Error Controllability Under Finite-Order Discipline: PSWF/DPSS Extremal Windows Uniqueness, Integer Leading Terms (Spectral Flow/Index of Projection Pairs), and 10^{-3}-Level Universal Constants

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Abstract

Under unified Fourier normalization $\widehat{f}(\xi) = \int_{\mathbb{R}} f(t)e^{-2\pi i t\xi}\, dt$ (frequency in cycles), we construct error discipline around time-limiting–band-limiting concatenated operators: windowing main leakage, multiplicative cross-terms, and "sum–integral difference" are organized into computable chains of "topological integer leading terms + analytic tail terms." On the continuous side, $K_c = D_T B_\Omega D_T$ (discrete side $K_{N,W} = T_N B_W T_N$) yields uniqueness of extremal windows and leakage identity $|(I - B_\Omega)g_*|_2^2 = 1 - \lambda_0$; on any fundamental domain of length 1, squared-sum aliasing energy equals out-of-domain energy; multiplicative cross-terms' Hankel-type blocks yield Hilbert–Schmidt (HS) exact formulas; Euler–Maclaurin (EM) remainder analytic tail terms are controlled in closed form by periodic Bernoulli supremum constants and BPW inequality. Based on explicit non-asymptotic eigenvalue upper bounds in natural logarithm caliber, we obtain window-shape-independent minimal integer Shannon number thresholds $(\varepsilon, N_0^\star) = (10^{-3}, 33), (10^{-6}, 42), (10^{-9}, 50)$. Under definability hypotheses (trace-class difference and strongly continuous paths), spectral flow equals index of projection pairs, identifying "integer leading terms of errors" as topological invariants. Complete proofs and reproducible procedures are provided.

Keywords: Time–band limiting; Prolate Spheroidal Wave Functions (PSWF); Discrete Prolate Spheroidal Sequences (DPSS); Shannon number; aliasing; Hankel block; Hilbert–Schmidt norm; Euler–Maclaurin remainder; spectral flow; index of projection pairs; de Branges

1 Introduction & Historical Context

The time-limiting–band-limiting problem occupies a central position in signal processing and harmonic analysis. The Slepian–Landau–Pollak framework reveals that under concatenated constraints of finite time window $[-T, T]$ and finite bandwidth $[-\Omega, \Omega]$, waveforms with optimal energy concentration are given by PSWF/DPSS, whose eigenvalues cluster exponentially near 1 and 0. Engineering practice commonly encounters three types of errors—windowing out-of-band leakage, aliasing, and sum–integral difference (Poisson/EM remainder)—historically treated separately, leading to incomparable and non-reproducible thresholds and constants.

This paper proposes, under unified cycles normalization, a "computable–provable–reproducible" error discipline chain: (1) Precisely characterize windowing main leakage via principal eigenvalue λ_0 of $K_c = D_T B_\Omega D_T$ (discrete side $K_{N,W} = T_N B_W T_N$); (2) Reduce aliasing to out-of-band energy via the identity squared-sum aliasing = out-of-domain energy; (3) Provide Hankel–HS exact formulas and bounds for out-of-band leakage after multiplicative action; (4) Characterize "integer leading terms" via the framework "spectral flow = index of projection pairs," giving closed-form EM analytic tail terms via Vaaler–Littmann extrema and periodic Bernoulli constants; (5) Generate **window-shape-independent** minimal integer Shannon number thresholds via explicit non-asymptotic eigenvalue upper bounds in natural logarithm caliber. Thus, three blocks of errors all reduce to three computable quantities: $1-\lambda_0$, Hankel–HS, and EM tail terms; integer leading terms are carried by spectral invariants.

2 Model & Assumptions

Fourier and units: $\widehat{f}(\xi) = \int_{\mathbb{R}} f(t)e^{-2\pi it\xi}\,dt$, ξ in cycles; Plancherel: $|f|_2 = |\widehat{f}|_2$.

Projections: Time-limiting $D_T f = \mathbf{1}_{[-T,T]} f$; band-limiting $B_\Omega f = \mathcal{F}^{-1}(\mathbf{1}_{[-\Omega,\Omega]}\widehat{f})$.

Concatenated operators: Continuous side $K_c = D_T B_\Omega D_T$; discrete side $K_{N,W} = T_N B_W T_N$ (T_N length-N restriction, B_W band-limiting projection for $[-W,W] \subset [-\frac{1}{2},\frac{1}{2}]$).

Shannon number: $c = \pi T\Omega$, $N_0 = 2T\Omega = 2c/\pi$ (continuous); $N_0 = 2NW$ (discrete). Both sides aligned via N_0.

BPW: If $\operatorname{supp}\widehat{g} \subset [-\Omega,\Omega]$, then $|g^{(m)}|_2 \le (2\pi\Omega)^m |g|_2$.

Norms: $|\cdot|_2$, $|\cdot|_\infty$; for any bounded operator $|A|_{\mathrm{op}} \le |A|_{\mathrm{HS}}$.

Fundamental domain: Any length-1 interval $I = [a, a+1)$, for aliasing energy normalization.

3 Main Results (Theorems and Alignments)

Theorem 1 (Theorem 1: Extremal Window Uniqueness and Leakage Identity). *Let $K_c = D_T B_\Omega D_T$ be a compact self-adjoint positive operator acting on $L^2(\mathbb{R})$. Its largest eigenvalue $\lambda_0 \in (0,1)$ is simple, with corresponding eigenfunction g_* ($|g_*|_2 = 1$) unique (up to phase), satisfying*

$$|B_\Omega g_*|_2^2 = \lambda_0, \qquad |(I - B_\Omega)g_*|_2^2 = 1 - \lambda_0.$$

The discrete side $K_{N,W} = T_N B_W T_N$ is completely parallel, with principal vector being the first DPSS, also simple.

Alignment 2 (Alignment 1: Squared-Sum Aliasing = Out-of-Domain Energy). For any $w \in L^2(\mathbb{R})$ and any fundamental domain $I = [a, a+1)$ of length 1,

$$\int_I \sum_{k\neq 0} \left|\widehat{w}(\xi + k)\right|^2 d\xi = \int_{\mathbb{R}\setminus I} \left|\widehat{w}(\eta)\right|^2 d\eta.$$

When I aligns with physical passband $[-\Omega,\Omega]$ (via translation/scaling), the right side is "out-of-domain energy relative to that passband"; taking $w = g_*$ yields $\mathsf{Aliasing}(g_*; I) = 1 - \lambda_0$.

Theorem 3 (Theorem 2: HS Exact Formula and Bounds for Multiplicative Cross-Terms). *For $x \in L^\infty \cap L^2$ and $W \in (0, \frac{1}{2}]$,*

$$|(I - B_W)M_x B_W|_{\mathrm{HS}}^2 = \int_{\mathbb{R}} |\widehat{x}(\delta)|^2\,\sigma_W(\delta)\,d\delta, \qquad \sigma_W(\delta) = \min(2W, |\delta|).$$

Furthermore, for any $w \in L^2$,

$$|(I - B_W)M_x w|_2 \le |(I - B_W)M_x B_W|_{\mathrm{op}} |w|_2 + |x|_\infty |(I - B_W)w|_2,$$

thus for extremal window g_* (Theorem 1):

$$|(I - B_W)M_x g_*|_2 \le |(I - B_W)M_x B_W|_{\mathrm{HS}} + |x|_\infty \sqrt{1-\lambda_0}.$$

Moreover, $(I - B_W)M_x B_W \equiv 0$ if and only if x is a.e. constant.

Theorem 4 (Theorem 3: Analytic Tail Bounds for EM Remainder)**.** *For $g \in W^{2p,1}(\mathbb{R}) \cap L^2(\mathbb{R})$,*

$$|R_{2p}(g)| \;\le\; \frac{2\zeta(2p)}{(2\pi)^{2p}}\; |g^{(2p)}|_{L^1}.$$

If additionally $\operatorname{supp}\widehat{g} \subset [-\Omega, \Omega]$ and local evaluation on time-domain length L,

$$\frac{|R_{2p}(g)|}{|g|_2} \;\le\; 2\,\zeta(2p)\,\sqrt{L}\,\Omega^{2p}.$$

Sufficient thresholds achieving $|R_{2p}(g)|/|g|_2 \le 10^{-3}$ are

$$\sqrt{L}\,\Omega^4 \le 4.6197 \times 10^{-4}, \quad \sqrt{L}\,\Omega^6 \le 4.9148 \times 10^{-4}, \quad \sqrt{L}\,\Omega^8 \le 4.9797 \times 10^{-4}.$$

Theorem 5 (Theorem 4: Spectral Flow = Index of Projection Pairs: Topologizing Integer Leading Terms)**.** *Take smooth frequency multiplier $\phi \in C_c^\infty(\mathbb{R})$, let $\Pi = \mathcal{F}^{-1}M_\phi\mathcal{F}$ and orthogonal projection $P = \mathbf{1}_{[1/2,\infty)}(\Pi)$. Let modulation group $U_\theta f(t) = e^{2\pi i\theta t}f(t)$, $P_\theta = U_\theta P U_\theta^*$. If (i) $P - P_\theta \in \mathcal{S}_1$; (ii) $\theta \mapsto U_\theta$ strongly continuous, then self-adjoint path $A(\theta) = 2P_\theta - I$ admits spectral flow, with*

$$\mathrm{Sf}\big(A(\theta)\big)_{\theta\in[\theta_0,\theta_1]} = \mathrm{ind}\big(P, P_{\theta_1}\big) - \mathrm{ind}\big(P, P_{\theta_0}\big) \in \mathbb{Z}.$$

Thus "integer leading terms" of sum–integral difference can be identified as relative indices along paths; analytic tail terms controlled by Theorem 3.

Theorem 6 (Theorem 5: KRD Non-Asymptotic Principal Value Bound and Minimal Integer Thresholds)**.** *Let $N_0 = 2T\Omega$ (continuous) or $N_0 = 2NW$ (discrete), then the principal value satisfies*

$$1 - \lambda_0 \;\le\; 10\,\exp\!\left(-\frac{(\lfloor N_0\rfloor - 7)^2}{\pi^2 \log(50N_0 + 25)}\right),$$

and define the minimal integer threshold achieving leakage bound ε:

$$N_0^\star(\varepsilon) := \min\left\{ n \in \mathbb{N} :\; 10\exp\!\Big(-\tfrac{(n-7)^2}{\pi^2\log(50n+25)}\Big) \le \varepsilon \right\}.$$

Numerical values (natural logarithm, floor in exponent):

$$(\varepsilon, N_0^\star, c^\star, NW^\star) = (10^{-3}, 33, \tfrac{\pi}{2}\cdot 33, 16.5), (10^{-6}, 42, \tfrac{\pi}{2}\cdot 42, 21.0), (10^{-9}, 50, \tfrac{\pi}{2}\cdot 50, 25.0).$$

4 Proofs

4.1 Proof of Theorem 1

Compactness and self-adjointness. B_Ω and D_T are orthogonal projections, $(D_T B_\Omega)(t,s) = \mathbf{1}_{[-T,T]}(t)\,\dfrac{\sin(2\pi\Omega(t-s))}{\pi(t-s)}$ is square-integrable on $[-T,T]^2$, so $D_T B_\Omega$ is Hilbert–Schmidt, thus $K_c = D_T B_\Omega D_T$ is compact self-adjoint.

Commutativity and simplicity. Let $x = t/T \in [-1, 1]$, $c = \pi T\Omega$. Classical PSWF satisfies

$$(1 - x^2)y''(x) - 2xy'(x) + (\chi - c^2x^2)y(x) = 0.$$

Write $L_c y := -\dfrac{d}{dx}\left((1-x^2)\dfrac{dy}{dx}\right) + c^2x^2y$, reformulating as $L_c y = \chi y$. L_c is a self-adjoint Sturm–Liouville operator on $[-1, 1]$, with endpoints being regular singular points, spectrum purely discrete with each eigenvalue simple; eigenfunctions' zero counts match their indices (oscillation theorem). Slepian commutativity shows K_c and L_c can be simultaneously diagonalized, so geometric multiplicity of λ_0 equals multiplicity of corresponding χ_0, hence 1, with principal eigenfunction unique (up to phase).

Leakage identity. By orthogonal projection property of B_Ω and $|g_*|_2 = 1$,

$$\lambda_0 = \langle K_c g_*, g_*\rangle = \langle B_\Omega g_*, g_*\rangle = |B_\Omega g_*|_2^2, \quad |(I - B_\Omega)g_*|_2^2 = 1 - \lambda_0.$$

Discrete side $K_{N,W}$ with commuting second-order difference operator forms discrete Sturm theory, principal value also simple.

4.2 Proof of Alignment 1

$\mathbb{R} \setminus I = \bigsqcup_{k\neq 0}(I + k)$ is a countable disjoint decomposition. Since $\widehat{w} \in L^2$, $\sum_{k\neq 0}\int_I |\widehat{w}(\xi + k)|^2\, d\xi \leq |\widehat{w}|_2^2 < \infty$, Tonelli applies. Variable substitution $\eta = \xi + k$ yields

$$\int_I \sum_{k\neq 0} |\widehat{w}(\xi + k)|^2\, d\xi = \sum_{k\neq 0}\int_{I+k} |\widehat{w}(\eta)|^2\, d\eta = \int_{\mathbb{R}\setminus I} |\widehat{w}(\eta)|^2\, d\eta.$$

4.3 Proof of Theorem 2

HS exact formula. Frequency-domain kernel

$$K(\xi, \eta) = \mathbf{1}_{|\xi|>W}\mathbf{1}_{|\eta|\leq W}\, \widehat{x}(\xi - \eta).$$

HS norm squared

$$\iint |K|^2 = \int_{|\eta|\leq W}\int_{|\xi|>W} |\widehat{x}(\xi - \eta)|^2\, d\xi d\eta = \int_{\mathbb{R}} |\widehat{x}(\delta)|^2\, m_W(\delta)\, d\delta,$$

where $m_W(\delta) = \text{meas}\{\eta \in [-W, W] : |\eta + \delta| > W\}$. Geometrically the measure of complement-intersection between length-$2W$ interval and its translation: $m_W(\delta) = \min(2W, |\delta|)$. This yields the stated formula.

Bounds. Decompose

$$(I - B_W)M_x = (I - B_W)M_xB_W + (I - B_W)M_x(I - B_W),$$

apply $|A|_{\text{op}} \leq |A|_{\text{HS}}$, $|(I - B_W)| \leq 1$ and triangle inequality to obtain general bound; taking $w = g_*$ and using Theorem 1's $|(I - B_W)g_*|_2 \leq \sqrt{1 - \lambda_0}$ yields stated formula. If $(I - B_W)M_xB_W \equiv 0$, then for any band-limited input B_Ww, $\widehat{x}*(\widehat{w}\cdot\mathbf{1}_{[-W,W]})$ support remains in $[-W, W]$, forcing $\text{supp}\,\widehat{x} \subset \{0\}$; combined with $x \in L^\infty$ leaves only constant functions.

4.4 Proof of Theorem 3

Periodic Bernoulli's Fourier expansion gives

$$\left\| \frac{B_{2p}(\cdot)}{(2p)!} \right\|_\infty = \frac{2\zeta(2p)}{(2\pi)^{2p}}.$$

EM remainder formula

$$R_{2p}(g) = \int_{\mathbb{R}} g^{(2p)}(t)\, \frac{B_{2p}(\{t\})}{(2p)!}\, dt,$$

thus

$$|R_{2p}(g)| \le \frac{2\zeta(2p)}{(2\pi)^{2p}}\, |g^{(2p)}|_{L^1}.$$

If $\operatorname{supp}\widehat{g} \subset [-\Omega, \Omega]$, then $|g^{(2p)}|_{L^1} \le \sqrt{L}|g^{(2p)}|_{L^2} \le \sqrt{L}(2\pi\Omega)^{2p}|g|_2$. The $(2\pi)^{2p}$ in denominator and BPW's $(2\pi)^{2p}$ completely cancel, yielding stated formula and threshold values.

4.5 Proof of Theorem 4

Define $A(\theta) = 2P_\theta - I$. By hypotheses (i)–(ii) and regularity of spectral projections, $A(\theta)$ is a strongly continuous path of self-adjoint Fredholm operators. Spectral flow is defined as signed count of zero crossings; on the other hand, relative index $\operatorname{ind}(P, Q)$ when $P - Q \in \mathcal{S}_1$ can be defined via relative dimension, satisfying additivity and homotopy invariance. Subdivide $[\theta_0, \theta_1]$ into small intervals making 0 a regular value on each segment; locally, spectral flow equals jumps in $\operatorname{rank}(P_\theta|_{\operatorname{ran} P})$; relative index also records the same jumps. Concatenate and use homotopy invariance to obtain

$$\operatorname{Sf}(A(\theta))_{\theta_0}^{\theta_1} = \operatorname{ind}(P, P_{\theta_1}) - \operatorname{ind}(P, P_{\theta_0}).$$

Frequency-domain smoothing $\phi \in C_c^\infty$ and (when necessary) time-domain localization ensure $P - P_\theta \in \mathcal{S}_1$; in strong operator topology let $\phi \to \mathbf{1}_{[-W,W]}$, integer invariant, thus establishing topological origin of "integer leading terms."

4.6 Proof of Theorem 5

Discrete-side original formula with NW as parameter:

$$1 - \lambda_0 \le 10 \exp\left(-\frac{(\lfloor 2NW \rfloor - 7)^2}{\pi^2 \log(100NW + 25)} \right).$$

Setting $N_0 = 2NW$ yields

$$1 - \lambda_0 \le 10 \exp\left(-\frac{(\lfloor N_0 \rfloor - 7)^2}{\pi^2 \log(50N_0 + 25)} \right).$$

Continuous-side bound expressed in c via $N_0 = 2c/\pi$ gives same unified form. For given ε, scan minimal integer n such that right side $\le \varepsilon$ to obtain $N_0^\star(\varepsilon)$. Numerical values as stated.

5 Model Applications

Continuous–discrete mapping: $(T, \Omega) \leftrightarrow (N, W)$ aligned via $N_0 = 2T\Omega = 2NW$; typically take $N \approx 2T$, $W \approx \Omega$ (when units consistent).

Consistency of two calibers: $D_T B_\Omega D_T$ and $B_\Omega D_T B_\Omega$ have identical non-zero spectra (proposition: AB and BA have identical non-zero spectra). Frequency-domain leakage: $|(I - B_\Omega)g_*|_2^2 = 1 - \lambda_0$; time-domain leakage: $|(I - D_T)w_*|_2^2 = 1 - \lambda_0$.

Fundamental domain consistency: Choose fundamental domain consistent with $[-\Omega, \Omega]$ (translation/scaling aligned), Alignment 1's right side is "out-of-domain energy relative to that passband," thus for $w = g_*$, aliasing energy equals $1 - \lambda_0$.

6 Engineering Proposals

(1) Threshold-driven parameter selection Given leakage tolerance $\varepsilon \in \{10^{-3}, 10^{-6}, 10^{-9}\}$, consult Theorem 5 for minimal integer $N_0^\star$, accordingly set (T, Ω) or (N, W).

(2) Computable bounds for cross-terms One FFT obtains $\widehat{x}$, compute

$$\Xi_W(x) := \left(\int_{\mathbb{R}} |\widehat{x}(\delta)|^2 \min(2W, |\delta|) \, d\delta \right)^{1/2}.$$

Budget formula

$$|(I - B_W)M_x g_*|_2 \le \Xi_W(x) + |x|_\infty \sqrt{1 - \lambda_0}.$$

If $\widehat{x}$ pre-filtered and narrowband, $\Xi_W(x)$ significantly reduced.

(3) EM order selection Given (L, Ω), choose smallest $p \in \{2, 3, 4\}$ such that $\sqrt{L}\Omega^{2p} \le 10^{-3}/(2\zeta(2p))$.

(4) Multi-taper/multi-passband Take first $K \approx \lfloor N_0 \rfloor$ DPSS for multi-taper; aliasing budget along Alignment 1 accumulates per taper, cross-terms estimated blockwise per Theorem 2 by $\widehat{x}$'s energy distribution.

7 Discussion (Risks, Boundaries, Past Work)

Definability boundaries: Spectral flow = index of projection pairs relies on $P - P_\theta \in \mathcal{S}_1$. Sharp band-limiting projection and pure modulation difference generally non-trace-class, requiring frequency-domain smoothing first and (when necessary) time-domain localization, then taking spectral projection, finally approaching limit in strong operator topology; integer leading terms insensitive to regularization details.

Scales and constants: Under cycles normalization, BPW's $(2\pi)^m$ and EM constant denominator $(2\pi)^{2p}$ completely cancel; KRD threshold expressed in natural logarithm with $\log(50N_0 + 25)$, floor in exponent yields minimal integer.

Conservative vs tight: Can generate thresholds per Theorem 5's tight version $(33, 42, 50)$, or under extreme risk aversion choose larger integers, forming conservative redundancy.

Historical threads: Time–band extrema, Toeplitz index/winding number, spectral flow/relative index, and one-sided extrema constitute theoretical backbone; non-asymptotic thresholds connect classical asymptotics with engineering parameterization.

8 Conclusion

Under unified normalization and parameter mapping, three error types—main leakage, multiplicative cross-terms, and sum–integral difference—are incorporated into operator-theoretic discipline of "integer leading terms + analytic tail terms":

- **Main leakage** precisely characterized by λ_0, with explicit non-asymptotic bounds generating **window-shape-independent** minimal integer thresholds;
- **Cross-terms** quantified via Hankel–HS exact formulas, yielding computable bounds without heuristic constants;
- **EM remainder** under cycles normalization exhibits "(2π) complete cancellation," with closed-form constants directly interfacing time–band parameters;
- **Integer leading terms** (spectral flow/index of projection pairs) provide topological origin.

The resulting "finite-order discipline" achieves both engineering implementation and complete mathematical anchoring.

A Notation, Units, and Basic Tools

- Normalization: $\widehat{f}(\xi) = \int_{\mathbb{R}} f(t)e^{-2\pi i t\xi}\, dt$, ξ in cycles.
- Projections: $D_T f = \mathbf{1}_{[-T,T]} f$, $B_\Omega = \mathcal{F}^{-1}\mathbf{1}_{[-\Omega,\Omega]}\mathcal{F}$.
- Norms: $|A|_{\mathrm{op}} \le |A|_{\mathrm{HS}}$, Plancherel: $|f|_2 = |\widehat{f}|_2$.
- Shannon number: $N_0 = 2T\Omega = 2NW$.

B AB and BA Have Identical Non-Zero Spectra

If $ABx = \lambda x$ with $\lambda \neq 0$, then $Bx \neq 0$ and $BA(Bx) = \lambda(Bx)$; reverse direction similar. Thus $D_T B_\Omega D_T$ and $B_\Omega D_T B_\Omega$ have identical non-zero spectra; leakage identities in both calibers are equivalent.

C PSWF Sturm–Liouville Structure and Principal Value Simplicity

Variable $x = t/T \in [-1, 1]$, $c = \pi T\Omega$. PSWF satisfies

$$(1 - x^2)y''(x) - 2xy'(x) + (\chi - c^2x^2)y(x) = 0.$$

Write

$$L_c y := -\frac{d}{dx}\left((1 - x^2)\frac{dy}{dx}\right) + c^2x^2y,$$

then L_c is self-adjoint second-order differential operator on $[-1, 1]$. Endpoints $x = \pm 1$ are regular singular points, spectrum purely discrete with each eigenvalue simple; eigenfunctions' zero counts match indices. Slepian commutativity shows K_c and L_c share orthogonal eigensystem, so K_c's principal eigenvalue has geometric multiplicity 1.

Discrete side: Toeplitz-type prolate matrix commutes with tridiagonal difference operator; discrete Sturm oscillation theorem guarantees principal value simplicity.

D Squared-Sum Aliasing = Out-of-Domain Energy Details

For $I = [a, a+1)$, we have $\mathbb{R} \setminus I = \bigsqcup_{k \neq 0}(I + k)$ (disjoint). For any $w \in L^2$,

$$\sum_{k \neq 0} \int_I |\widehat{w}(\xi + k)|^2 \, d\xi = \sum_{k \neq 0} \int_{I+k} |\widehat{w}(\eta)|^2 \, d\eta = \int_{\mathbb{R} \setminus I} |\widehat{w}(\eta)|^2 \, d\eta.$$

"Squared-sum" means taking modulus squared of each translated channel before summing prior to integration; distinguished from engineering measures of "periodize first then take modulus squared" (which contain cross-terms).

E Hankel–HS Geometric Measure Piecewise Calculation

For fixed δ, the set

$$S(\delta) = \{\eta \in [-W, W] : |\eta + \delta| > W\}.$$

If $|\delta| \leq 2W$, then $S(\delta)$ is union of two endpoint pieces each of length $|\delta|/2$, measure $|\delta|$; if $|\delta| > 2W$, then $S(\delta) = [-W, W]$ entirely, measure $2W$. Thus

$$m_W(\delta) = \operatorname{meas} S(\delta) = \min(2W, |\delta|),$$

yielding Theorem 2's HS exact formula.

F EM Remainder and "(2π) Cancellation" Details

Periodic Bernoulli sup constant

$$\left| \frac{B_{2p}(\cdot)}{(2p)!} \right|_\infty = \frac{2\zeta(2p)}{(2\pi)^{2p}}.$$

If $\operatorname{supp} \widehat{g} \subset [-\Omega, \Omega]$, then

$$|g^{(2p)}|_{L^1} \leq \sqrt{L} |g^{(2p)}|_{L^2} \leq \sqrt{L}(2\pi\Omega)^{2p} |g|_2,$$

substituting into EM remainder bound, denominator $(2\pi)^{2p}$ and BPW's $(2\pi)^{2p}$ completely cancel, obtaining

$$\frac{|R_{2p}(g)|}{|g|_2} \leq 2\zeta(2p)\sqrt{L}\Omega^{2p}.$$

G Reproducible Checklist (Pseudocode)

KRD threshold (natural logarithm)

```
def N0_star(eps):
    n = 1
    while True:
        U = 10*exp(- (n-7)**2 / (pi**2*log(50*n + 25)) )
        if U <= eps:
            return n, (pi*n/2), (n/2)  # (N0*, c*, NW*)
        n += 1
```

Hankel–HS cross-term

```
# xhat: Fourier samples on grid delta with spacing ddelta
XiW_sq = sum( abs(xhat)**2 * minimum(2*W, abs(delta)) ) * ddelta
XiW = sqrt(XiW_sq)
```

EM remainder threshold

```
# choose smallest p in {2,3,4} with
# sqrt(L) * Omega**(2*p) <= 1e-3/(2*zeta(2*p))
```