NOTE ON EINSTEIN'S THEORY OF GRAVITATION

By H. B. Phillips

1. It is the object of this paper to discuss the gravitational field about a mass particle by direct use of the equivalence principle without the tensor calculus.[1] In this way one can obtain a more concrete physical picture than that presented by the more mathematical method. Such a discussion, however, requires concrete interpretations of the fundamental hypotheses which may seem to include additional assumptions.

2. **Fundamental hypotheses.** In developing his theory of gravitation Einstein used three fundamental hypotheses.

(A) **The equivalence hypothesis.** Acceleration of the reference system is equivalent to gravitation.

This means that if an observer and his reference system, or laboratory, are accelerated, events seem to occur as in a gravitational field and in no way can he determine whether he is in a gravitational field or whether his reference system is merely accelerated. Thus, if the laboratory is accelerated upward objects press with greater force upon the floor as if subject to a gravitational force acting downward.

It is to be noticed that the acceleration of the reference system and the equivalent gravitation are oppositely directed. If the laboratory falls freely, its acceleration, having the same direction, annuls the action of the field and objects cease to have any apparent weight. Also it is in measurements on objects that do not directly experience his acceleration that the moving observer obtains the same results as the fixed observer on objects at rest in a gravitational field.

(B) **The infinitesimal hypothesis.** In the infinitesimal neighborhood of a world point (x, y, z, t) it is possible to choose a reference system with respect to which the restricted relativity is valid.

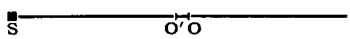
[1] A complete exposition of Einstein's theory is contained in A. S. Eddington's Report on the Relativity Theory of Gravitation, published by the Physical Society of London.

Such a reference system is called Galilean since Galileo first used the restricted form of relativity. It is assumed that a free particle moves at each point of its path like a point in a Galilean system at that point.

(C) **The general principle of relativity.** The laws of physics can be stated in a form that is the same for all reference systems.

This condition is fulfilled by expressing the laws as relations between tensors. No use of it is made in this paper.

3. **The observer falling from infinity.** Consider a mass concentrated at a point S and a fixed observer F or series of fixed observers along a line through S.



Suppose a Galilean observer G starts from infinity and falls freely toward S. I assume that he will move along a straight line which we may take on the line of the fixed observers. When F and G are at the same point O, we wish to determine the relation between their measurements made on events occurring on the line near O.

The observer G is actually accelerated. The observer F is subject to gravitation which affects his measurements on fixed objects as if he were accelerated. Hence in passing from O to O' the measurements of F should be affected by gravitation in the same way those of G are affected by acceleration. At infinity both are Galilean observers at rest and so obtain the same measurements. Hence we conclude that, at all points of the line, F *and* G *obtain the same measure of a fixed length and the same interval of time at a fixed point.* It is to be noticed that this equality applies only to measurements made on objects in the fixed system; for, as stated above, the equivalence is with respect to objects that do not have the acceleration of the moving observer.

Let dr be the distance between two points of the line and dt the interval of time between events at those points as measured by F and dR, dT the corresponding quantities measured by G. I assume that these pairs of measurements will be connected by linear equations,

$$dr = AdR + BdT, \qquad dT = Cdr + Ddt. \tag{1}$$

To measure a fixed object the moving observer must determine the distance between the positions occupied by its ends at the same time ($dT=0$). Hence, from (1)

$$dr = AdR.$$

But in this case we must also have

$$dr = dR.$$

Hence

$$A = 1. \tag{2}$$

To measure an interval of time at a fixed point we must place $dr=0$. Hence

$$dT = Ddt.$$

But in this case

$$dT = dt,$$

and so

$$D = 1. \tag{3}$$

Substituting (2) and (3) in (1) and solving for dR, we get

$$\begin{aligned} dR &= (1-BC)\,dr - Bdt, \\ dT &= Cdr + dt. \end{aligned} \tag{4}$$

Choose units of measurement such that the velocity of light in the moving system is 1. Then

$$\begin{aligned} ds^2 = dT^2 - dR^2 = (1-B^2)dt^2 &+ 2[B + C(1-B^2)]drdt \\ &+ [C^2 - (1-BC)^2]dr^2. \end{aligned}$$

If events in the fixed system are symmetrical with respect to past and future time, this must have the same value whether dt is positive or negative. The coefficient of $drdt$ is then zero; that is,

$$B + C(1-B^2) = 0,$$

whence

$$C = -\frac{B}{1-B^2}.$$

Let

$$B = -\sqrt{1-\gamma}$$

where γ is a function to be determined later. Then

$$C = \frac{\sqrt{1-\gamma}}{\gamma}.$$

Substituting these values of B and C in (4), we get

$$dR = \frac{1}{\gamma}\,dr + \sqrt{1-\gamma}\,dt,$$

$$dT = \frac{\sqrt{1-\gamma}}{\gamma}\,dr + dt. \tag{5}$$

Solving for dr and dt,

$$dr = dR - \sqrt{1-\gamma}\,dT,$$

$$dt = \frac{1}{\gamma}\,dT - \frac{1}{\gamma}\sqrt{1-\gamma}\,dR. \tag{6}$$

4. **The function γ.** So far we have treated the moving system as a point. Actually the Galilean system which verifies the equivalence principle in the neighborhood of a point to terms of the second order is a three-dimensional one each point of which falls separately toward S. Let O, P be fixed points at distance r from S as estimated by the falling observer, and O', P' fixed points at distance $r+dr$ from S. To the fixed observer OO' and PP' are fixed lines intersecting in S. The point in the moving system where G estimates S to be should appear to F to be on both these lines. Consequently that point should appear to F to be the fixed point S.

Substituting $dR = r$, $dt = 0$ in the equation

$$dR = \frac{1}{\gamma}\,dr + \sqrt{1-\gamma}\,dt,$$

we find that the distance to S as estimated by F is γr.

If, now, we move from O to O', we have

$$OO' = dr = d(\gamma r),$$

whence

$$\gamma r = r - C, \qquad \gamma = 1 - \frac{C}{r}, \tag{7}$$

the negative sign being used since $\gamma < 1$.

From (5), a fixed point $(dr=0)$ appears to the moving observer to have the velocity

$$V=\frac{dR}{dT}=\sqrt{1-\gamma}.$$

Hence,

$$\gamma=1-V^2. \tag{8}$$

At considerable distance from the point S, the motion is represented approximately by Newton's equation

$$V^2=\frac{2m}{r}, \tag{9}$$

where m is the mass at S. Comparing (7), (8) and (9) we see that $C=2m$, and so

$$\gamma=1-\frac{2m}{r}. \tag{10}$$

It is interesting to note that the velocity measured by the moving observer always has the value

$$V^2=\frac{2m}{r},$$

as in Newton's theory.

5. **Value of** ds. Consider events occurring in a plane through S. Let O, P be points at distance r from S and let $d\phi$ be the angle OSP as measured by the falling observer. Since this observer makes measurements as in restricted relativity,

$$arc\ OP=r\,d\phi.$$

I assume that this is also the distance OP measured by the fixed observer.

In the plane OSP, the Minkowski interval ds between two events is given by

$$ds^2=dT^2-r^2d\phi^2-dR^2.$$

Replacing dR and dT by their values from (5), this becomes

$$ds^2=\gamma dt^2-r^2d\phi^2-\frac{1}{\gamma}dr^2, \tag{11}$$

We have found that two radii appear to the fixed observer at O to converge to a point at distance γr from O. If the observer is at the point $r=2m$,

$$\gamma r=\left(1-\frac{2m}{2m}\right)r=0.$$

Hence to the fixed observer the sun is not the point $r=0$ but the point $r=2m$. This is also indicated by the expression for ds^2. For when $\gamma=0$, dr must be zero if ds is to be finite. No measurement beyond $\gamma=0$ exists.

6. **Motion of a free particle.** By the equivalence hypothesis the motion of the falling observer should neutralize evidences of gravitation in his immediate neighborhood. Thus a free particle which instantaneously coincides with the falling observer G should appear to be free from gravitation and so to be moving in a straight line with constant velocity. If a plane is passed through S and the initial direction of motion of the free particle, its acceleration is in this plane. Hence the particle should continue to move in this plane.

As seen by G the motion of the free particle at each point of its path should satisfy the condition

$$\delta\int ds=\delta\int\sqrt{dT^2-r^2d\phi^2-dR^2}=0.$$

Hence, the path should be one for which the integral

$$\int ds=\int\sqrt{\gamma dt^2-r^2d\phi^2-\frac{1}{\gamma}dr^2}$$

has a stationary value. Let

$$L=\sqrt{\gamma\left(\frac{dt}{ds}\right)^2-r^2\left(\frac{d\phi}{ds}\right)^2-\frac{1}{\gamma}\left(\frac{dr}{ds}\right)^2}$$

$$=\sqrt{\gamma\left(\dot{t}\right)^2-r^2\left(\dot{\phi}\right)^2-\frac{1}{\gamma}\left(\dot{r}\right)^2}.$$

Then the condition to be satisfied is

$$\delta\int L ds=0.$$

Differentiating and replacing h by 1, the Euler conditions

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{t}}\right)-\frac{\partial L}{\partial t}=0,$$

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{\phi}}\right)=\frac{\partial L}{\partial \phi}=0,$$

give

$$\gamma\frac{dt}{ds}=C, \qquad r^2\frac{d\phi}{ds}=h, \tag{12}$$

where C and h are constants. As a third equation we use the identical relation

$$ds^2=\gamma dt^2-r^2 d\phi^2-\frac{1}{\gamma}dr^2. \tag{13}$$

By solving the differential equations (12) and (13) we can determine the motion of the particle.

7. **Particle falling from rest.** In this case $h=0$. Let dT, dR be values measured by the observer falling with the particle. The equations expressing measurements dt, dr of the fixed observer, in terms of dT, dR, have the form

$$dt=A\,dT+B\,dR,$$

$$dr=E\,dT+F\,dR. \tag{14}$$

At a point in the moving system $dR=0$ and $dT=ds$. Hence, from (12) and (14),

$$A=\frac{dt}{dT}=\frac{dt}{ds}=\frac{C}{\gamma},$$

$$E=\frac{dr}{dT}=\frac{dr}{ds}=\sqrt{C^2-\gamma},$$

the last expression being obtained by substituting

$$dt=\frac{Cds}{\gamma}$$

in (13) with $d\phi=0$. Using these values of A and E and substituting in the general equation

$$ds^2=dT^2-dR^2=\gamma dt^2-\frac{1}{\gamma}dr^2,$$

we then find

$$F=C, \quad B=-\frac{1}{\gamma}\sqrt{C^2-\gamma}.$$

Hence equations (14) become

$$\begin{aligned} dt&=\frac{C}{\gamma}dT-\frac{1}{\gamma}\sqrt{C^2-\gamma}\,dR,\\ dr&=CdR-\sqrt{C^2-\gamma}\,dT. \end{aligned} \tag{15}$$

Solving for dT and dR, we get

$$\begin{aligned} dT&=Cdt+\frac{1}{\gamma}\sqrt{C^2-\gamma}\,dr,\\ dR&=\frac{C}{\gamma}dr+\sqrt{C^2-\gamma}\,dt. \end{aligned} \tag{16}$$

In case of a particle falling from infinity, $C=1$ and equations (15), (16) reduce to (5), (6).

In the measurement of a fixed distance ($dT=0$), from (15),

$$dr=C\,dR.$$

In the measurement of time at a fixed point ($dr=0$), from (16),

$$dT=C\,dt.$$

These measurements made by the two observers have a constant ratio. They are equal only in the case $C=1$ when the moving observer falls from infinity.

The velocity of the falling observer ($dR=0$) as seen by the fixed one is

$$\frac{dr}{dt}=-\frac{\gamma\sqrt{C^2-\gamma}}{C}.$$

This is zero when $C=\sqrt{\gamma}$ and so the motion starts from the point where

$$C^2=\gamma=1-\frac{2m}{r}.$$

At this point equations (16) become

$$dT=Cdt=\sqrt{\gamma}\,dt,$$

$$dR=\frac{C}{\gamma}dr=\frac{1}{\sqrt{\gamma}}dr. \tag{17}$$

Although both observers are here at rest they do not obtain the same measurements. This shows the effect of gravitation on measurements of length and time. The falling observer may be considered to measure correctly. The fixed observer in a gravitational field obtains measurements of time increased in the ratio

$$\frac{1}{\sqrt{\gamma}}$$

and of radial distance diminished in the ratio

$$\sqrt{\gamma}.$$

The factors $\sqrt{\gamma}$ in the expression

$$ds^2=\left(\sqrt{\gamma}dt\right)^2-r^2d\phi^2-\left(\frac{dr}{\sqrt{\gamma}}\right)^2$$

have the effect of changing the gravitational measures dr, dt into Galilean measures.

8. **Measurement of time at a fixed point.** Let dT, dR be measurements made by the free observer at O who is accelerated but has zero velocity. Let w be the velocity, as seen by this observer, of a free particle passing through O. For this particle, using (17), we find

$$ds^2=dT^2-R^2d\phi^2-dR^2=(1-w^2)dT^2=(1-w^2)\gamma dt^2,$$

whence, from (12),

$$1-w^2=\frac{1}{\gamma}\left(\frac{ds}{dt}\right)^2=\frac{\gamma}{C^2}. \tag{18}$$

If the observer on the passing particle makes a measurement of time at the point O of the system instantaneously at rest, since both systems are Galilean, he will find w to be the velocity of that system and so

$$(1-w^2)dT'^2=dT^2=\gamma dt^2. \tag{19}$$

Comparing (18) and (19), we see that

$$dT'=C\,dt.$$

Hence an observer on a free particle finds measurements of time at fixed points that have a constant ratio to those of the gravitational observers at those points.

9. **Orbit of a planet.**[2] Eliminating ds and dt from equations (12) and (13), we get

$$\left(\frac{h}{r^2}\frac{dr}{d\phi}\right)^2+\frac{h^2}{r^2}=C^2-1+\frac{2m}{r}+\frac{2mh^2}{r^3}.$$

Writing $u=\frac{1}{r}$, this becomes

$$\left(\frac{du}{d\phi}\right)^2=\frac{C^2-1}{h^2}+\frac{2mu}{h^2}+2mu^3-u^2. \tag{20}$$

Let $u_1<u_2<u_3$ be the roots of the cubic

$$2mu^3-u^2+\frac{2mu}{h^2}+\frac{C^2-1}{h^2}=0.$$

Then

$$u_1+u_2+u_3=\frac{1}{2m}, \tag{21}$$

$$u_1u_2+u_2u_3+u_3u_1=\frac{1}{h^2}. \tag{22}$$

[2] The method used by Eddington and others, in which approximations have been made in the differential equations, has been criticised. The following treatment is here given to meet these criticisms.

In case of the sun,

$$m=1.47 \text{ km.},$$

and for the planets u is of the order

$$10^{-8} \text{ km.}$$

At aphelion and perihelion $\frac{du}{d\phi}$ is zero. Hence u passes through two of the roots in description of the orbit. These roots, u_1 and u_2 are then of the order 10^{-8} km., and from (21) u_3 is about one hundred million times larger than either of these.

We can write (20) in the form

$$\left(\frac{du}{d\phi}\right)^2=2m(u-u_1)(u-u_2)(u-u_3)$$

whence

$$\phi=2\int_{u_1}^{u_2}\frac{du}{\sqrt{2m(u-u_1)(u_2-u)(u_3-u)}} \tag{23}$$

is the angle turned through in a complete description of the orbit. Taking out a factor u_3 and expanding in series, this becomes

$$\phi=\frac{2}{\sqrt{2mu_3}}\int_{u_1}^{u_2}\frac{du}{\sqrt{(u-u_1)(u_2-u)\left(1-\frac{u}{u_3}\right)}}$$

$$=\frac{2}{\sqrt{2mu_3}}\int_{u_1}^{u_2}\frac{du}{\sqrt{(u-u_1)(u_2-u)}}\left(1+\frac{1}{2}\frac{u}{u_3}+\ldots\right)$$

$$=\frac{1}{\sqrt{2mu_3}}\left[2\pi+\frac{\pi}{2}\frac{u_1+u_2}{u_3}+\ldots\right].$$

Using

$$u_3=\frac{1}{2m}-(u_1+u_2),$$

by a further expansion we obtain

$$\phi=2\pi\left[1+\frac{3}{2}m(u_1+u_2)+\ .\ .\ .\right],$$

the neglected terms in each case being of the order

$$\left(\frac{u_1+u_2}{u_3}\right)^2.$$

Dividing (22) by (21), we have to the same order of approximation

$$u_1+u_2=\frac{2m}{h^2},$$

whence

$$\phi=2\pi\left[1+\frac{3m^2}{h^2}+.\ .\ .\right]. \tag{24}$$

In a complete description of the orbit, the perihelion advances the fraction

$$\frac{3m^2}{h^2}$$

of a revolution approximately, the approximation being of the order of 1 part in 10^8.

10. **Deflection of a ray of light.** In case of a ray of light,

$$ds=0.$$

The equations

$$\gamma\frac{dt}{ds}=C, \qquad r^2\frac{d\phi}{ds}=h$$

show that C and h are infinite. The ratio

$$\frac{C}{h}=\frac{\gamma dt}{r^2 d\phi}=k \tag{25}$$

is, however, in general finite. Substituting these values in (20) we have

$$\left(\frac{du}{d\phi}\right)^2=k^2+2mu^3-u^2 \tag{26}$$

to determine the path of a ray of light.

Suppose the nearest approach to the sun is

$$r=\frac{1}{u}=a.$$

At that point

$$\left(\frac{du}{d\phi}\right)^2=k^2+\frac{2m}{a^3}-\frac{1}{a^2}=0.$$

Subtracting from (26), we have

$$\left(\frac{du}{d\phi}\right)^2=2m\left(u^3-\frac{1}{a^3}\right)-\left(u^2-\frac{1}{a^2}\right).$$

In a complete description of the path from infinity back to infinity, the total change of direction is

$$\phi=2\int_0^{\frac{1}{a}}\frac{du}{\sqrt{\frac{1}{a^2}-u^2-m\left(\frac{1}{a^3}-u^3\right)}}$$

$$=2\int_0^{\frac{1}{a}}\frac{du}{\sqrt{\left(\frac{1}{a^2}-u^2\right)\left(1-2m\dfrac{\frac{1}{a^2}+\frac{u}{a}+u^2}{\frac{1}{a}+u}\right)}}$$

$$=2\int_0^{\frac{1}{a}}\frac{du}{\sqrt{\frac{1}{a^2}-u^2}}\left(1+m\frac{\frac{1}{a^2}+\frac{u}{a}+u^2}{\frac{1}{a}+u}+\dots\right)$$

$$=\pi+4\frac{m}{a}$$

approximately. In case of a ray passing close to the sun the neglected terms are of the order

$$\left(2m\frac{\frac{1}{a^2}+\frac{u}{a}+u^2}{\frac{1}{a}+u}\right)^2<\left(3\frac{m}{R}\right)^2<\left(\frac{1}{150{,}000}\right)^2$$

11. **Path of light near the center of the sun.** Suppose the sun concentrated within a radius less than $3m$. The expression

$$\left(\frac{du}{d\phi}\right)^2=k^2+2mu^3-u^2$$

has a minimum value at

$$u=\frac{1}{3m}, \qquad r=3m.$$

If, then, a ray of light penetrates beyond

$$r=3m,$$

$\frac{du}{d\phi}$ can never become zero and so it can never recede. It will continue to penetrate inward. The equation

$$r^2\frac{d\phi}{dt}=\gamma\frac{h}{C}$$

shows that an infinite time will be required to reach the position

$$\gamma=0, \qquad r=2m.$$

If the ray is inclined so as just to reach $r=3m$,

$$k^2+2m\left(\frac{1}{3m}\right)^3-\left(\frac{1}{3m}\right)^2=0,$$

and so

$$\left(\frac{du}{d\phi}\right)^2=\left(u-\frac{1}{3m}\right)^2\left(2mu+\frac{1}{3}\right).$$

The total angle turned through in reaching $r=3m$ is

$$\phi=\int_0^{\frac{1}{3m}}\frac{du}{\left(\frac{1}{3m}-u\right)\sqrt{2mu+\frac{1}{3}}}=\infty.$$

The ray will describe a spiral of an infinite number of terms and require an infinite time to reach $r=3m$.

If the ray is directed along the circle $r=3m$, $\frac{du}{d\phi}$ will remain zero, the ray describing a circular orbit.