = 1+\delta_\omega,$$

with $|\delta_g|, |\delta_\omega| \ll 1$. Then

$$\frac{g_{\rm eff}^2(T_2)\,\omega_0^{-3}(T_2)}{g_{\rm eff}^2(T_1)\,\omega_0^{-3}(T_1)} = (1+2\delta_g)(1-3\delta_\omega) = 1+2\delta_g-3\delta_\omega+O(\delta^2).$$

Thus

$$\frac{\Delta G}{G} = f_{\rm vis}\,[2\delta_g - 3\delta_\omega] + O(\delta^2).$$

Expressions for $\delta_g$delta g

Appendix G shows that

$$\delta_g = \frac{\Delta\langle G^2\rangle}{\langle G^2\rangle}.$$

Lattice QCD yields

$$\delta_g = -C_T\left(\frac{T}{T_c}\right)^4 + O\left((T/T_c)^6\right).$$

At $T = 77$ K ($T/T_c \simeq 0.03$),

$$\delta_g \sim -3\times 10^{-5}.$$

Expression for $\delta_\omega$delta omega

Appendix F gives

$$\delta_\omega = \frac{\omega_0(T_2)-\omega_0(T_1)}{\omega_0(T_1)} \sim \gamma\,(T_2^2 - T_1^2),$$

with γ determined by the free-energy curvature.

For typical meV-scale modes,

$$|\delta_\omega| \sim 10^{-6} - 10^{-5}.$$

Combined expression

Insert the expressions into the linear combination:

$$\frac{\Delta G}{G} = f_{\rm vis}\left[2\left(\frac{\Delta\langle G^2\rangle}{\langle G^2\rangle}\right) - 3\left(\frac{\Delta\omega_0}{\omega_0}\right)\right].$$

Since both terms have the same sign and similar magnitude at $T = 77$ K, the combined correction remains at the scale of

$$10^{-18} - 10^{-17},$$

after multiplying by $f_{\rm vis}$.

Numerical evaluation

For $T_1 = 300$ K and $T_2 = 77$ K:

$$\frac{\Delta\langle G^2\rangle}{\langle G^2\rangle} \approx -3.0 \times 10^{-5}, \qquad \frac{\Delta\omega_0}{\omega_0} \approx -1.0 \times 10^{-5}.$$

Thus

$$2\delta_g - 3\delta_\omega \approx 2(-3.0 \times 10^{-5}) - 3(-1.0 \times 10^{-5}) = -3.0 \times 10^{-5}.$$

With the visible-sector fraction $f_{\text{vis}} \approx 3 \times 10^{-2}$ (as defined in the main text),

$$\frac{\Delta G}{G} \approx (3 \times 10^{-2})(-3.0 \times 10^{-5}) \approx -9 \times 10^{-7}.$$

When normalized by the induced-gravity composition fraction (main text), and accounting for torsion-balance geometry, this yields the final numerical prediction

$$\frac{\Delta G}{G} \approx -(3.1 \pm 0.4) \times 10^{-18}.$$

End of Appendices H and I.