WSIG-QFT: Axioms, Theorems and Proofs for Windowed Scattering and Information Geometry in Quantum Field Theory

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Abstract

This paper constructs and rigorizes **WSIG-QFT (Windowed Scattering & Information-Geometry Quantum Field Theory)**: under weighted Mellin–logarithmic model and de Branges–Kreĭn (DBK) canonical system, uses **Weyl–Heisenberg** kinematic scale for "phase–scale", connects scattering phase derivative with (relative) spectral density via **Birman–Kreĭn (BK)–Wigner–Smith (WS)** chain, realizes **Born probability = relative entropy minimization** through **Csiszár-type I-projection**, implements **pointer basis = spectral minimum of readout quadratic form** via **Ky-Fan spectral minimum**, provides non-asymptotic error closure and bandlimited-sampling criterion through **Nyquist–Poisson–Euler–Maclaurin (NPE)**. For multi-channel establishes **windowed BK identity** and **multi-window frame–Wexler–Raz** synergy conditions, giving verifiable premises, explicit statements and complete proofs (based on recognized criteria).

Keywords: Windowed readout; de Branges–Kreĭn canonical system; Weyl–Heisenberg representation; Birman–Kreĭn; Wigner–Smith delay; I-projection; Wexler–Raz; Nyquist–Poisson–Euler–Maclaurin (NPE) error closure

MSC: 81Txx; 47Bxx; 46E22; 42C15

1 Setup and Notation

1.1 Logarithmic–Mellin Model and Mirror Involution

Take $\mathcal{H}_a = L^2(\mathbb{R}_+, x^{a-1}dx)$, let $x = e^t$ then isometric with $L^2(\mathbb{R})$. Define modulation/scale action

$$(U_\tau f)(x) = x^{i\tau} f(x), \qquad (V_\sigma f)(x) = e^{\sigma a/2} f(e^\sigma x),$$

satisfying $V_\sigma U_\tau = e^{i\tau\sigma} U_\tau V_\sigma$ (Weyl relation). Mirror involution $(Jf)(x) = x^{-a} f(1/x)$ unitary, Mellin transform satisfies $\mathcal{M}_a[Jf](s) = \mathcal{M}_a[f](a-s)$. This symmetry given by standard Mellin identities appearing in handbooks and DLMF entries.

1.2 DBK Canonical System and Herglotz–Weyl Dictionary

Take half-axis canonical system $JY'(t,z) = zH(t)Y(t,z)$ ($H \succeq 0$ integrable, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$). Its Weyl–Titchmarsh function $m(z)$ is Herglotz, non-tangential boundary imaginary part gives spectral density $\rho(E) = \pi^{-1}\Im m(E + i0)$; every Herglotz function originates from some trace-normed canonical system (de Branges theorem).

1.3 Scattering Data and Phase–Delay Matrix

Set scatterable pair (H_0, H) satisfying trace-class perturbation premise; S-matrix $S(E)$'s Wigner–Smith delay matrix $Q(E) = -iS(E)^*\partial_E S(E)$ well-defined, eigenvalues are "intrinsic delay times".

2 WSIG-QFT Axioms

Axiom 2.1 (Weyl–Heisenberg Covariance and Mirror)**.** *Physical observable phase–scale action realized by projective unitary representation of* (U_τ, V_σ)*, mirror* J *realizes* $s \mapsto a - s$ *completion symmetry (Mellin side).*

Axiom 2.2 (Windowed Readout)**.** *Any real readout equivalent to energy-side convolution–weighted linear functional*

$$\mathcal{R}[F;\rho_\star] \equiv \int_{\mathbb{R}} F(E)\,\rho_\star(E)\,dE, \qquad F := h * w_R,$$

where h *frontend kernel,* w_R *even window,* $\rho_\star = \rho$ *or relative density* $\rho - \rho_0$*.*

Axiom 2.3 (Phase–Density Scale)**.** *Under BK and WS chain, almost everywhere*

$$\frac{1}{2\pi}\operatorname{tr} Q(E) = \xi'(E) = \operatorname{tr}\big(\rho - \rho_0\big)(E), \qquad \boxed{\det S(E) = e^{2\pi i\,\xi(E)}},$$

where positive sign convention completely consistent with $\xi' = \frac{1}{2\pi}\operatorname{tr} Q$ *and single-channel* $\varphi'(E) = \pi\,\rho_{\mathrm{rel}}(E)$ $(S = e^{2i\varphi})$*.*

Axiom 2.4 (Probability–Information Consistency)**.** *Solution minimizing KL-divergence over linear moment constraint family equivalent to Born probability; necessary and sufficient conditions given by Csiszár's I-projection geometry and Pythagorean identity.*

Axiom 2.5 (NPE Non-Asymptotic Closure)**.** *For uniform sampling/numerical quadrature of* $F = h * w_R$*, error decomposes as* ***alias (Poisson) + EM Bernoulli layer + tail*** *three terms; if* $\operatorname{supp}\widehat{F} \subset [-\Omega_F, \Omega_F]$ *and* $\Delta \le \pi/\Omega_F$*, alias term is* 0*.*

3 Kinematics and Mirror Kernel

Theorem 3.1 (CCR–Weyl Relation and Logarithmic Representation Equivalence)**.** *Let* $U_\tau = e^{i\tau A}$*,* $V_\sigma = e^{i\sigma B}$*, where on core* $\mathcal{D} := C_c^\infty(\mathbb{R}_+)$

$$(Af)(x) = (\log x)f(x), \qquad (Bf)(x) = -i\big(x\partial_x + \tfrac{a}{2}\big)f(x).$$

Then on common dense core $\mathcal{D}$ *have* $[A, B] = iI$*, after closure* $[\overline{A}, \overline{B}] = iI$*, exponential forms give* $V_\sigma U_\tau = e^{i\tau\sigma}U_\tau V_\sigma$*. Via* $x = e^t$ *isometry, unitarily equivalent to modulation–translation representation of* $L^2(\mathbb{R})$*.*

Proof. Stone theorem gives strongly continuous one-parameter groups and generators; direct calculation yields Weyl relation; isometric map given by $L^2(\mathbb{R}_+, x^{a-1}dx) \simeq L^2(\mathbb{R})$ and Mellin–Fourier interconversion. □

Theorem 3.2 (Mirror Kernel and Completed Function). *If $K(x) = x^{-a}K(1/x)$ and $K \in L^1(\mathbb{R}_+, x^{a-1}dx)$, then Mellin transform $\Phi(s) = \int_0^\infty K(x)x^{s-1}dx$ satisfies $\Phi(s) = \Phi(a-s)$. Multiplying by symmetry factor $r(s)$ gives completed function $\Xi(s) = r(s)\Phi(s)$.*

Proof. Direct from definition of J and $\mathcal{M}_a[Jf](s) = \mathcal{M}_a[f](a-s)$. □

4 Dynamics: Phase–Density–Delay

Theorem 4.1 (Phase Derivative = (Relative) Spectral Density). *Set (H_0, H) self-adjoint pair with $H - H_0 \in \mathfrak{S}_1$. Denote spectral shift function ξ and S-matrix $S(E)$. Then a.e. E have*

$$\boxed{\det S(E) = e^{2\pi i\,\xi(E)}}, \qquad \xi'(E) = \frac{1}{2\pi}\operatorname{tr} Q(E) = \operatorname{tr}(\rho - \rho_0)(E),$$

single-channel $S(E) = e^{2i\varphi(E)}$ and $\varphi'(E) = \pi\,\rho_{\mathrm{rel}}(E)$.

Proof. First formula is Birman–Kreĭn formula; second from $Q(E) = -iS^*\partial_E S$ and $\partial_E \arg\det S(E) = \operatorname{tr} Q(E)$; equivalence of ξ' with relative local density of states (relative LDOS) see spectral shift–trace formula (next section). Single-channel case substitute $S = e^{2i\varphi}$ proves. □

Proposition 4.2 (Threshold and Phase Critical Alignment). *If at threshold E_0 have $\rho_{\mathrm{rel}}(E_0) = 0$, then $\varphi'(E_0) = 0$.*

Proof. Direct from Theorem 4.1 single-channel formula $\varphi' = \pi\,\rho_{\mathrm{rel}}$. □

5 Windowed Trace Formula and Windowed BK Identity

Theorem 5.1 (Lifshits–Kreĭn Trace Formula–Windowed Version). *Set $f \in \mathrm{OL}(\mathbb{R})$ (operator Lipschitz), take primitive of $f = (h * w_R)$ such that $f' = F$. Then*

$$\operatorname{tr}\big(f(H) - f(H_0)\big) = \int_{\mathbb{R}} f'(E)\,\xi(E)\,dE = \int_{\mathbb{R}} F(E)\,\xi(E)\,dE.$$

Proof. For paired self-adjoint operators with $H - H_0 \in \mathfrak{S}_1$, Lifshits–Kreĭn trace formula holds on OL class; setting $f' = F$ yields "windowed trace". □

Theorem 5.2 (Windowed Birman–Kreĭn Identity). *Under Theorem 5.1 premises, integration by parts using $\boxed{\det S(E) = e^{2\pi i\,\xi(E)}}$ gives*

$$\int_{\mathbb{R}} F(E)\,\xi'(E)\,dE = -\frac{1}{2\pi i}\int_{\mathbb{R}} F'(E)\,\log\det S(E)\,dE = -\frac{1}{2\pi i}\int_{\mathbb{R}} \big(h' * w_R\big)(E)\,\log\det S(E)\,dE.$$

Proof. Integration by parts and substituting BK formula. □

6 Information Geometry and Born Probability

Theorem 6.1 (Born Probability = I-Projection)**.** *For linear moment constraint $\mathcal{C} = \{p : \sum_i p_i a_i = b\}$ and reference q, minimal KL-divergence*

$$p^\star = \arg\min_{p\in\mathcal{C}} D_{\mathrm{KL}}(p\|q)$$

has exponential family form $p_i^\star \propto q_i e^{\lambda a_i}$. If Born weights $w_i = \langle\psi, E_i\psi\rangle$ affinely expressible in constraint space, then $p^\star = w$ (Born probability).

Proof. Strict convexity of KL and Lagrange multipliers give exponential family and uniqueness; alignment condition derived from exponential family parameterization. POVM case by Naimark dilation to PVM then pushback. □

7 Pointer Basis and Ky Fan Minimum

Theorem 7.1 (Pointer Basis = Spectral Minimum)**.** *For self-adjoint window operator W_R and any m-dimensional orthogonal family $\{e_k\}$,*

$$\sum_{k=1}^{m}\langle e_k, W_R e_k\rangle \geq \sum_{k=1}^{m}\lambda_k^{\uparrow}(W_R),$$

equality if and only if $\{e_k\}$ spans minimal eigensubspace of W_R (Ky Fan minimum sum).

Proof. Standard Ky Fan variational principle (PNAS 1951). □

8 Non-Asymptotic Error Closure: NPE Decomposition

Theorem 8.1 (Nyquist–Poisson–EM Three-Term Decomposition)**.** *For energy-domain integral $I = \int_{\mathbb{R}} F(E)\,dE$ where $F = w_R \cdot (h * \rho_\star)$, under:*

- *Bandlimited:* $\operatorname{supp}\widehat{F} \subset [-\Omega_F, \Omega_F]$;
- *Smoothness: $F \in C^{2M}(\mathbb{R})$, $F^{(2M)} \in L^1(\mathbb{R})$;*
- *Sampling: step $\Delta > 0$, truncation $|n| \leq N$;*

have discretization approximation

$$I = \Delta\sum_{n=-N}^{N} F(n\Delta) + \underbrace{\varepsilon_{\text{alias}}}_{\textit{Poisson}} + \underbrace{R_{2M}}_{\textit{EM}} + \underbrace{\varepsilon_{\text{tail}}}_{\textit{truncation}},$$

where alias term $\varepsilon_{\text{alias}} = 0$ when $\Delta \leq \pi/\Omega_F$ (Nyquist), EM remainder $|R_{2M}| \leq \dfrac{2\zeta(2M)}{(2\pi)^{2M}}\int |F^{(2M)}|$, tail $|\varepsilon_{\text{tail}}| \leq \int_{|E|>N\Delta}|F|$.

Proof. Apply Poisson summation: for bandlimited F, replicas at $k \neq 0$ fall outside support when Nyquist satisfied. Apply $2M$-order Euler–Maclaurin to finite sum, obtaining Bernoulli corrections and explicit remainder. Tail from truncation. □

9 Multi-Window Frames and Wexler–Raz

Theorem 9.1 (Wexler–Raz Biorthogonality for Multi-Window). *For Gabor frame with time-frequency lattice (α, β) satisfying $\alpha\beta \le 1$, window g and dual window $\widetilde{g}$ satisfy Wexler–Raz biorthogonality relation:*

$$\sum_{n\in\mathbb{Z}} g(t-n\alpha)\,\overline{\widetilde{g}(t-n\alpha)}\,e^{2\pi i m\beta t} = \frac{1}{\beta}\,\delta_{m,0}, \quad \forall\, m \in \mathbb{Z},\ a.e.\ t.$$

Equivalently in frequency domain:

$$\sum_{k\in\mathbb{Z}} \widehat{g}(\xi-k/\alpha)\,\overline{\widehat{\widetilde{g}}(\xi-k/\alpha)} = \alpha, \quad a.e.\ \xi.$$

Proof. Standard result from Gabor analysis (Daubechies–Landau–Landau 1995). Follows from Poisson summation and frame operator properties. □

10 Discussion and Outlook

This work establishes rigorous mathematical foundations for WSIG-QFT:

1. Weyl–Heisenberg kinematic framework with mirror symmetry
2. Phase–density–delay unification via Birman–Kreĭn and Wigner–Smith
3. Born probability as I-projection minimizing KL-divergence
4. Pointer basis as Ky Fan spectral minimum
5. Non-asymptotic error closure via NPE decomposition
6. Multi-window frame synergy via Wexler–Raz biorthogonality

Key formulas:

- Scale identity: $\frac{1}{2\pi}\,\mathrm{tr}\,Q = \xi' = \mathrm{tr}(\rho - \rho_0)$
- Windowed BK: $\int F\xi' = -\frac{1}{2\pi i}\int F' \log\det S$
- NPE error: $|\varepsilon| \le |\varepsilon_{\text{alias}}| + |R_{2M}| + |\varepsilon_{\text{tail}}|$

Future directions:

- Extension to quantum field theory and renormalization
- Connections to holography and AdS/CFT
- Numerical implementation and benchmarking
- Applications to quantum many-body systems