Menger Curvature Consistency and GH–Continuity in the RTLI Coherence Index

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Abstract

In a previous note, I introduced the RTLI coherence index η as a scalar proxy for the onset of geometric coherence and as a practical screening tool for Gromov–Hausdorff (GH) convergence of discrete metric spaces to smooth manifolds. Here I give more explicit, distance–based definitions for the components of η, with particular emphasis on a Menger–curvature–based curvature consistency term κ_{cons}. I also formulate a natural GH–continuity conjecture and a scale–bound heuristic for κ_{cons}, which together suggest a path toward a rigorous continuity theory for the RTLI index.

1 Background and Motivation

Let (X, d) be a compact metric space, typically arising from a point cloud, graph metric or numerical sampling of a manifold. In the RTLI framework, the coherence index $\eta(X)$ is defined schematically by

$$\eta(X) \;=\; \frac{L_c(X)^2 \cdot \kappa_{\text{cons}}(X)}{c_1\,\mathrm{Var}(R_c)(X) + c_2\,\Delta_{\text{TI}}(X)}, \tag{1}$$

where:

- L_c is a characteristic correlation length,
- κ_{cons} measures curvature consistency across the space,
- $\mathrm{Var}(R_c)$ quantifies local distance variance,
- Δ_{TI} measures deviations from the triangle inequality,
- $c_1, c_2 > 0$ are fixed dimensionless weights.

Empirical tests in [1] show that η acts as an internal coordinate for the transition from a disorder–dominated regime to a geometry–dominated regime, with a visible "knee" near $\eta \approx 1.7$ under a fixed normalization.

The purpose of this note is to record more precise definitions for these components, to propose a concrete Menger–curvature formulation of κ_{cons}, and to state a GH–continuity conjecture that isolates where the mathematical difficulty truly lies.

2 Metric Preliminaries and Menger Curvature

We work throughout with compact metric spaces (X, d) of finite diameter $\mathrm{diam}(X) \le D < \infty$. For clarity we first present the finite setting, where X is a finite point set with $|X| = n$.

Definition 1 (Menger curvature)**.** *Let $x, y, z \in X$ be three distinct points, and set*

$$a = d(y, z), \quad b = d(x, z), \quad c = d(x, y).$$

The Menger curvature $k_M(x, y, z)$ is defined as the reciprocal of the circumradius

$$k_M(x, y, z) = \frac{1}{R(x, y, z)},$$

where $R(x, y, z)$ is the radius of the unique circle passing through the three points in the Euclidean comparison triangle with side lengths (a, b, c). Using Heron's formula, the area A of this triangle is

$$A = \frac{1}{4}\sqrt{4a^2b^2 - (a^2 + b^2 - c^2)^2},$$

and the circumradius is

$$R(x, y, z) = \frac{abc}{4A}.$$

If the three points are collinear (degenerate triangle, $A = 0$), we set $k_M(x, y, z) = 0$.

Menger curvature depends only on the pairwise distances between the three points. This makes it naturally compatible with GH convergence, which is defined purely in terms of distortions of distances under correspondences.

2.1 Triplet space and scale selection

In practice, we only consider triplets whose edge lengths lie near a chosen relational scale. The RTLI framework uses the correlation length $L_c(X)$ to define that scale.

Definition 2 (Correlation length)**.** *Let (X, d) be a finite metric space with n points. Denote by*

$$\langle d \rangle = \frac{1}{\binom{n}{2}} \sum_{x \neq y} d(x, y), \qquad \langle d^2 \rangle = \frac{1}{\binom{n}{2}} \sum_{x \neq y} d(x, y)^2$$

the mean distance and mean squared distance over unordered pairs. The correlation length is

$$L_c(X) := \frac{\sqrt{\langle d^2 \rangle}}{\langle d \rangle}.$$

In this note we set the local geometric scale to

$$r := L_c(X),$$

which captures the typical relational scale of the space.

Definition 3 (Triplet space at scale r)**.** *For a finite metric space (X, d) and $r > 0$, define the triplet set*

$$\mathcal{T}_r(X) = \Big\{(x, y, z) \in X^3 : x, y, z \text{ distinct}, \ \max\{d(x, y), d(y, z), d(z, x)\} \le r\Big\}.$$

In applications we additionally exclude nearly degenerate triplets by imposing

$$\min\{d(x, y), d(y, z), d(z, x)\} \ge \alpha r,$$

for a fixed $\alpha \in (0, 1)$ (e.g. $\alpha = 0.1$).

The restriction to $\mathcal{T}_r(X)$ avoids both ultra–local noise (very small triangles) and global averages (very large triangles), and the nondegeneracy condition controls the stability of Menger curvature with respect to small changes in the metric.

3 Curvature Consistency and Local Variance Terms

We now give explicit, distance–based definitions of the components entering the RTLI index.

3.1 Curvature consistency via Menger curvature

In many applications the target smooth limit manifold M is not known a priori. For that reason, we formulate curvature consistency relative to an *intrinsic* reference curvature.

Definition 4 (Intrinsic reference curvature)**.** *Given a metric space (X, d) and scale r, the intrinsic reference curvature is defined as*

$$\bar{k}(X; r) := \mathrm{median}\{k_M(x, y, z) : (x, y, z) \in \mathcal{T}_r(X)\}.$$

To measure how consistent local curvature is across the space, we use a robust dispersion statistic.

Definition 5 (Menger curvature consistency)**.** *Fix $\epsilon_0 > 0$ (a regularization parameter) and scale $r = L_c(X)$. Define the median absolute deviation (MAD) of Menger curvature at scale r by*

$$\mathrm{MAD}_r(X) := \mathrm{median}\left\{|k_M(x, y, z) - \bar{k}(X; r)| : (x, y, z) \in \mathcal{T}_r(X)\right\}.$$

The curvature consistency invariant is then

$$\kappa_{\mathrm{cons}}(X) := [\mathrm{MAD}_r(X) + \epsilon_0]^{-1}. \tag{2}$$

High curvature consistency (small dispersion around $\bar{k}$) produces a large value of κ_{cons}, in line with the interpretation of η as a coherence index. The use of MAD rather than raw variance makes the definition robust to isolated noisy or nearly degenerate triplets.

3.2 Local distance variance

We next formalize the local variance term.

Definition 6 (Local distance variance)**.** *Given (X, d) and scale $r > 0$, define for each $x \in X$ the local neighborhood*

$$B_r(x) = \{y \in X : d(x, y) \le r\}.$$

Let

$$\bar{d}_r(x) = \frac{1}{|B_r(x)|} \sum_{y \in B_r(x)} d(x, y)$$

be the local mean distance. The local variance at x is

$$V_r(x) = \frac{1}{|B_r(x)|} \sum_{y \in B_r(x)} \left(d(x, y) - \bar{d}_r(x)\right)^2.$$

The global local–variance functional is

$$\mathrm{Var}(R_c)(X) := \frac{1}{|X|} \sum_{x \in X} V_r(x). \tag{3}$$

Intuitively, $\mathrm{Var}(R_c)$ is small when local neighborhood scales are homogeneous across the space, and large when the metric is highly irregular or contains multiple incompatible scales.

3.3 Triangle inequality deviation

Finally, we measure how far the metric is from being "triangularly consistent" at scale r.

Definition 7 (Triangle inequality deviation)**.** *Given a finite set of triplets* $\{(x_i, y_i, z_i)\}_{i=1}^N \subset X^3$*, define the normalized slack*

$$\delta_{\mathrm{TI}}(x_i, y_i, z_i) := \frac{d(x_i, y_i) + d(y_i, z_i) - d(x_i, z_i)}{d(x_i, z_i) + \epsilon_0}.$$

The triangle inequality deviation functional is defined as a high percentile of this slack:

$$\Delta_{\mathrm{TI}}(X) := \mathrm{Percentile}_p\big(\{\delta_{\mathrm{TI}}(x_i, y_i, z_i)\}_{i=1}^N\big), \tag{4}$$

for a fixed $p \in (0, 100)$ *(e.g.* $p = 90$*). In practice the triplets are sampled at scale* r*, and the percentile formulation makes* Δ_{TI} *robust to rare outliers.*

When (X, d) arises from a clean Riemannian manifold sampling, triangle inequality slack is typically small and concentrated, so Δ_{TI} is close to zero. Large Δ_{TI} indicates metric defects or numerical inconsistencies.

4 The RTLI Coherence Index: Final Form

With the above ingredients, we can now write the RTLI index in a fully explicit form.

Definition 8 (RTLI coherence index)**.** *Let* (X, d) *be a finite metric space of diameter at most* D*. Fix:*

- *a scale* $r = L_c(X)$*,*
- *weights* $c_1, c_2 > 0$*,*
- *a small regularization parameter* $\epsilon_0 > 0$ *with units of distance squared, for instance* $\epsilon_0 = 10^{-6}\,\mathrm{diam}(X)^2$*.*

Define:

- $L_c(X)$ *by the correlation length definition above,*
- $\kappa_{\mathrm{cons}}(X)$ *by* (2)*,*
- $\mathrm{Var}(R_c)(X)$ *by* (3)*,*
- $\Delta_{\mathrm{TI}}(X)$ *by* (4)*.*

The RTLI coherence index is

$$\eta(X) := \frac{L_c(X)^2 \cdot \kappa_{\mathrm{cons}}(X)}{c_1\,\mathrm{Var}(R_c)(X) + c_2\,\Delta_{\mathrm{TI}}(X) + \epsilon_0}. \tag{5}$$

Empirically, $\eta \approx 1$ corresponds to a critical balance between coherence and disorder, while an effective geometric regime sets in once η exceeds a shifted threshold η_{geo} in the range 1.6–1.8 under current normalization choices, with a typical value near 1.7 [1, 2].

5 GH–Continuity and a Scale–Bound Conjecture

A natural next step in formalizing the RTLI framework is to understand how η behaves under GH convergence.

Definition 9 (GH–continuity)**.** *A functional F defined on compact metric spaces is said to be GH–continuous if*

$$d_{\mathrm{GH}}(X_n, X) \to 0 \quad \Longrightarrow \quad F(X_n) \to F(X).$$

For many simple invariants of a metric space, such as diameter or average distance, GH–continuity is straightforward to establish: they are continuous functions of pairwise distances, and d_{GH} controls the distortion of all pairwise distances under correspondences.

The challenging component in η is the curvature consistency term κ_{cons}, which is built from Menger curvature. It is natural to conjecture a quantitative scale–bound.

Conjecture 1 (Scale–bound for curvature consistency)**.** *Let (X, d_X) and (Y, d_Y) be compact metric spaces with $\mathrm{diam}(X), \mathrm{diam}(Y) \le D$, and assume mild nondegeneracy conditions (no volume collapse, and exclusion of extremely skinny triangles at scale r). Then there exist constants $C > 0$ and $\alpha \in (0, 1]$ (expected $\alpha = 1/2$) such that*

$$\left|\kappa_{\mathrm{cons}}(X) - \kappa_{\mathrm{cons}}(Y)\right| \;\le\; C(D)\, d_{\mathrm{GH}}(X, Y)^{\alpha}.$$

Heuristically, the proof would proceed by:

1. Using an ε–correspondence between X and Y to map triplets in $\mathcal{T}_r(X)$ to triplets in $\mathcal{T}_{r+O(\varepsilon)}(Y)$.
2. Controlling the change in side lengths of corresponding triangles by $O(\varepsilon)$.
3. Using stability of Heron's formula and the circumradius to bound the difference in Menger curvature, $|k_M(\Delta_X) - k_M(\Delta_Y)|$.
4. Propagating these bounds through the MAD statistic defining κ_{cons}.

A full proof would require careful handling of near–degeneracies and of the sampling measure on triplets.

Assuming Conjecture 1 and the simpler GH–continuity of L_c, $\mathrm{Var}(R_c)$ and Δ_{TI} (which are direct functionals of distances), one would then obtain GH–continuity of η away from the idealized limit where the denominator of (5) vanishes.

Remark 1 (Singular limit and divergence of η)**.** *The denominator in* (5)*,*

$$B(X) = c_1\, \mathrm{Var}(R_c)(X) + c_2\, \Delta_{\mathrm{TI}}(X) + \epsilon_0,$$

measures local metric irregularity. In the ideal continuous limit of a perfectly smooth manifold, one would expect $B(X) \to 0$, so that $\eta(X)$ diverges. In practical discrete settings, the onset of a geometry–dominated regime appears at finite $\eta \approx 1.7$, corresponding to a sharp but finite suppression of local variance and triangle inequality deviations under realistic asymmetries.

6 Outlook

The explicit Menger–curvature formulation of κ_{cons} recorded here has three advantages for the RTLI program:

- It is distance–only and thus compatible with GH convergence.
- It uses robust statistics (MAD, percentiles), making it numerically stable.
- It isolates the main technical challenge in proving GH–continuity of η: a quantitative stability theory for Menger curvature under GH perturbations.

Future work will aim to:

1. Establish GH–continuity of the simpler components L_c, $\mathrm{Var}(R_c)$ and Δ_{TI} under mild geometric assumptions.
2. Prove or refine Conjecture 1, possibly first in special cases (e.g. spaces converging to constant curvature manifolds).
3. Explore connections between high κ_{cons} and existing synthetic curvature frameworks such as Alexandrov and $\mathrm{RCD}(K, N)$ spaces.

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References

[1] E. De Jesús, *RTLI Coherence Index as a Practical Proxy for Gromov–Hausdorff Convergence*, Technical Note, Zenodo (2025).

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