WSIG-QM: Windowed Scattering & Information Geometry for Quantum Mechanics

A Unified Framework of Quantum Concept Definitions and Criterion System (with Complete Proofs)

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Abstract

Using the **de Branges–Kreĭn (DBK) canonical system** and **weighted Mellin space** as carriers, we explicitly incorporate the **finite bandwidth/time window** of real instruments into spectral measures, forming a **windowed readout** framework; we characterize "commit (collapse/commit)" through **KL/Bregman information geometry**; use **scattering phase–spectral density–Wigner–Smith delay** as energy scale; close non-asymptotic errors via **Nyquist–Poisson–Euler–Maclaurin (three-term decomposition)**; ensure realizability and stability through **variational optimization of frame/sampling density and window/kernel**. The core unified formula is

$$\boxed{\varphi'(E) = -\pi\,\rho_{\mathrm{rel}}(E) = \tfrac{1}{2}\,\mathrm{tr}\,\mathsf{Q}(E)} \quad \text{(a.e.)}$$

unifying (single/multi-channel) scattering phase derivative, relative local density of states (LDOS) and Wigner–Smith delay; under information geometry we obtain **Born probability = minimal-KL projection (I-projection)**, and "pointer basis" is **spectral minimum of windowed readout operator**. Above criteria consistent with Herglotz–Weyl, Birman–Kreĭn, Wigner–Smith, Ky Fan, Poisson/EM and other standard results, directly interchangeable and implementable.

1 Notation & Baseplates

1.1 DBK Canonical System and Herglotz–Weyl Dictionary

Consider first-order symplectic canonical system $JY'(t,z) = zH(t)Y(t,z)$ $(H \succeq 0)$, whose Weyl–Titchmarsh function $m : \mathbb{C}^+ \to \mathbb{C}^+$ belongs to Herglotz class, with representation

$$m(z) = a\,z + b + \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big)\, d\mu(t),\ \ a \ge 0,\ \ b \in \mathbb{R},$$

and $\Im m(E+i0) = \pi\,\rho(E)$ (a.e.). This gives absolutely continuous density ρ of continuous spectrum and is compatible with DBK framework.

1.2 Weyl–Heisenberg (Phase–Scale) Representation (Mellin Version)

On weighted Mellin space $\mathcal{H}_a = L^2(\mathbb{R}_+, x^{a-1}dx)$ define

$$(U_\tau f)(x) = x^{i\tau} f(x),\ (V_\sigma f)(x) = e^{\sigma a/2} f(e^\sigma x),$$

satisfying Weyl relation $V_\sigma U_\tau = e^{i\tau\sigma} U_\tau V_\sigma$. Via power-log isomorphism $x = e^t$, $\mathcal{H}_a$ and $L^2(\mathbb{R})$ Weyl–Heisenberg/Gabor framework are isometric and parallel, serving as phase–scale kinematic baseplate.

1.3 Finite-Order EM / Poisson Three-Term Decomposition Discipline

Parent map and all discrete–continuous reordering adopt **finite-order Euler–Maclaurin (EM)**, closing errors via "**alias (Poisson) + Bernoulli layer (EM) + tail term**" three-term decomposition; "bandlimited + Nyquist" makes alias term zero. Poisson summation and sampling criteria adopt **angular frequency convention** (Ω unit: rad/unit(E); **unit conversion**: if using Hertz B, have $B = \Omega/(2\pi)$, then switch to $T_s \le 1/(2B)$; for time t as independent variable unit is rad/s).

1.4 Convention and Notation

Fourier/Parseval Convention Table:

Item	Formula	Note
Fourier transform	$\widehat{f}(\xi) = \int f(t) e^{-it\xi}\, dt$	ξ angular frequency (rad/unit(t))
Parseval relation	$\int_{\mathbb{R}} \|f\|^2 = \frac{1}{2\pi} \int_{\mathbb{R}} \|\widehat{f}\|^2$	Energy conservation, with $\frac{1}{2\pi}$ factor
Convolution theorem	$\widehat{f * g} = \widehat{f} \cdot \widehat{g}$	Time convolution = frequency product
Product transform	$\widehat{f \cdot g} = \frac{1}{2\pi}\, \widehat{f} * \widehat{g}$	Time product = frequency convolution/2π

Birman–Kreĭn adopts

$$\det S(E) = e^{-2\pi i\, \xi(E)}.$$

This paper fixes above formula. **Equivalence chain bridge** ($\det S$–ξ–Q triple relation):

$$\operatorname{tr} \mathsf{Q}(E) = -i\, \partial_E \ln \det S(E) = -2\pi\, \xi'(E);$$

thus for any channel number N, have $\operatorname{tr} \mathsf{Q}(E) = -2\pi\, \xi'(E)$; single-channel $S = e^{2i\varphi}$ gives $\operatorname{tr} \mathsf{Q}(E) = 2\varphi'(E)$, thus $\varphi'(E) = -\pi\, \xi'(E)$ (consistent with $\rho_{\mathrm{rel}} = \xi'$).

1.5 Projection Operator Notation Unification

This paper fixes **frequency-domain projection** as $P_B^{(\xi)} : \widehat{f} \mapsto \chi_B\, \widehat{f}$ (where χ_B characteristic function of frequency band $B = [-\Omega, \Omega]$), its integral kernel realization in energy domain denoted Π_B, i.e.,

$$(\Pi_B f)(E) = \int_{\mathbb{R}} k_B(E - E')\, f(E')\, dE', \quad k_B(t) = \frac{\sin(\Omega t)}{\pi t}.$$

In T6 variational equations always use $P_B^{(\xi)}$ (frequency projection); in A4/T3 kernel–trace class/Hilbert–Schmidt arguments use Π_B (energy integral operator). This distinction avoids confusion between "first convolve then project" and "projection is convolution".

2 Axioms

Axiom 2.1 (Carrier and Covariance)**.** *Quantum states placed in $\mathcal{H}(E)$ or $\mathcal{H}_a$; phase–scale covariance realized by projective unitary representation of (U_τ, V_σ) Weyl–Heisenberg (Stone theorem: strongly continuous one-parameter unitary group $\Leftrightarrow$ self-adjoint generator; Stone–von Neumann: irreducible representation of Weyl relation essentially unique).*

Axiom 2.2 (Observables and Windowed Readout)**.** *Instrument window w_R and* ***bandlimited response kernel*** *$h \in L^1$ act on continuous spectral density of state, defining* ***windowed readout***

$$\boxed{\langle K_{w,h}\rangle_\rho = \int_{\mathbb{R}} w_R(E)\,\big[h * \rho_\star\big](E)\,dE}\,.$$

*where $\rho_\star$ can be $\rho_{\rm abs}$ (absolutely continuous spectral density, i.e., density of $\mu_\rho^{\rm ac}$) or $\rho_{\rm rel} = \rho_{\rm abs} - \rho_{0,\rm abs}$ (relative density). In "**no-blur hard limit**" $h \to \delta$ recovers $\int w_R(E)\rho_\star(E)\,dE$. Readout controlled by three-term decomposition error.*

Axiom 2.3 (Probability–Information Consistency)**.** *Commit (collapse/commit) =* ***minimal-KL projection (I-projection)*** *on apparatus/window constraint; PVM hard limit returns to Born.*

Axiom 2.4 (Pointer Basis)**.** *"Pointer basis" defined as* ***spectral projection subspace corresponding to minimal spectral value*** *of* ***window operator*** *$W_R = \int w_R\,dE_A$ (Ky Fan "minimum sum"; if minimal spectral value not attained, take limit subspace as $\varepsilon \downarrow 0$);* ***existence and verifiable condition****: if $w_R \in L^\infty$ then W_R bounded self-adjoint; if further $w_R \in L^2(\mathbb{R})$* ***(e.g., finite support window)****, let $k_B(t) = \sin(\Omega t)/(\pi t)$, have:*

Under bandlimited projection Π_B, ***uniformly adopt*** *$\Pi_B M_{w_R}$ notation. Its integral kernel $K(x,y) = k_B(x-y)\,w_R(y)$. By L3.3a know $\|k_B\|_{L^2}^2 = \Omega/\pi < \infty$;* ***HS kernel verification one-liner****:*

$$\|K\|_{L^2(\mathbb{R}^2)}^2 = \int_{\mathbb{R}^2} |k_B(x-y)|^2\,|w_R(y)|^2\,dxdy = \|k_B\|_{L^2}^2\,\|w_R\|_{L^2}^2 = \frac{\Omega}{\pi}\,\|w_R\|_{L^2}^2 < \infty,$$

*thus by Fubini–Tonelli theorem $\Pi_B M_{w_R}$ is Hilbert–Schmidt, hence $\Pi_B M_{w_R} \Pi_B$ also Hilbert–Schmidt/compact (**HS kernel theorem**: if integral kernel $K \in L^2(\mathbb{R}^2)$ then corresponding integral operator is Hilbert–Schmidt, hence compact). Instrument kernel h only affects readout and error, does not change spectral structure of W_R.*

Axiom 2.5 (Phase–Density–Delay Scale)**.** *If (H, H_0) satisfy relative trace class/smooth scattering standard regularity (see L3.5), then* ***on absolutely continuous spectrum almost everywhere***

$$\boxed{\varphi'(E) = -\pi\,\rho_{\rm rel}(E) = \tfrac{1}{2}\,\mathrm{tr}\,\mathsf{Q}(E)} \qquad (a.e.\ \ on\ \sigma_{\rm ac})$$

where $\rho_{\rm rel}(E) = \xi'(E)$, $\mathsf{Q} := -iS^\dagger \frac{dS}{dE}$ (unit system $\hbar = 1$). ***Sign convention****: this paper uniformly adopts $\det S(E) = e^{-2\pi i\,\xi(E)}$, thus $\mathrm{tr}\,\mathsf{Q}(E) = \partial_E \arg\det S(E) = -2\pi\,\xi'(E)$,* *hence $\varphi'(E) = -\pi\,\rho_{\mathrm{rel}}(E)$. Above equality holds almost everywhere on absolutely continuous spectrum;* ***near threshold or resonance, interpreted by*** $\lim_{\epsilon\downarrow 0}$ ***non-tangential limit or distributional sense (principal value + singular part)****, consistent with Herglotz boundary value $\Im m(E+i0) = \pi\rho(E)$. Under lossless assumption $S(E)$ unitary.*

Axiom 2.6 (Window/Kernel Optimization and Multi-Window Synergy)**.** *Window $w \in \mathrm{PW}_\Omega^{\mathrm{even}}$; objective to minimize three-term decomposition error upper bound; necessary condition is* ***frequency-domain "polynomial multiplier + convolution kernel" bandlimited projection-KKT equation****; multi-window version characterized by* ***generalized Wexler–Raz biorthogonality*** *and frame operator for Pareto frontier and stability.*

Axiom 2.7 (Threshold and Singularity Stability)**.** *Under "finite-order EM + Nyquist–Poisson–EM" discipline, windowing/reordering generates no new singularities; zero count stable and verifiable within Rouché radius.*

3 Basic Definitions

Definition 3.1 (State)**.** Pure state $\psi \in \mathcal{H}$ ($|\psi| = 1$); mixed state $\rho \succeq 0$, $\operatorname{tr}\rho = 1$.

Definition 3.2 (Observable)**.** Self-adjoint A and spectral projection E_A.

Definition 3.3 (Windowed Readout and Regularity Conditions)**.** $\langle K_{w,h}\rangle_\rho = \int w_R\,[h * \rho_\star]\,dE$, where $\rho_\star$ can be ρ_{abs} (μ_ρ absolutely continuous part density) or ρ_{rel} (relative density). Measure perspective writable as $d(h*\mu_\rho) = h * d\mu_\rho$ ($h \in \mathrm{PW}_\Omega \cap L^1$, standard convolution for Radon measures). **Regularity and integrability sufficient conditions**: to ensure three-term decomposition (Poisson–EM–Tail) remainder bounds hold and measure convolution legal, take

$$w_R \in \mathrm{PW}_\Omega \cap W^{2M,1}(\mathbb{R}), \quad h \in \mathrm{PW}_\Omega \cap W^{2M,1}(\mathbb{R}),$$

and choose one of:

(c) w_R **compactly supported** and $w_R \in W^{2M,1}(\mathbb{R})$, thus $F = w_R(h * \rho_\star) \in L^1$ and $F^{(2M)} \in L^1$ still hold (**engineering first choice**);

(b) $\rho_\star \in L^1(\mathbb{R})$ (or $L^1 \cap L^\infty$ or sufficient weighted integrability) and $h \in \mathcal{S}$, then $h * \rho_\star \in L^1 \cap L^\infty$ and higher derivatives integrable.

Definition 3.4 (Commit/Collapse)**.** Given apparatus constraint, observation probability p is I-projection of reference q to feasible set; softmax softening $\to$ Born hard limit.

Definition 3.5 (Pointer Basis)**.** Basis spanning spectral projection subspace corresponding to minimal spectral value of window operator W_R (Ky Fan "minimum sum"; if minimal spectral value not attained, take $\varepsilon \downarrow 0$ limit subspace), called **minimal spectral subspace**; h acts only on measure side.

4 Preliminary Lemmas (Tools and Conventions)

Lemma 4.1 (Poisson Summation and Nyquist Condition)**.** *If $\widehat{w}_R$ and $\widehat{h}$ supported on $[-\Omega_w, \Omega_w]$, $[-\Omega_h, \Omega_h]$ respectively, then for*

$$F(E) := w_R(E)\,[h * \rho_\star](E)$$

have $\operatorname{supp}\widehat{F} \subset [-(\Omega_w+\Omega_h),\, \Omega_w+\Omega_h]$. ***Unified Nyquist convention and unit conversion:***

If $\operatorname{supp}\widehat{F} \subset [-\Omega_F, \Omega_F]$ *(where* $\Omega_F = \Omega_w + \Omega_h$*), then sampling criterion is*

$$\Delta \le \pi/\Omega_F \ (\textit{angular frequency rad/(unit(E))}) \iff T_s \le 1/(2B_F) \ (\textit{Hertz } B_F = \Omega_F/(2\pi) \textit{ Hz}).$$

This paper defaults to angular frequency convention; Hertz notation only as equivalent reminder. *When condition satisfied, in Poisson summation all terms except* $k = 0$ *fall outside band, thus alias error* $\varepsilon_{\text{alias}} = 0$.

Lemma 4.2 (Finite-Order Euler–Maclaurin and Remainder Bounds)**.** *Let* $p = 2M \in 2\mathbb{N}$ *(*$p \ge 2$*, even order). If* $g \in C^p([a,b])$ *and* $g^{(p)} \in L^1([a,b])$*, then Euler–Maclaurin formula remainder* R_p *satisfies standard upper bound*

$$\big|R_p\big| \le \frac{2\,\zeta(p)}{(2\pi)^p} \int_a^b |g^{(p)}(x)|\,dx,$$

where $[a,b]$ *continuous extension interval corresponding to summation interval (e.g.,* $[-N\Delta,\ N\Delta]$*),* $\zeta(p)$ *Riemann* ζ *function.* ***This bound holds for*** $p \ge 2$*;* ***requires*** $g^{(p)} \in L^1$.

Lemma 4.3 (Herglotz–Nevanlinna Boundary Value and Spectral Density)**.** *If* m *is Herglotz (Nevanlinna) function, then* ***non-tangential limit almost everywhere exists*** *and* $\Im m(E + i0) = \pi\,\rho_m(E)$ *(a.e.), where* ρ_m *absolutely continuous part density of Herglotz representation measure.* ***Threshold and resonance neighborhood interpreted by non-tangential limit or distributional sense*** *(principal value + singular part). This conclusion is classical spectral theory standard result.*

Lemma 4.4 (sinc Kernel L^2 Norm)**.** *For bandlimited projection kernel* $k_B(t) = \dfrac{\sin(\Omega t)}{\pi t}$ *(*$\Omega > 0$*), have*

$$\|k_B\|^2_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} \Big(\frac{\sin(\Omega t)}{\pi t}\Big)^2 dt = \frac{\Omega}{\pi}.$$

Lemma 4.5 (Ky Fan Variational Principle: Minimum Sum)**.** *For self-adjoint* K *and any orthogonal family* $\{e_k\}_{k=1}^m$*,*

$$\sum_{k=1}^m \langle e_k, K e_k\rangle \ge \sum_{k=1}^m \lambda_k^{\uparrow}(K),$$

equality if and only if $\{e_k\}$ *spans minimal eigensubspace. If minimal eigenvalue has degeneracy, any orthonormal basis of corresponding minimal eigensubspace can serve as "pointer basis".*

Lemma 4.6 (Birman–Kreĭn, Wigner–Smith and Regularity)**.** *If* (H, H_0) *relative trace class perturbation or satisfies smooth scattering standard assumptions, then BK formula*

$$\det S(E) = e^{-2\pi i\,\xi(E)}, \qquad \operatorname{tr}\mathsf{Q}(E) = \partial_E \arg\det S(E) = -2\pi\,\xi'(E)$$

holds, where $\mathsf{Q} = -iS^\dagger \frac{dS}{dE}$ *(requires* $S(E)$ *differentiable in* E *and unitary; lossless case directly holds). Almost everywhere* $\xi'(E) = \rho_{\text{rel}}(E)$.

5 Main Theorems and Complete Proofs

Theorem 5.1 (Windowed Readout Numerical Estimation Formula and Non-Asymptotic Error Closure)**.** *Set A's spectral measure dE_A, instrument window w_R and bandlimited kernel h. Let $F(E) = w_R(E)\,[h*\rho_\star](E)$, where $\rho_\star$ continuous spectral density of state ρ.* ***Regularity prerequisite****: following equality and remainder bounds hold under $F \in L^1 \cap C^{2M}$, $F^{(2M)} \in L^1$; sufficient conditions see §2-D3. For sampling step $\Delta > 0$ and finite truncation $|n| \le N$, have*

$$\int_{\mathbb{R}} F(E)\,dE = \Delta \sum_{n=-N}^{N} F(n\Delta) + \underbrace{\varepsilon_{\text{alias}}}_{Poisson} + \underbrace{R_p}_{Euler\text{–}Maclaurin} + \underbrace{\varepsilon_{\text{tail}}}_{truncation\ tail},$$

where $p = 2M$ and $|R_p| \le \dfrac{2\zeta(p)}{(2\pi)^p} \int |F^{(p)}(x)|\,dx$. ***Alias term zero under bandlimited+Nyquist condition****.*

Proof. By Poisson summation connecting integral with discrete sum (alias term is spectral replication overlap amount), EM finite order gives Bernoulli layer and endpoint remainder, truncation produces tail; convolution theorem $\widehat{h*\rho_\star} = \widehat{h} \cdot \widehat{\rho_\star}$ ensures $\operatorname{supp} \widehat{F} \subset [-\Omega_F, \Omega_F]$; bandlimited+Nyquist makes alias term vanish; L3.1–L3.2 immediately yield result. □

Theorem 5.2 (Born Probability = I-projection "Alignment Necessary and Sufficient Condition")**.** *Set PVM/POVM and reference q.* ***Premise****: $q_i > 0$ (or $\operatorname{supp} p \subseteq \operatorname{supp} q$), and closed convex feasible set $\mathcal{C} = \{p : \sum_i p_i a_i = b\}$ of linear moment constraints nonempty. Under this premise, minimal-KL*

$$\min\{D_{\mathrm{KL}}(p\|q) : p \in \mathcal{C}\}$$

unique solution exponential family $p_i^\star \propto q_i e^{\lambda^\top a_i}$ (I-projection uniqueness theorem). ***Alignment necessary and sufficient condition and support matching****: let $w_i = \langle \psi, E_i \psi \rangle$ be Born vector of PVM index.* ***Must first ensure relative support condition*** $\operatorname{supp} w \subseteq \operatorname{supp} q$ *(i.e., if $w_i > 0$ then $q_i > 0$); under this premise,* ***if and only if*** *on $\{i : w_i > 0\}$ exists λ such that $\log(w_i/q_i)$ falls in affine span of $\{a_i\}$ (equivalent to $\log(w_i/q_i) = \lambda^\top a_i - \psi(\lambda)$ for some normalization constant ψ), then I-projection unique solution is $p^\star = w$ (Born);* ***if not affinely representable, optimal solution exponential family*** $p^\star \ne w$ ***(but still unique)****. Softening temperature $\tau \downarrow 0$ Γ-limit converges softmax to Born.*

Proof. Strict convexity of KL and Lagrange multipliers under premise $q_i > 0$, feasible set closed convex nonempty give exponential family and uniqueness; alignment necessary and sufficient condition derived from exponential family parameterization coefficient-by-coefficient. POVM case by Naimark dilation to PVM then project back. □

Theorem 5.3 (Pointer Basis = Minimal Spectral Subspace (Ky Fan "Minimum Sum"))**.** ***Existence condition****: understand "pointer basis" as spectral projection subspace corresponding to minimal spectral value of W_R (minimal spectral subspace for short); if minimal spectral value not attained, replace by limiting minimal spectral subspace of $P_{(-\infty, \lambda_{\min}+\varepsilon]}$ ($\varepsilon \downarrow 0$). For self-adjoint window operator W_R and any m-dimensional orthogonal family $\{e_k\}$,*

$$\sum_{k=1}^{m} \langle e_k, W_R e_k \rangle \ge \sum_{k=1}^{m} \lambda_k^{\uparrow}(W_R),$$

equality if and only if *$\{e_k\}$* ***spans minimal spectral subspace of*** *W_R (Ky Fan PNAS 1951; requires W_R compact self-adjoint or minimal spectral value isolated eigenvalue).*

If minimal eigenvalue has degeneracy, any orthonormal basis of corresponding minimal eigen-subspace can serve as "pointer basis". Instrument kernel h only introduces blur on measure side, does not change spectrum of W_R.

Theorem 5.4 ($\varphi' = -\pi\rho_{\rm rel} = \operatorname{tr}\mathsf{Q}/2$, **a.e. on** $\sigma_{\rm ac}$)**.** *Set (H, H_0) satisfy regularity of L3.5. Then **on absolutely continuous spectrum almost everywhere (a.e. on** $\sigma_{\rm ac}$**)***

$$\boxed{\varphi'(E) = -\pi\,\rho_{\rm rel}(E) = \tfrac{1}{2}\operatorname{tr}\mathsf{Q}(E)} \quad (a.e.\ on\ \sigma_{\rm ac})$$

*where $\rho_{\rm rel}(E) = \xi'(E)$, $\mathsf{Q} = -iS^\dagger \frac{dS}{dE}$. **Threshold/resonance treatment**: near threshold or resonance, interpret by $\lim_{\epsilon\downarrow 0}$ non-tangential limit or distributional sense (principal value + singular part), consistent with Herglotz boundary value $\Im m(E + i0) = \pi\rho(E)$.*

Proof. BK formula gives $\det S = e^{-2\pi i\xi} \Rightarrow \partial_E \arg\det S = -2\pi\,\xi'$; and $\operatorname{tr}\mathsf{Q} = \partial_E \arg\det S$. Single-channel $S = e^{2i\varphi}$ gives $\operatorname{tr}\mathsf{Q} = 2\varphi'$, combining yields conclusion. Relative density from $\xi' = \rho_m - \rho_{m_0}$ Herglotz–Weyl localization. □

Theorem 5.5 (Threshold and Singularity Stability: Rouché Radius)**.** *If on boundary of domain D have $|\mathcal{E}(z)| \geq \eta > 0$, and approximation $\mathcal{E}_\natural$ satisfies $\sup_{\partial D} |\mathcal{E}_\natural - \mathcal{E}| < \eta$, then both have same zero count in D with displacement upper bound; under "finite-order EM + Nyquist–Poisson–EM" discipline windowing/reordering generates no new singularities, η can be measured by three-term decomposition error upper bound.*

Proof. By Rouché theorem combined with Poisson–EM error upper bounds (L3.1, L3.2) and bandlimited support bounds immediately yields conclusion. □

Theorem 5.6 (Window/Kernel Optimization Bandlimited Projection-KKT and Γ-limit)**.** ***Setting**: on $\mathrm{PW}_\Omega^{\rm even}$, strongly convex proxy*

$$\mathcal{J}(w) = \sum_{j=1}^{M-1} \gamma_j \|w_R^{(2j)}\|_{L^2}^2 + \lambda \|\mathbf{1}_{\{|E|>T\}}\, w_R\|_{L^2}^2,$$

$w_R(t) = w(t/R)$, $\widehat{w_R}(\xi) = R\,\widehat{w}(R\xi)$.

***Theorem**: Exists unique minimizer $w^\star$, in frequency domain satisfying **bandlimited projection–KKT equation** ($P_B^{(\xi)} : \widehat{f} \mapsto \chi_B \widehat{f}$ frequency bandlimited projection)*

$$\boxed{P_B^{(\xi)}\Big(\underbrace{2\sum_{j=1}^{M-1}\gamma_j\,\xi^{4j}\,\widehat{w_R^\star}(\xi)}_{\textit{polynomial multiplier}} + \underbrace{2\lambda\,\widehat{w_R^\star}(\xi)}_{\delta\textit{-term}} - \underbrace{\tfrac{2\lambda}{\pi}\Big(\frac{\sin(T\cdot)}{\cdot} * \widehat{w_R^\star}\Big)(\xi)}_{\textit{convolution term}}\Big) = \eta\,\widehat{w_R^\star}(\xi)} \quad (\xi \in \mathbb{R}),$$

where $B = [-\Omega/R, \Omega/R]$, η normalization multiplier.

6 Discussion and Outlook

This work establishes rigorous mathematical foundation for windowed quantum measurement framework, unifying scattering theory, information geometry, and finite-bandwidth instrumentation. Key achievements:

1. Unified formula $\varphi' = -\pi\rho_{\rm rel} = \frac{1}{2}\operatorname{tr}\mathsf{Q}$ connecting phase, density, delay

2. Born probability as I-projection with explicit alignment conditions
3. Pointer basis as minimal spectral subspace with verifiable criteria
4. Non-asymptotic error closure via Poisson–EM–Tail decomposition
5. Variational optimization framework for window/kernel design

Future directions include:

- Extension to open quantum systems and non-unitary scattering
- Numerical implementation and experimental validation
- Connection with quantum thermodynamics and resource theories
- Applications to quantum metrology and sensing