From Theorem 7.1, the ML degree of a bipyramid is

$$\left|\sum_{s=3}^{n}(-1)^{s-3}(s-3)!\left(\frac{1}{(s-1)!}\sum_{i=0}^{s-1}(-1)^{s-1-i}\binom{s-1}{i}((i-1)^{n-2}+(-1)^{n-2}(i-1))\right.\right.$$
$$\left.\left.+\frac{1}{(s-2)!}\sum_{j=0}^{s-2}(-1)^{s-j}\binom{s-2}{j}((j-1)^{n-2}+(-1)^{n-2}(j-1))\right)\right|.$$

The summand corresponding to $s=n$ simplifies to $(-1)^{n-3}(n-3)!$, so it suffices to show that the following expression is zero for all $n\geq 4$:

$$\begin{aligned}&(-1)^n(n-4)+(-1)^{n-3}(n-3)!\\&+\sum_{s=3}^{n-1}(-1)^{s-3}\Bigg(\sum_{i=0}^{s-1}\frac{(-1)^{s-1-i}}{(s-1)(s-2)}\binom{s-1}{i}((i-1)^{n-2}+(-1)^{n-2}(i-1))\\&+\sum_{j=0}^{s-2}\frac{(-1)^{s-j}}{s-2}\binom{s-2}{j}((j-1)^{n-2}+(-1)^{n-2}(j-1))\Bigg).\end{aligned}$$

Let H_n denote the harmonic numbers and let $S(n,i)$ denote the Stirling numbers of the second kind. We record the following facts.

1. $\displaystyle\sum_{k=0}^{n}(-1)^k k!S(n,k)=(-1)^n$. *Proof:* Evaluate $\sum_{k=0}^{n}S(n,k)x^{\underline{k}}=x^n$, where $x^{\underline{k}}$ is the falling factorial of x of order k, at $x=-1$. ■

2. $\displaystyle\sum_{k=0}^{n}(-1)^n k!H_i S(n,k)=n(-1)^n$. *Proof:* Differentiate $\sum_{k=0}^{n}S(n,k)x^{\underline{k}}=x^n$ by x, then set $x=-1$. ■

3. $\displaystyle\sum_{j=0}^{n}(-1)^j\binom{n}{j}j^n=(-1)^n n!$. *Proof:* Consider the partial fraction decomposition of $\frac{n!(-y)^n}{y(y+1)\cdots(y+n)}$. Since the denominator is square free and the numerator has lower degree than the denominator, the rational function is equal to $\sum_{j=0}^{n}\frac{\alpha j}{y+j}$, where $\alpha_j=(\frac{\text{numerator}(y)}{\text{denominator}'(y)})|_{y=-j}$ (Lagrange's identity). A calculation confirms that $\alpha_j=(-1)^j\binom{n}{j}j^n$. ■

4. $\displaystyle\sum_{j=0}^{n}(-1)^j\binom{n+1}{j+1}j^n=(-1)^n$. *Proof:* Argue as before, now using $\frac{(n+1)!(y+1)^n}{y(y+1)\cdots(y+n+1)}$. ■

5. $\displaystyle\sum_{j=1}^{n}(-1)^j\frac{1+j+j^2-nj}{(j+1)^2}\binom{n}{j}=H_{n+1}-2$. *Proof:* Use computer algebra (Zeilberger's algorithm) to construct a recurrence for the sum, then check that the right hand side satisfies the same recurrence and that the initial values match. ■

6. $\displaystyle\sum_{j=0}^{n}\frac{(-1)^j}{j+1}\binom{n+1}{j+1}\binom{j}{k}=(-1)^k H_{n+1}-(-1)^k H_k$. *Proof:* Use computer algebra (Zeilberger's algorithm) to construct two recurrences for the sum (one with respect to n and one with respect to k), then check that the right hand side satisfies the same recurrences and that the initial values match. ■

We start by simplifying the expression. We can let the second inner sum run to $s-1$ instead of to $s-2$ because $\binom{s-2}{s-1}=0$ and simplify using the identity $\binom{s-2}{j}=\frac{s-j-1}{s-1}\binom{s-1}{j}$. Then, the two inner sums can be merged

$$\sum_{s=3}^{n-1}\sum_{j=0}^{s-1}\frac{(-1)^{s-1-j}}{(s-1)(s-2)}\frac{s-j}{s}\binom{s}{j}+\frac{(-1)^{s-j}}{s-2}\frac{s-j-1}{s-1}\binom{s-1}{j}$$

and simplified via the identity $\binom{s}{j}=\frac{s}{s-j}\binom{s-1}{j}$ to obtain

$$(-1)^n(n-4)+(-1)^{n-3}(n-3)!-\sum_{s=3}^{n-1}\sum_{j=0}^{s-1}(-1)^j\frac{s-j-2}{(s-1)(s-2)}\binom{s-1}{j}((j-1)^{n-2}+(-1)^n(j-1)).$$

Now, replace n by $n+3$ and s by $s+2$ (and observe that $((j-1)^{n+1}-(-1)^n(j-1))=0$ when $j=0,1$):

$$(-1)^{n+1}(n-1)+(-1)^n n!-\sum_{s=1}^{n}\sum_{j=2}^{s+1}(-1)^j\frac{s-j}{s(s+1)}\binom{s+1}{j}((j-1)^{n+1}-(-1)^n(j-1)).$$

The claim is that this is equal to zero for all $n\geq 1$.

Since $\binom{s+1}{j}$ is zero for $j>s+1$, we can let the sum over j run from 2 to ∞ and then exchange the order of summation to obtain

$$(-1)^{n+1}(n-1)+(-1)^n n!-\sum_{j=2}^{\infty}\Bigg(\sum_{s=1}^{n}(-1)^j\frac{s-j}{s(s+1)}\binom{s+1}{j}\Bigg)((j-1)^{n+1}-(-1)^n(j-1)).$$

Computer algebra (Gosper's algorithm) applies to the inner sum and allows us to simplify the whole expression to

$$(-1)^{n+1}(n-1)+(-1)^n n!-\sum_{j=2}^{\infty}(-1)^{j-1}\frac{1-j+j^2+n-nj}{j^2}\binom{n}{j-1}((j-1)^n-(-1)^n).$$

Replace j by $j+1$ to get

$$(-1)^{n+1}(n-1)+(-1)^n n!-\sum_{j=1}^{\infty}(-1)^j\frac{1+j+j^2-nj}{(j+1)^2}\binom{n}{j}(j^n-(-1)^n).$$

Using Fact 5, this simplifies further to

$$(-1)^{n+1}(n+1)-(-1)^{n+1}H_{n+1}+(-1)^n n!-\sum_{j=1}^{\infty}(-1)^j\frac{1+j+j^2-nj}{(j+1)^2}\binom{n}{j}j^n.$$

The summand is zero for $j=0$, so it doesn't matter whether we let the sum start from 0 or from 1. We can use Fact 3 to simplify further to

$$(-1)^{n+1}(n+1)-(-1)^{n+1}H_{n+1}-\sum_{j=0}^{\infty}(-1)^j\frac{-j-jn}{(j+1)^2}\binom{n}{j}j^n.$$

Rewrite this to

$$(-1)^{n+1}(n+1)-(-1)^{n+1}H_{n+1}-\sum_{j=0}^{\infty}(-1)^j\Big(\frac{1}{j+1}-1\Big)\binom{n+1}{j+1}j^n$$

and use Fact 4 to obtain

$$(-1)^{n+1}n-(-1)^{n+1}H_{n+1}-\sum_{j=0}^{\infty}\frac{(-1)^j}{j+1}\binom{n+1}{j+1}j^n.$$

At this point, we translate j^k to $j^{\underline{k}}=k!\binom{j}{k}$ (at the cost of introducing Stirling numbers), obtaining

$$(-1)^{n+1}n-(-1)^{n+1}H_{n+1}-\sum_{k=0}^{n}\Bigg(\sum_{j=0}^{\infty}\frac{(-1)^j}{j+1}\binom{n+1}{j+1}k!\binom{j}{k}\Bigg)S(n,k).$$

Using Fact 6, we get

$$(-1)^{n+1}n-(-1)^{n+1}H_{n+1}-\sum_{k=0}^{n}(-1)^k k!(H_{n+1}-H_k)S(n,k).$$

From here, Fact 1 brings us to

$$(-1)^{n+1}n+\sum_{k=0}^{n}(-1)^k k!H_kS(n,k),$$

which, according to Fact 2, is indeed zero. This completes the proof.