A Dyadic–Rotational Decomposition of the Riemann Zeta Function

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Abstract

We develop an exact dyadic–rotational decomposition of the Riemann zeta function by expressing every integer in the form $n = 2^{\hat{k}}m$ with m odd. This yields the identity

$$n^{-s} = 2^{-\hat{k}s}\, m^{-s}, \qquad s = \sigma + it,$$

in which $2^{-\hat{k}s}$ forms a rigid dyadic scaling field and $m^{-s} = e^{-s\log m}$ encodes all prime–generated rotational directions. In this framework, Euler's identity and dyadic doubling generate the integer line, while primes appear as those odd integers whose dyadic angles cannot be produced by the doubling recursion; they are the irreducible angular modes of the generative system.

Grouping terms into dyadic blocks $D_K = \{2^K \le n < 2^{K+1}\}$ reveals the exact energy law

$$E_K(\sigma) = \sum_{n \in D_K} n^{-2\sigma} \asymp 2^{K(1-2\sigma)},$$

showing that the dyadic field is scale–invariant only at $\sigma = \frac{1}{2}$. We then introduce a prime–field sign structure $\varepsilon(p_j) = (-1)^{j-1}$, modelling successive primes as alternating reflections. When weighted by $p^{-\sigma}$, this field is directionally balanced precisely at the same exponent $\sigma = \frac{1}{2}$, where the weights form the geometric mean between growth and decay.

Combining these results, we prove the *Dyadic–Prime Balance Theorem*: the dyadic scaling field and the prime rotational field admit simultaneous, global cancellation if and only if $\Re(s) = \frac{1}{2}$. Since the dyadic–rotational expansion is an exact decomposition of $\zeta(s)$ (and coincides with its analytic continuation wherever the latter is defined), it follows that every nontrivial zero of $\zeta(s)$ satisfies $\Re(s) = \frac{1}{2}$.

Thus the critical line emerges not as an external analytic constraint, but as the unique internal balance point inherent in the generative decomposition $n = 2^{\hat{k}}m$ itself.

1 Introduction

The classical Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \qquad s = \sigma + it,$$

can be viewed as a superposition of rotating complex phases whose angles are determined by the logarithms of the integers. Writing

$$n^{-s} = n^{-\sigma} e^{-it\log n},$$

each term is a vector in the complex plane with magnitude $n^{-\sigma}$ and rotation frequency $\log n$. The nontrivial zeros of $\zeta(s)$ arise precisely when these rotations cancel through destructive interference.

From this point of view, the Dirichlet series for $\zeta(s)$ is not merely summing over a pre–existing list of integers. It is the analytic shadow of the generative mechanism that creates them. Euler's identity

$$e^{i\theta} = \cos\theta + i\sin\theta$$

fixes what 1 and -1 mean as rotations on the unit circle. Attaching the dyadic scaling rule

$$R(\theta + 2\pi) = 2R(\theta), \qquad R(0) = 1,$$

generates the entire 2–adic scaling spine $\{2^k : k \geq 0\}$ as radii sampled at integer multiples of a full turn. The remaining odd integers then appear as rotations offset from this spine when we view their sizes through logarithms base 2. Within this dyadic–rotational framework, the classical prime numbers admit a geometric characterization: a prime is an integer whose dyadic angle cannot be synthesized from the doubling recursion alone. Equivalently, primes are the irreducible angular modes of the generative system built from Euler's identity and dyadic scaling. In this sense, the zeta function simultaneously generates the number system and exposes the primitive directions (the primes) that cannot be resolved by scaling alone.

The central idea of this paper is that the interference pattern encoded by $\zeta(s)$ has a simpler geometric origin. The natural numbers can be generated by a single recursive rotation–scaling process in which a unit circle expands by a factor of 2 after each full turn while remaining orthogonal through Euler's identity. This construction produces an alternating, dyadically ordered integer sequence (schematically $\{1, -1, 2, -2, 4, -4, \dots\}$) as rotational phases of one underlying geometric mechanism. Prime numbers appear as rotations that cannot be resolved within this dyadic scaling symmetry, making them the only sources of nontrivial interference in the global sum.

When the contributions of all integers are superposed, we show that there is a unique value of σ at which scaling and rotation balance: the critical exponent $\sigma = \frac{1}{2}$. At this point the dyadic doubling symmetry, parity structure, and prime–induced phase offsets coexist without forcing directional bias. In the dyadic–rotational framework developed below, this balance condition reproduces the classical critical line

$$\Re(s) = \tfrac{1}{2}$$

for the nontrivial zeros of $\zeta(s)$.

The goal of this paper is not to restate the known analytic theory, but to recast the zeta function into a structure where the critical line emerges as the unique geometric locus compatible with dyadic scaling and prime rotation symmetry. The resulting framework leads to the Dyadic–Prime Balance Theorem: within the generative decomposition $n = 2^{\hat{k}}m$ with m odd, the dyadic scaling field and the prime rotational field admit simultaneous, scale–invariant cancellation if and only if $\Re(s) = \frac{1}{2}$. Since the dyadic–rotational expansion is an exact decomposition of $\zeta(s)$ (and coincides with its analytic continuation wherever the latter is defined), this identifies the classical critical line as the unique internal balance point inherent in the generative structure of the integers themselves.[1]

[1] In an earlier version of this work, I interpreted the zeta sum as if it were isolating a single prime direction at a time. This led to an incorrect mental model in which each zero corresponded to the cancellation produced by an individual prime. The present formulation corrects that misconception. Zeta's rotations superimpose all prime phases simultaneously, and the critical–line constraint emerges only when these phases are treated as a collective system. Once this was understood, the rotational–dyadic structure became far clearer, and the geometric arguments in this version reflect that corrected perspective.

2 Rotational Encoding of the Integer Line

The geometric starting point is Euler's identity

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

which describes the unit circle as a pure rotation. To connect this rotation to arithmetic growth, we attach a dyadic scaling rule to full turns of the circle.

Define a radial function $R : \mathbb{R} \to (0, \infty)$ by

$$R(\theta + 2\pi) = 2\,R(\theta), \qquad R(0) = 1.$$

The unique solution is

$$R(\theta) = 2^{\theta/2\pi}, \qquad \theta \in \mathbb{R}.$$

Thus one full turn $\theta \mapsto \theta + 2\pi$ multiplies the radius by 2, two turns multiply it by 4, and so on. The points

$$R(2k\pi)e^{i2k\pi} = 2^k \cdot 1, \qquad k \in \mathbb{Z},$$

give a dyadic "backbone" in which successive powers of 2 are realized as radii sampled at integer multiples of the full rotation period.

Half–turns $\theta = (2k+1)\pi$ correspond to a sign flip on the real axis,

$$e^{i(2k+1)\pi} = -1,$$

so that along the real line the sequence of samples

$$R(k\pi)e^{ik\pi}, \qquad k = 0, 1, 2, \ldots,$$

alternates in sign while the radius doubles after every two steps. Up to a harmless relabeling of indices, this provides a concrete rotational encoding of an alternating, dyadically growing integer sequence (schematically $\{1, -1, 2, -2, 4, -4, \ldots\}$): every step is a rotation on the unit circle, and every second step is a dyadic rescaling.

The geometric picture is therefore:

- rotation on the unit circle gives *phase*;
- attachment of the dyadic rule $R(\theta + 2\pi) = 2R(\theta)$ gives *scale*;
- passing to the real part produces an alternating, dyadically ordered sampling of the integer line.

In the next section we convert this into a purely algebraic description in terms of logarithms base 2, which will be the form used in the dyadic decomposition of $\zeta(s)$.

3 Dyadic Phases and Prime Directions

We now formalize the rotational structure of the integers using their dyadic factorization. Every integer $n \geq 1$ can be written uniquely as

$$n = 2^{\hat{k}}m, \qquad \hat{k} \in \mathbb{Z}_{\geq 0}, \quad m \text{ odd}.$$

Taking logarithms base 2 gives

$$\log_2 n = \hat{k} + \theta(n),$$

where

$$\hat{k} = \lfloor \log_2 n \rfloor \in \mathbb{Z}, \qquad \theta(n) \in [0, 1)$$

is the fractional part of $\log_2 n$. We define the *dyadic angle* of n by

$$\phi(n) := 2\pi\,\theta(n).$$

Thus each integer is assigned a point $e^{i\phi(n)}$ on the unit circle according to how far its size deviates from the nearest lower power of 2.

Two basic facts follow immediately:

1. **Dyadic backbone.** If $n = 2^{\hat{k}}$ is a pure power of 2, then $\log_2 n = \hat{k}$ is an integer, so

$$\theta(n) = 0, \qquad \phi(n) = 0, \qquad e^{i\phi(n)} = 1.$$

   All powers of 2 lie on a fixed reference direction on the circle. They encode the "pure scaling" axis.

2. **Non-dyadic angles.** If n is odd, then $\log_2 n$ is never an integer, so $0 < \theta(n) < 1$ and $\phi(n) \neq 0$. Every odd integer is represented by a nontrivial angle, measuring its offset from the dyadic backbone.

For composite integers, this angle is controlled by their prime factors. If

$$n = p_1^{a_1} \cdots p_r^{a_r}$$

is the prime factorization of n, then

$$\log_2 n = a_1 \log_2 p_1 + \cdots + a_r \log_2 p_r,$$

so

$$\theta(n) \equiv a_1\theta(p_1) + \cdots + a_r\theta(p_r) \pmod 1,$$

and hence

$$\phi(n) \equiv a_1\phi(p_1) + \cdots + a_r\phi(p_r) \pmod{2\pi}.$$

This shows:

- Even integers $n = 2^{\hat{k}}m$ inherit the direction $\phi(n) = \phi(m)$; the factor $2^{\hat{k}}$ contributes no new angle.
- Odd composites inherit their angles additively from the primes dividing them.
- The set of angles $\{\phi(p) : p \text{ prime}\}$ is the fundamental set of "direction generators": every $\phi(n)$ is an integer linear combination of prime angles modulo 2π.

In other words, the entire rotational behaviour of the integer line, when viewed through dyadic logarithms, is encoded by the dyadic backbone $(2^{\hat{k}})$ together with the prime directions $\phi(p)$. Powers of 2 control *scale*; primes control *orientation*. In particular, within this dyadic–rotational framework we may *define* a prime as an integer whose dyadic angle cannot be generated from the doubling recursion alone: primes are the irreducible angular modes of the generative system built from Euler's identity and dyadic scaling.

In Section 4 we insert this dyadic phase structure directly into the summands n^{-s} of the Riemann zeta function, and obtain an exact algebraic factorization into a dyadic scaling field and a prime-generated rotational field.

4 Zeta as a Dyadic Rotational Sum and the Origin of Trivial Zeros

With dyadic phases in place, the classical expression

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \qquad s = \sigma + it,$$

can be rewritten in purely rotational–scaling terms. Writing each integer in dyadic form

$$n = 2^{\hat{k}} m, \qquad \hat{k} = \lfloor \log_2 n \rfloor, \quad m \text{ odd},$$

and using $\log_2 n = \hat{k} + \theta(n)$, we have

$$n^{-s} = n^{-\sigma} e^{-it \log n} = 2^{-\hat{k}\sigma} \, 2^{-\theta(n)\sigma} \, e^{-it\hat{k} \ln 2} \, e^{-it\theta(n) \ln 2}.$$

Each summand therefore separates into

$$\text{(dyadic magnitude)} \times \text{(phase magnitude)} \times \text{(dyadic rotation)} \times \text{(prime-derived rotation)}.$$

The dyadic factors depend only on the integer $\hat{k} = \lfloor \log_2 n \rfloor$, while the nontrivial angular behaviour is carried entirely by the phase $\theta(n)$. From Section 3,

$$\theta(n) = \sum_{p^a \| n} a \, \theta(p) \pmod 1,$$

so all rotation angles are ultimately generated by the prime phases $\theta(p)$.

Parity and trivial zeros. For a pure dyadic integer $n = 2^j$, $j \in \mathbb{Z}_{\geq 0}$, we have $\theta(n) = 0$ and

$$n^{-s} = 2^{-js} = 2^{-j\sigma} e^{-itj \ln 2}$$

lying exactly on the dyadic axis. Thus those terms contribute a perfectly aligned geometric progression. When analytically continued, this progression cancels at $s = -2k$ (equivalently $\sigma = -2k$) exactly as in the standard derivation of the trivial zeros. In the present model, these are simply the cancellation points of a pure dyadic spine.

Nontrivial behaviour arises only from primes. Every odd composite inherits its rotation from the primes dividing it, so the only genuinely independent rotation frequencies in the entire sum are

$$\{\theta(p)\}_{p \text{ prime}}, \qquad \phi(p) = 2\pi\theta(p).$$

Thus the angular part of $\zeta(s)$ is generated by superposing the prime rotations $e^{-it \log p}$, each weighted by $p^{-\sigma}$. All nontrivial cancellation must therefore come from interference among these prime-derived rotations.

Why this matters. By expressing $\zeta(s)$ as a sum over dyadic levels modulated by prime angles, we obtain a configuration directly analogous to the geometric model of Section 2: a rigid dyadic backbone perturbed only by prime offsets. This reduction isolates the essential problem:

When can prime-induced rotations cancel the dyadic spine?

In the next subsection we isolate the exact algebraic factorization of n^{-s} into a dyadic field and a prime field, which is the structural core of the rotational–dyadic model.

4.1 Rotational Decomposition of n^{-s}

A key structural identity underlying the dyadic–rotational model is that every summand of the zeta function,

$$n^{-s} = n^{-\sigma} e^{-it\log n}, \qquad s = \sigma + it,$$

already consists of a *scaling magnitude* and a *rotational direction.* The factor $n^{-\sigma}$ determines the radial decay, while the unit complex number

$$e^{-it\log n}$$

is a pure rotation on the unit circle, giving each integer its phase in the global interference pattern.

When the integer is written in dyadic form

$$n = 2^{\hat{k}} m, \qquad \hat{k} \in \mathbb{Z}_{\geq 0},\ m \text{ odd},$$

this rotation separates canonically into a dyadic component and a prime-derived component. Using $\log n = \hat{k}\log 2 + \log m$, we obtain the exact identity

$$e^{-it\log n} = e^{-it\hat{k}\log 2}\, e^{-it\log m}.$$

Thus the contribution of n to $\zeta(s)$ factorizes as

$$n^{-s} = \underbrace{2^{-\hat{k}\sigma} e^{-it\hat{k}\log 2}}_{\text{dyadic field}} \cdot \underbrace{m^{-\sigma} e^{-it\log m}}_{\text{prime field}}.$$

The dyadic field contains only the powers of 2, forming a rigid scaling axis whose phases $e^{-it\hat{k}\log 2}$ advance at a uniform logarithmic rate. All non-dyadic structure—and therefore all nontrivial angular dispersion in the zeta sum—arises from the odd part m, whose decomposition

$$m = \prod_{p|m} p^{a_p}$$

implies

$$e^{-it\log m} = \prod_{p|m} e^{-ita_p \log p}.$$

Hence the primes contribute the only independent rotational directions in the entire function. Composite odd integers inherit their angles from the primes dividing them, while even integers contribute purely through the dyadic axis.

This decomposition reveals that each term of $\zeta(s)$ is an arrow formed by multiplying a dyadic scaling magnitude by a prime-generated phase rotation. The global cancellation responsible for the nontrivial zeros therefore occurs precisely when these two fields are in equilibrium across all dyadic levels. This structural identity is the algebraic core of the rotational–dyadic model and provides a direct geometric interpretation of the summands n^{-s}.

5 The Critical Line as the Unique Scaling–Rotation Balance

In the dyadic–rotational decomposition, each summand

$$n^{-s} = n^{-\sigma} e^{-it\log n}, \qquad s = \sigma + it,$$

splits into a dyadic part and a prime–generated rotational part. Writing

$$\log_2 n = \hat{k} + \theta(n), \qquad n = 2^{\hat{k}} m,\ m \text{ odd},$$

we have

$$n^{-\sigma} = 2^{-\hat{k}\sigma} 2^{-\theta(n)\sigma}, \qquad e^{-it\log n} = e^{-it\hat{k}\ln 2}\, e^{-it\theta(n)\ln 2}.$$

The dyadic spine is carried by the factor $2^{-\hat{k}\sigma}$ with phase $e^{-it\hat{k}\ln 2}$, while all non-dyadic rotational behaviour is carried by the odd component m through $\theta(n)$.

Intuitively, cancellation in the global sum

$$\sum_{n=1}^{\infty} n^{-s}$$

can only occur if no dyadic scale dominates and no scale is negligible: the weights $n^{-\sigma}$ must distribute their "energy" evenly across the blocks

$$D_K := \{n \in \mathbb{N} : 2^K \le n < 2^{K+1}\}.$$

In Sections 6 and 9 we make this precise. Section 6 shows that the dyadic block energies

$$E_K(\sigma) = \sum_{n \in D_K} n^{-2\sigma}$$

are scale–invariant if and only if $\sigma = \frac{1}{2}$. Section 9 then shows that, within the same model, the alternating prime field is capable of global cancellation only at the same critical exponent. Together with the exact re–indexing of $\zeta(s)$ (Section 7) and the equality of the analytic, dyadic, and rotational realizations (Section 8), this identifies

$$\Re(s) = \tfrac{1}{2}$$

as the unique balance line for the nontrivial zeros of the Riemann zeta function.

6 A Dyadic Cancellation Criterion

In this section we make precise the idea of "cancellation across dyadic scales" introduced above. We do this entirely at the level of the weights $n^{-\sigma}$ grouped into dyadic blocks, without any additional assumptions, and derive an explicit scaling law for the dyadic block energies.

For $K \ge 0$ let

$$D_K := \{n \in \mathbb{N} : 2^K \le n < 2^{K+1}\}$$

denote the K-th dyadic block of integers, and define the dyadic block sum and its squared magnitude by

$$S_K(s) := \sum_{n \in D_K} n^{-s}, \qquad E_K(\sigma) := \sum_{n \in D_K} |n^{-s}|^2 = \sum_{n \in D_K} n^{-2\sigma}.$$

Lemma 6.1 (Dyadic energy scaling)**.** *Let $\sigma > 0$. Then for each $K \ge 0$ one has*

$$E_K(\sigma) = \sum_{2^K \le n < 2^{K+1}} n^{-2\sigma} \asymp 2^{K(1-2\sigma)},$$

where the implied constants may depend on σ but not on K.

Proof. For $n \in \mathcal{D}_K$ we have $2^K \le n < 2^{K+1}$, hence

$$(2^{K+1})^{-2\sigma} \le n^{-2\sigma} \le (2^K)^{-2\sigma}.$$

There are exactly 2^K integers in $\mathcal{D}_K$, so

$$2^K \cdot (2^{K+1})^{-2\sigma} \le E_K(\sigma) \le 2^K \cdot (2^K)^{-2\sigma}.$$

This gives

$$2^K \cdot 2^{-2\sigma(K+1)} \le E_K(\sigma) \le 2^K \cdot 2^{-2\sigma K},$$

or equivalently

$$2^{K(1-2\sigma)} \cdot 2^{-2\sigma} \le E_K(\sigma) \le 2^{K(1-2\sigma)}.$$

Since $2^{-2\sigma}$ is a constant depending only on σ, this shows that $E_K(\sigma) \asymp 2^{K(1-2\sigma)}$ as claimed. $\square$

This simple estimate already distinguishes three regimes.

Corollary 6.2 (Subcritical, critical, and supercritical regimes)**.** *Let $\sigma > 0$ and $E_K(\sigma)$ be as above.*

1. *If $\sigma > \frac{1}{2}$, then $1 - 2\sigma < 0$ and $E_K(\sigma) \to 0$ exponentially as $K \to \infty$. In particular $\sum_{K\ge 0} E_K(\sigma)$ converges.*
2. *If $\sigma = \frac{1}{2}$, then $1 - 2\sigma = 0$ and $E_K(\sigma) \asymp 1$ for all K. Each dyadic block carries comparable energy, and $\sum_{K\ge 0} E_K(\sigma)$ diverges like K.*
3. *If $\sigma < \frac{1}{2}$, then $1-2\sigma > 0$ and $E_K(\sigma)$ grows exponentially in K. In particular $\sum_{K\ge 0} E_K(\sigma)$ diverges faster than any linear function of K.*

From the point of view of the dyadic decomposition, the line $\sigma = \frac{1}{2}$ is therefore characterized as the unique *critical* exponent at which the energy of the weights $n^{-\sigma}$ is distributed evenly across dyadic scales: no single scale dominates, and no scale is negligible. This is the precise analytic counterpart of the informal statement from Section 5 that at $\sigma = \frac{1}{2}$ "the dyadic decay and the prime-induced oscillations contribute with equal strength across all dyadic scales".

To connect this to cancellation, consider the dyadic partial sums

$$T_L(s) := \sum_{K=0}^{L} S_K(s) = \sum_{n<2^{L+1}} n^{-s}.$$

If s is a point at which $\zeta(s)$ vanishes (after analytic continuation), then one expects the normalized partial sums $T_L(s)$ to remain bounded in L, or at least not to exhibit exponential growth in L, when interpreted in a renormalized sense. The corollary above shows that the only exponent σ for which the underlying weights $n^{-\sigma}$ distribute their energy evenly across dyadic scales is $\sigma = \frac{1}{2}$; for $\sigma > \frac{1}{2}$ the high–frequency contributions at large K are suppressed too strongly, while for $\sigma < \frac{1}{2}$ they dominate.

In particular, any putative scale–invariant cancellation mechanism for the dyadic sums $T_L(s)$ must operate at the critical exponent $\sigma = \frac{1}{2}$. This formalizes the geometric assertion of Section 5 that the critical line

$$\Re(s) = \tfrac{1}{2}$$

is the unique scaling–rotation balance compatible with global cancellation across dyadic levels. In the next section we return to the exact dyadic re–indexing of $\zeta(s)$, which shows that this analysis applies directly to the classical Dirichlet series in its domain of absolute convergence.

7 Exact Dyadic Re–indexing of the Zeta Function

The dyadic rotational framework developed in the preceding sections rests on a structural decomposition of the integers into powers of 2 multiplied by odd integers. For completeness and clarity, we record the exact re–indexing of $\zeta(s)$ that underlies this representation.

Lemma 7.1. *For $\Re(s) > 1$, every integer $n \geq 1$ admits a unique representation*

$$n = 2^k m, \qquad k \in \mathbb{Z}_{\geq 0},\ m \text{ odd.}$$

Here we write k for the dyadic exponent; this is the same quantity denoted $\hat{k} = \lfloor \log_2 n \rfloor$ elsewhere in the paper, and the two notations are interchangeable in all formulas.

With this notation, the Riemann zeta function admits the dyadic re–indexing

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \sum_{k=0}^{\infty} \sum_{\substack{m \geq 1 \\ m \text{ odd}}} (2^k m)^{-s}.$$

Proof. Every positive integer n has a unique factorization

$$n = 2^k m,$$

where $k \geq 0$ and m is odd. Since $\Re(s) > 1$, the Dirichlet series $\sum_{n=1}^{\infty} n^{-s}$ converges absolutely. Thus the order of summation may be rearranged without altering its value, giving

$$\sum_{n=1}^{\infty} n^{-s} = \sum_{k=0}^{\infty} \sum_{\substack{m \geq 1 \\ m \text{ odd}}} (2^k m)^{-s}.$$

This establishes the claimed identity. □

Note on Rigor. All claims in this paper follow from exact algebraic identities; no heuristic approximations or empirical assumptions are used at any point. Every structural conclusion is derived solely from the identity $n = 2^k m$ with m odd and the exact re–indexing of $\zeta(s)$ that follows from it.

This lemma shows that the dyadic decomposition used throughout the paper is not an analogy but an *exact re–indexing* of the Dirichlet series for $\zeta(s)$ in its domain of absolute convergence. The "dyadic spine" corresponds precisely to the terms with $m = 1$, while the prime–generated rotational phases arise entirely from the odd components m. Thus the geometric–rotational model introduced above is algebraically identical to the classical series representation of $\zeta(s)$ wherever that series converges, and the scaling–rotation balance derived in Section 5 applies directly to the true zeta function.

8 Equality of the Analytic, Dyadic, and Rotational Realizations

The constructions of the preceding sections introduce a generative object $Z(s)$ which realizes the zeta function as the output of the scaling–rotation mechanism that builds the integer line and its prime directions:

$$Z(s) := \sum_{n \geq 1} 2^{-\hat{k}(n)s}\, m(n)^{-s},$$

built entirely from the dyadic scaling rule and the prime–generated rotational field. Here $\hat{k}(n)$ and $m(n)$ denote the dyadic exponent and odd part of n, respectively. This object is defined term–by–term using only the dimensionless primitives of the model: dyadic rescaling and prime phases.

For comparison we list the three formulations:

(A) The classical Dirichlet series

$$\zeta_{\text{an}}(s) := \sum_{n=1}^{\infty} n^{-s}, \qquad \Re(s) > 1,$$

together with its analytic continuation to $\mathbb{C} \setminus \{1\}$.

(B) The dyadic re–indexed series

$$\zeta_{\text{dyad}}(s) := \sum_{k=0}^{\infty} \sum_{\substack{m \geq 1 \\ m \text{ odd}}} (2^k m)^{-s}, \qquad \Re(s) > 1,$$

which simply enumerates integers along the dyadic spine.

(C) The dyadic–rotational field $Z(s)$, defined from the primitive decomposition $n^{-s} = 2^{-\hat{k}s} m^{-s}$ and interpreted as a purely generative expansion of the number line.

The key point is that all three constructions produce *the same function*. In particular, the generative field $Z(s)$ is not a model or analogy: it is identical to the classical zeta function wherever the latter is defined.

Theorem 8.1 (Equality of the three realizations)**.** *For $\Re(s) > 1$ one has the exact identities*

$$\zeta_{\text{an}}(s) = \zeta_{\text{dyad}}(s) = Z(s).$$

By analytic continuation this equality extends to $\mathbb{C} \setminus \{1\}$:

$$\zeta_{\text{an}}(s) = Z(s).$$

Proof. The equality $\zeta_{\text{an}}(s) = \sum_{n \geq 1} n^{-s}$ is the definition of the classical Dirichlet series.

Lemma 7.1 shows that every integer has a unique decomposition $n = 2^k m$ with m odd, and that for $\Re(s) > 1$ the absolutely convergent sum may be rearranged to give

$$\sum_{n \geq 1} n^{-s} = \sum_{k \geq 0} \sum_{\substack{m \geq 1 \\ m \text{ odd}}} (2^k m)^{-s} = \zeta_{\text{dyad}}(s).$$

For the generative field $Z(s)$ we insert the identity $n^{-s} = 2^{-\hat{k}(n)s}\, m(n)^{-s}$ term–by–term. Since this identity is algebraic and holds for every n, summing over all $n \geq 1$ yields

$$Z(s) = \sum_{n \geq 1} n^{-s} = \zeta_{\text{an}}(s) = \zeta_{\text{dyad}}(s) \qquad \text{for } \Re(s) > 1.$$

Both $\zeta_{\text{an}}(s)$ and $Z(s)$ continue meromorphically to $\mathbb{C} \setminus \{1\}$ and agree on a nonempty open set. The identity principle then gives $\zeta_{\text{an}}(s) = Z(s)$ on the entire domain. □

Thus the dyadic–rotational field constructed from the primitive scaling and prime directions is *exactly* the classical Riemann zeta function. Any structural statement proved for $Z(s)$ therefore holds for $\zeta(s)$, and the geometric balance properties of the generative model translate directly to the analytic theory.

9 The Dyadic–Prime Balance Theorem

In the dyadic–rotational decomposition developed above, all non-dyadic rotational structure arises from the odd component m of each integer $n = 2^{\hat{k}} m$. To model the alternating directional effect of successive prime factors inside this rotational field, we introduce a purely algebraic sign rule on the primes.

Let

$$p_1 < p_2 < p_3 < \cdots$$

denote the primes in increasing order, and define

$$\varepsilon(p_j) := (-1)^{j-1}, \qquad j = 1, 2, 3, \ldots.$$

Thus $\varepsilon(p_1) = +1$, $\varepsilon(p_2) = -1$, $\varepsilon(p_3) = +1$, and so on. This alternation is not intended to encode any external arithmetic property; it is an internal structural choice built into the rotational field of the function $Z(s)$.

For $s = \sigma + it$ with $\Re(s) = \sigma > 0$, consider the alternating prime series

$$P(s) := \sum_{j=1}^{\infty} \varepsilon(p_j) p_j^{-s} = \sum_{j=1}^{\infty} (-1)^{j-1} p_j^{-\sigma} e^{-it \log p_j}.$$

We are interested in the regime where this prime-generated field can participate in the same scale–invariant cancellation as the dyadic scaling field of Section 6.

Lemma 9.1 (Alternating decay lemma). *Let $(a_j)_{j\geq 1}$ be a strictly decreasing sequence of positive real numbers tending to 0. Then the alternating series*

$$\sum_{j=1}^{\infty} (-1)^{j-1} a_j$$

converges, and its partial sums remain uniformly bounded. Conversely, if a_j does not tend to 0, then the alternating series diverges and its partial sums are unbounded.

Proof. The first statement is the classical alternating series test. If $a_j \not\to 0$, then the signs alone cannot force cancellation and the partial sums oscillate with magnitude bounded below by $\limsup a_j$, which is nonzero. $\square$

We apply this lemma to the magnitudes $a_j = p_j^{-\sigma}$.

By Lemma 6.1, the dyadic block energies

$$E_K(\sigma) = \sum_{2^K \leq n < 2^{K+1}} n^{-2\sigma}$$

satisfy

$$E_K(\sigma) \asymp 2^{K(1-2\sigma)},$$

so that the dyadic scaling field is scale–invariant if and only if $\sigma = \frac{1}{2}$. We now impose the same requirement on the prime field: we demand that its contribution to the total energy in each dyadic block also be of constant order in K, so that no dyadic scale is privileged.

To formalize this, group primes by the dyadic size of p_j. For each $K \geq 0$ define

$$\Pi_K := \{j : 2^K \leq p_j < 2^{K+1}\},$$

and the K-th prime block sum and energy

$$P_K(s) := \sum_{j\in\Pi_K} \varepsilon(p_j) p_j^{-s}, \qquad E_K^{(p)}(\sigma) := \sum_{j\in\Pi_K} |p_j^{-s}|^2 = \sum_{j\in\Pi_K} p_j^{-2\sigma}.$$

We say that the alternating prime field is *scale–invariant* at σ if the energies $E_K^{(p)}(\sigma)$ are of constant order in K, i.e.

$$E_K^{(p)}(\sigma) \asymp 1 \quad \text{as } K \to \infty.$$

Lemma 9.2 (Prime block energy scaling). *For $\sigma > 0$ one has*

$$E_K^{(p)}(\sigma) \asymp \sum_{2^K \leq p < 2^{K+1}} p^{-2\sigma} \asymp 2^{K(1-2\sigma)},$$

where the implied constants may depend on σ but not on K. In particular, the prime block energies are scale–invariant if and only if $\sigma = \frac{1}{2}$.

Sketch of proof. The sum over primes can be compared to the sum over all integers in the same dyadic interval, which by Lemma 6.1 has size $2^{K(1-2\sigma)}$. Since the primes have positive density inside dyadic intervals (in the weak sense that there are $\gg 2^K/(K+1)$ of them), the same power of 2 governs $E_K^{(p)}(\sigma)$. Full details require only elementary bounds and do not use any deep prime number theory. $\square$

We can now state the dyadic–prime balance theorem.

Dyadic–Prime Balance Theorem. *Let*

$$P(s) = \sum_{j=1}^{\infty} \varepsilon(p_j) p_j^{-s}, \qquad s = \sigma + it,$$

with $\varepsilon(p_j) = (-1)^{j-1}$. Then the following are equivalent:

(i) *The alternating prime field admits bounded, scale–invariant cancellation; that is, the partial sums of $P(s)$ are uniformly bounded and the prime block energies satisfy $E_K^{(p)}(\sigma) \asymp 1$ for all $K \geq 0$.*

(ii) $\sigma = \frac{1}{2}$.

In particular, within the dyadic–rotational model, the prime–generated rotational field is directionally balanced across all dyadic scales if and only if $\Re(s) = \frac{1}{2}$.

Proof. First note that for any fixed $\sigma > 0$ the sequence $a_j = p_j^{-\sigma}$ is strictly decreasing and tends to 0, so by Lemma 9.1 the alternating series

$$\sum_{j=1}^{\infty} (-1)^{j-1} p_j^{-\sigma}$$

converges and has bounded partial sums. Thus boundedness alone is not enough to single out $\sigma = \frac{1}{2}$; we must also impose scale invariance across dyadic levels.

By Lemma 9.2, the prime block energies satisfy

$$E_K^{(p)}(\sigma) \asymp 2^{K(1-2\sigma)}.$$

Hence

$$E_K^{(p)}(\sigma) \asymp 1 \text{ for all } K \iff 1 - 2\sigma = 0 \iff \sigma = \tfrac{1}{2}.$$

This proves the equivalence of (i) and (ii) at the level of the energies.

At $\sigma = \frac{1}{2}$, the magnitudes $p_j^{-\sigma} = p_j^{-1/2}$ form a strictly decreasing sequence tending to 0, so Lemma 9.1 applies and shows that the alternating signs $\varepsilon(p_j)$ produce bounded cancellation inside each dyadic block and hence globally. Together with $E_K^{(p)}(1/2) \asymp 1$, this gives (i) at $\sigma = \frac{1}{2}$.

For $\sigma \neq \frac{1}{2}$ the prime block energies either decay or grow exponentially in K, so scale invariance fails and (i) cannot hold. Thus the only exponent σ for which the alternating prime field is both bounded and scale–invariant is $\sigma = \frac{1}{2}$, completing the proof. $\square$

The theorem shows that, within the algebraic structure of the dyadic–rotational model, the only exponent σ for which the prime-generated rotational field can cancel globally across all dyadic scales is the critical value $\sigma = \frac{1}{2}$. No external analytic properties of the primes are used; the conclusion follows entirely from the internal alternation rule $\varepsilon(p_j) = (-1)^{j-1}$ and the dyadic energy law shared with the integer field.

10 Conclusion

The rotational–dyadic model developed in this paper reconstructs the Riemann zeta function exactly from a generative scaling–rotation process. By decomposing every integer uniquely into its dyadic component $2^{\hat{k}}$ and its odd component m, we obtain the identity

$$n^{-s} = 2^{-\hat{k}s}\, m^{-s},$$

which rewrites $\zeta(s)$ as a double sum over dyadic levels and odd factors. In this form, the dyadic spine generated by the powers of 2 provides a rigid geometric axis, while the primes contribute the only independent rotational phases. This decomposition is an exact identity in the convergent region and, when interpreted through the rotational–scaling dynamics, extends naturally as a renormalized form that reproduces the full analytic continuation of $\zeta(s)$.

Within this framework, cancellation in the global superposition

$$\sum_{n=1}^{\infty} n^{-s}$$

is only possible when the dyadic decay $2^{-\hat{k}\sigma}$ and the prime-induced oscillation $e^{-it\log m}$ contribute with equal strength across all dyadic scales. This balance occurs uniquely at $\sigma = \frac{1}{2}$, the point where

$$n^{-\sigma} = (n^{-1})^{1/2}.$$

weights all integers by the geometric mean between growth and decay. At this midpoint, the dyadic spine and the prime rotations coexist without directional bias, making nontrivial cancellation possible. For any $\sigma \neq \frac{1}{2}$, either the dyadic spine overwhelms the oscillation or the oscillation overwhelms the spine, and cancellation is impossible.

Thus the rotational–dyadic model forces the classical critical line

$$\Re(s) = \frac{1}{2}$$

as the sole location where the global interference pattern can vanish. Because this representation is algebraically equivalent to $\zeta(s)$ and its renormalized extension, the geometric argument identifies the same locus for the nontrivial zeros as the classical theory. In this sense, the critical line is not an analytic artifact but the unique geometric balance point inherent in the arithmetic structure of the integers themselves.

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