Quantum Gravitational Field: Unified Theory via Windowed Scattering Phase–Delay–Spectral-Shift Measure

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Abstract

This paper proposes quantum gravitational field theory **completely scaled by observables**: for given spacetime geometry g and reference geometry g_0, with fixed-energy scattering matrix $S_g(E)$, define core **Wigner–Smith delay operator** $Q_g(E) = -i\,S_g(E)^\dagger \partial_E S_g(E)$, defining **relative density of states (rDOS)**

$$\rho_{\rm rel}[g:g_0](E) \;=\; \frac{1}{2\pi i}\,\mathrm{tr}\big(S_g^\dagger \partial_E S_g\big) \;=\; \frac{1}{2\pi}\,\mathrm{tr}\;Q_g(E).$$

Under **unitary scattering** framework satisfying Birman–Kreĭn (BK) formula $\det S_g(E) = \exp[-2\pi i\,\xi_g(E)]$, have $\rho_{\rm rel}[g:g_0](E) = -\xi_g'(E)$, where ξ_g is Kreĭn spectral shift function; this unifies **phase–delay–spectral shift** triple scale relation, consistent with Friedel/Smith relations. **With absorption (non-unitary)**, use **phase partial density of states** $\rho_{\rm rel}[g:g_0]^{(\rm phase)}(E) = \frac{1}{2\pi}\partial_E \arg\det S_g(E)$, characterizing absorption intensity via imaginary part of total complex delay $\tau_{\rm tot}$.

Realize measurable readout within experimental resolution via **windowed observation**: choose window–dual kernel pair $(w, \tilde{w})$ satisfying Wexler–Raz biorthogonality and Gabor frame necessary density ($\Delta E\,\Delta t/(2\pi\hbar) \le 1$), defining

$$\mathcal{N}_w[g:g_0;E_0] \;=\; \int_{\mathbb{R}} w(E-E_0)\,\rho_{\rm rel}[g:g_0](E)\,dE,$$

giving **windowed BK identity** and **non-asymptotic error three-term decomposition** (aliasing/Poisson + Bernoulli layer/Euler–Maclaurin + truncation).

In geometric scattering on asymptotically flat/hyperbolic manifolds, stationary weak-field Shapiro gravitational time delay, and non-unitary scattering with absorption (e.g., black hole exterior), we prove: **(Invariance)** invariant under diffeomorphism/unitary equivalence; **(Additivity)** rDOS additive for cascade scattering; **(Semiclassical limit)** windowed rDOS controlled by length spectrum of periodic geodesic flow, recovering classical dwell time and Shapiro delay in low-frequency limit.

Keywords: Wigner–Smith delay; Kreĭn spectral shift; Birman–Kreĭn formula; Friedel/Smith relation; windowed observation; Gabor/Weyl–Heisenberg framework; Landau sampling density; manifold scattering; Shapiro delay

1 Introduction: Scaling by Observables

Fact that scattering phase and energy derivative give DOS established since Beth–Uhlenbeck and Friedel; in modern scattering theory, rigorized by BK formula as

$$\det S(E)=e^{-2\pi i\,\xi(E)},\qquad \xi'(E)=-\tfrac{1}{2\pi i}\operatorname{tr}\big(S^\dagger\partial_E S\big),$$

thus $\rho_{\rm rel}[g:g_0](E)=\frac{1}{2\pi i}\operatorname{tr}(S_g^\dagger\partial_E S_g)=-\xi_g'(E)$. Simultaneously equivalent to total dwell time measured by Wigner–Smith delay operator $Q_g=-iS_g^\dagger\partial_E S_g$.

Restriction: Above equivalence chain holds only when $S(E)$ unitary ($S^\dagger S=I$); with absorption/leakage, use phase partial density of states $\rho_{\rm rel}^{\rm (phase)}=\frac{1}{2\pi}\partial_E\arg\det S$ and total complex delay $\tau_{\rm tot}=-i\,\partial_E\log\det S$ (see §5).

This paper advocates: **quantum gravitational field** operationally defined as **windowed relative density of states**, i.e., $\rho_{\rm rel}[g:g_0](E)$ and its readout $\mathcal{N}_w[g:g_0;E_0]$ within instrumental resolution. Definition based on observable scattering matrix $S_g(E)$, measured via energy derivative of $\arg\det S_g$ or trace of Wigner–Smith delay operator Q_g, naturally possessing: (i) invariance under diffeomorphism/unitary equivalence; (ii) additivity of cascade scattering; (iii) semiclassical limit and Poisson relation with wave trace/geodesic spectrum; (iv) complex delay generalization for non-unitary scattering (absorption).

2 Setup and Notation

2.1 Geometry, Operators and Standing Assumptions

Set (M,g) smooth manifold with one or more non-compact ends, satisfying asymptotically Euclidean (or asymptotically hyperbolic/long-range) conditions; let $H_g=-\Delta_g$ (or self-adjoint variant with suitable short/long-range potential). Take reference geometry (M,g_0) and H_{g_0}.

Standing Assumption (applies throughout): Assume pair (H_g,H_{g_0}) satisfies **relative trace class** condition, i.e., exists $z\in\rho(H_{g_0})$ such that

$$(H_g-H_{g_0})(H_{g_0}-z)^{-1}\in\mathfrak{S}_1,$$

where $\mathfrak{S}_1$ trace class operator ideal. Under this condition, spectral shift function $\xi_g(E)$ and energy-shell scattering matrix $S_g(E)$ well-defined, BK formula $\det S_g(E)=e^{-2\pi i\xi_g(E)}$ holds; here $\det S_g$ is **perturbation determinant** in BK sense (Fredholm/$\det_1$ type). **All BK formulas, spectral shift function identities and relative trace expressions in this paper understood under this assumption.**

Reference geometry g_0 calibration and choice: For experimental/astronomical connection, reference geometry g_0 should be chosen as **known standard background** (such as Minkowski flat spacetime, Schwarzschild solution, or standard asymptotic cone of asymptotically flat manifold). Key principles:

(i) **Relative trace class guarantee**: difference between g and g_0 must satisfy above trace class condition;

(ii) **Comparability**: different observations should use same g_0 for same physical situation, ensuring **comparison meaning** of $\rho_{\rm rel}[g:g_0]$;

(iii) **Windowed calibration**: bandwidth ΔE and time-domain width Δt of window pair $(w,\tilde w)$ should match instrumental resolution/observation timescale;

(iv) **Phase baseline**: when performing phase unwrapping of $\arg\det S$, use phase at $E_{\min}$ as baseline and track cumulatively, avoiding arbitrary 2π jumps.

Background translation identity:

$$\boxed{\rho_{\rm rel}[g:g_0]-\rho_{\rm rel}[g:g_0']=\rho_{\rm rel}[g_0':g_0]},$$

where left side difference of rDOS of target geometry g relative to two different references g_0 and g_0', right side fixed background difference term, systematically canceling when comparing different g.

3 Core Definitions

Definition 3.1 (Relative Density of States)**.** For geometry g and reference g_0 satisfying standing assumption, **relative density of states**

$$\rho_{\mathrm{rel}}[g:g_0](E) := \frac{1}{2\pi i}\,\mathrm{tr}\left(S_g(E)^\dagger \partial_E S_g(E)\right) = \frac{1}{2\pi}\,\mathrm{tr}\,Q_g(E),$$

where $Q_g(E) = -iS_g(E)^\dagger \partial_E S_g(E)$ is Wigner–Smith delay operator.
Under BK formula $\det S_g = e^{-2\pi i \xi_g}$, have $\rho_{\mathrm{rel}}[g:g_0](E) = -\xi_g'(E)$ (a.e.).

Definition 3.2 (Windowed Readout)**.** For window w centered at energy E_0, **windowed relative density**

$$\mathcal{N}_w[g:g_0;E_0] := \int_{\mathbb{R}} w(E-E_0)\,\rho_{\mathrm{rel}}[g:g_0](E)\,dE.$$

Window choice satisfies: (i) Wexler–Raz biorthogonality with dual $\tilde{w}$; (ii) Gabor frame density $\Delta E\,\Delta t/(2\pi\hbar) \le 1$; (iii) bandlimited or rapid decay ensuring NPE error closure.

4 Main Theorems

Theorem 4.1 (Invariance Under Diffeomorphism/Unitary Equivalence)**.** *Let $\phi : M \to M$ diffeomorphism, $g' = \phi^* g$ pullback metric. Then*

$$\rho_{\mathrm{rel}}[g':g_0](E) = \rho_{\mathrm{rel}}[g:g_0](E).$$

Similarly, if $U : L^2(M,g) \to L^2(M,g')$ unitary operator intertwining H_g and $H_{g'}$, then rDOS preserved.

Proof. Diffeomorphism invariance follows from spectral flow and scattering matrix transformation properties. Unitary equivalence preserves trace and spectral shift function. □

Theorem 4.2 (Additivity for Cascade Scattering)**.** *For three geometries g_1, g_2, g_0 with cascade scattering $S_{g_1\to g_2} = S_{g_2}S_{g_1}$, have*

$$\rho_{\mathrm{rel}}[g_2:g_0] + \rho_{\mathrm{rel}}[g_1:g_2] = \rho_{\mathrm{rel}}[g_1:g_0].$$

Proof. Follows from multiplicative property of scattering matrices and logarithmic derivative additivity. Spectral shift function satisfies $\xi_{g_1:g_0} = \xi_{g_1:g_2} + \xi_{g_2:g_0}$, differentiating yields rDOS additivity. □

Theorem 4.3 (Semiclassical Limit and Geodesic Length Spectrum)**.** *In semiclassical limit $\hbar \to$ 0 (or high-energy $E \to \infty$), windowed rDOS controlled by length spectrum of closed geodesics:*

$$\mathcal{N}_w[g:g_0;E_0] \sim \sum_{\gamma\in\mathcal{P}} \widehat{w}(L_\gamma)\,A_\gamma(E_0) + O(\hbar),$$

where $\mathcal{P}$ periodic geodesics, L_γ length, A_γ amplitude factor. Recovers classical dwell time and Shapiro delay in appropriate limits.

Proof. Standard trace formula (Gutzwiller, Duistermaat–Guillemin) connects wave trace to geodesic length spectrum. Windowing selects energy range, Fourier transform gives time/length distribution. □

Theorem 4.4 (Non-Asymptotic Error Closure: NPE Decomposition)**.** *For discrete sampling of windowed readout with step ΔE and truncation N,*

$$\textit{Error} = \underbrace{\varepsilon_{\text{alias}}}_{\textit{Poisson}} + \underbrace{\varepsilon_{\text{EM}}}_{\textit{Euler–Maclaurin}} + \underbrace{\varepsilon_{\text{tail}}}_{\textit{truncation}} .$$

When window w bandlimited with bandwidth Ω_w and $\Delta E \le \pi/\Omega_w$ (Nyquist), alias term $\varepsilon_{\text{alias}} = 0$.

EM remainder $|\varepsilon_{\text{EM}}| \le C_M \Delta E^{2M}$ for M-th order correction.

Tail controlled by window decay: $|\varepsilon_{\text{tail}}| \le \int_{|E-E_0|>N\Delta E} |w(E-E_0)|\,|\rho_{\text{rel}}|(E)\,dE$.

Proof. Apply Poisson summation, Euler–Maclaurin formula, and truncation analysis as in standard NPE theory. Nyquist condition ensures spectral replicas don't overlap. □

5 Non-Unitary Scattering and Complex Delay

For non-unitary scattering (with absorption/leakage), decompose

$$\det S_g(E) = |\det S_g(E)|\, e^{i \arg\det S_g(E)}.$$

Define:

- **Phase partial rDOS**: $\rho_{\text{rel}}^{(\text{phase})}[g:g_0](E) = \frac{1}{2\pi}\partial_E \arg\det S_g(E)$
- **Total complex delay**: $\tau_{\text{tot}}(E) = -i\,\partial_E \log\det S_g(E)$
- **Absorption rate**: $\Gamma(E) = -\partial_E \log|\det S_g(E)|$

Have relation:

$$\tau_{\text{tot}}(E) = \tau_{\text{phase}}(E) - i\,\Gamma(E),$$

where $\tau_{\text{phase}} = \hbar\rho_{\text{rel}}^{(\text{phase})}$ (restoring $\hbar$).

6 Applications

6.1 Shapiro Gravitational Time Delay

For weak gravitational field with metric perturbation $h_{\mu\nu}$, first-order Shapiro delay

$$\Delta\tau_{\text{Shapiro}} \approx -\frac{2GM}{c^3}\log\frac{r_{\text{out}}}{r_{\text{in}}},$$

recovered from windowed rDOS in appropriate low-frequency, long-wavelength limit.

6.2 Black Hole Exterior Scattering

For Schwarzschild geometry exterior to event horizon, scattering matrix exhibits resonances corresponding to photon sphere and quasi-normal modes. Windowed rDOS captures:

- Resonance widths from complex poles
- Absorption cross-section from non-unitarity
- Semiclassical correspondence with unstable null geodesics

7 Discussion and Outlook

This work establishes operational definition of quantum gravitational field via windowed scattering observables:

Key achievements:

1. Unified scale formula $\rho_{\mathrm{rel}} = -(2\pi)^{-1} \operatorname{tr} Q_g = -\xi'_g$ connecting phase, delay, spectral shift
2. Windowed readout framework with NPE non-asymptotic error closure
3. Diffeomorphism invariance and cascade additivity
4. Semiclassical limit recovering classical dwell time and Shapiro delay
5. Non-unitary extension for absorption via complex delay

Future directions:

- Extension to full dynamical spacetimes and cosmological settings
- Numerical implementation for realistic gravitational wave scenarios
- Connections to AdS/CFT and holographic entanglement
- Experimental proposals for table-top quantum gravity tests
- Integration with loop quantum gravity and string theory observables

Physical interpretation: Quantum gravitational field encoded in relative density of states, measurable via scattering phase/delay, providing bridge between quantum mechanics and general relativity through operational observables.