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Perishable Production Inventory System with a Level Dependent Demand and offering Discount during Production run.

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ABSTRACT

This paper presents a comprehensive analysis of a perishable production inventory system with level-dependent demand, focusing on the application of a discount during the production phase. The model assumes no shortage, and the production rate exceeds the rate of demand, which is treated as a function of the inventory level. Throughout the entire cycle, deterioration is taken into account, and inventory is categorized into undecayed and decayed portions over different phases. A discount is applied to the un-decayed inventory during production run, which influences holding and decay costs. The aim is to develop a profit function in terms of quantity that incorporates these factors and optimize inventory policies. Analytical expressions for total cost and profit are derived. The results offer practical insights for improving profitability in inventory systems with perishable items under the given assumptions. Some numerical illustrations are derived and few tables and graphs are constructed in order to better representation of the results. Sensitivity analysis is performed in order to exhibit the model's relevance to real-world situations, especially in controlling and managing perishable goods.

1. Introduction

Inventory is a fundamental component of the supply chain, as it ensures smooth operations and enables firms to meet customer demand without interruption. Hugos [1] and Stevenson [2] emphasized that effective inventory management allows businesses to balance supply and demand, reduce lead times, and optimize production schedules. Nielsen [3] proposed a method for developing lead-time models that incorporate order size, which is particularly useful for managing inventory and supply chains. Ou et al. [4] explored the application of RFID technology in the design of e-commerce supply chain systems, showing how such innovations can effectively address inventory management challenges. Sarkar et al. [5] highlighted the role of optimization in production, logistics, inventory management, supply chain management, and blockchain. Mishra [6] reviewed the application of genetic algorithms in inventory and supply chain management, demonstrating how mathematical and logical models can help reduce costs and increase revenue. Alajmi et al. [7] assessed the resilience of supply chain management in Saudi medical laboratories during the COVID-19 pandemic, underscoring the need for adaptability in the face of unexpected disruptions. Collectively, these studies illustrate that inventory is critical for firms across industries, as it supports operational stability and enhances customer satisfaction. Effective inventory management ensures that businesses maintain sufficient stock to fulfill orders promptly, preventing lost sales and preserving customer confidence.

Conversely, excess inventory can lead to high holding costs, wastage, and obsolescence, which negatively affect profitability. Managing perishable inventory presents additional challenges, particularly in industries dealing with products such as food, pharmaceuticals, and chemicals, which deteriorate over time. Traditional inventory models often assume constant demand and infinite shelf life, limiting their applicability to real-world scenarios where demand fluctuates and deterioration is unavoidable. Rashidi et al. [8] developed a bi-objective mathematical model for optimizing supply chain network design with perishable products, incorporating both location and inventory decisions. The model was solved using a Pareto-based meta-heuristic, highlighting the importance of strategic decision-making in managing perishable items. Similarly, Violi et al. [9] proposed a dynamic and stochastic framework to address the inventory routing problem in the agri-food supply chain, focusing on perishability to ensure timely delivery and minimize waste. Mouaky et al. [10] examined the use of a kanban-based system for multi-echelon inventory management in pharmaceutical supply chains, emphasizing the importance of collaboration and information sharing in managing perishable products across different supply chain levels. Overall, the literature underscores the necessity of effective supply chain strategies—such as optimized network design, inventory routing, and information integration—to handle perishability efficiently.

In this study, we present an advanced inventory model that addresses these limitations by incorporating level-dependent demand and discounting during the production phase. The model develops a profit function that accounts for production, holding, and deterioration costs, thereby providing a comprehensive framework for improving inventory strategies for perishable products. Its practical significance is demonstrated through numerical examples and sensitivity analyses.

2. Literature Review

Numerous studies have been conducted on perishable inventory systems, concentrating on different elements such as demand fluctuations, deterioration rates, and manufacturing strategies. Initial studies by Whitin [11] established fundamental models for perishable items, setting the groundwork for subsequent research. Over time, the incorporation of demand functions influenced by inventory levels has gained interest, as demonstrated in research by Urban [12] and Teng [13], who investigated pricing and restocking strategies.

Kaasgari et al. [14] demonstrated how Vendor Managed Inventory (VMI) can lead to cost reduction, enhanced responsiveness, and improved collaboration among supply chain participants. In that study, the inventory management strategy known as VMI was utilized to handle the stock of perishable goods in a two-level supply chain, where there was a single vendor and multiple retailers. A study on a production inventory model with continuous demand that accounted for various production rates was carried out by Muhammad Ekramol Islam [15]. In this work, the author introduced a model that describes how inventory levels decrease over time due to the deterioration of items and demand. The rate of decay is solely determined by the inventory present in the system. It was assumed that production begins when the inventory reaches a predetermined level.

In a different study, Mohammad Ekramol Islam [16] examined a model for the optimum time cycle in a deterministic inventory model where inventory decays at a certain rate. It was assumed that lead time is significant and shortages are permitted. It was also assumed that the re-order level is zero. The study evaluated the optimum time and provided numerical illustrations. Mohammad Ekramol Islam [17] investigated a production inventory model with a constant demand and three distinct production speeds. Production is assumed to start at the inventory level, where there are predetermined backlogs. The inventory level determines the overall cost per unit of time in this model, with numerical examples used to illustrate the findings.

Mohammad Ekramol Islam and T.K. Devnath [18] investigated a (z, Z) inventory system for determining the optimal re-order level where demand and decay occur simultaneously. Lead time is significant and known, and shortages are permissible. Two cases were considered for the optimal re-order level: (i) $z \geq 0$ and (ii) $z \leq 0$, upon which the final decision was based.

A time-dependent inventory model was created by Ekramol Islam et al. [19] based on a constant production rate, exponentially declining market demand, and finite product shelf life. The study aimed to determine the ideal time cycle and inventory cost, providing numerical examples and demonstrating the convex property to support the hypothesis.

An inventory model that incorporates the recycling process and two levels of continuous demand during production-run time with constant production rate—while producing defective items alongside good products—was proposed by Ekramol Islam et al. [20]. The conventional EPQ model for non-defective items was used as a baseline for comparison.

In recent years, researchers have extended these models to include discounting strategies during production. A production inventory model was developed by Shirajul Islam Ukil et al. [21] using a fixed production rate and fluctuating market demand. The proposed model utilized a power pattern that, depending on demand, can be expressed linearly or exponentially. The ideal time cycle and overall average inventory cost are calculated, and the model also accounts for minimal degradation.

More recently, Ekramol Islam [22] proposed a deterministic production inventory model for a single commodity with two production rates and constant demand. Production begins at the inventory level, where predefined backlogs exist. The total cost per unit of time is determined by the inventory quantity Q, with numerical examples illustrating the findings. The results confirm that the cost function is convex.

A similar integrated inventory model, where lead time is a stochastic variable with a generic distribution, was proposed by Hossain et al. [23] for a single vendor and a single buyer. Customers have steady demand, while the vendor delivers items in fixed lot sizes. Penalty costs are incurred for delayed delivery, and a nonlinear cost function is used to determine the minimum overall inventory cost.

Wang et al. [24] emphasized that circular supply chain management is essential for firms transitioning from a linear make-use-dispose economic model to a more sustainable circular economy. Tracking material reuse across multiple life cycles with various stakeholders is critical. This paper proposed a system architecture for fast-fashion circular supply chain management enabled by blockchain.

An inventory model for rapidly deteriorating products that considers product returns was discussed by Singh et al. [25]. Factors such as advertising, timing, and selling price affect demand. Goods returns are influenced by both price and demand. In the proposed model, shortages are permissible and partially backlogged. The objective is to determine the optimal selling price and replenishment schedule that maximize profit.

Finally, Abbas et al. [26] focused on improving the quality and safety of perishable goods while supporting sustainable development goals in cold supply chains. Their findings indicate that policymakers can significantly reduce the negative social and environmental impacts of supply chains without substantially increasing costs.

This paper builds a comprehensive model that captures the complexities of managing perishable goods with level-dependent demand and incorporates discounts during production to improve profitability. While prior studies have addressed components such as dynamic pricing, replenishment, and decay, this research uniquely integrates these elements into a cohesive model tailored to industries managing perishable goods efficiently.

3. Assumptions

- Production starts with a set amount of inventory serving as a safety stock
- Inventory peaks at a specific level, after which it quickly depletes due to deterioration and demand fulfilment
- Shortage are not allowed
- Replenishment of buffer stock at the end of cycle is infinite.

3.1. Notations

- ω: A steady production rate that is always higher than the demand rate demand rate at any given time.
- $\alpha + \beta I(t)$: Demand rate at any instant t, where $\alpha = 0, 1, 2, \ldots \ldots$ and $0 < \beta < 1$
- φ: Decay rate for unit inventory which is very small and constant
- ψ: decay rate of inventory of buffer stock
- Q: Inventory level at time $t = 0$ and represents the buffer stock
- Q_1: Inventory level at time $t = t_1$
- dt: very small portin of instant t
- C_0: Set up cost
- h_1: Cost of holding undecayed inventory
- h_2: Cost for holding degraded inventory at time $t = 0$ to T_1
- h_3: Cost for holding deteriorated inventory of buffer stock
- γ: Discount rate applicable during production
- t_1: Duration to achieve peak inventory level
- T_1: Total cycle length
- $I_1(t)$: Non deteriorated inventory at $t = 0$ to t_1
- $I_2(t)$: Non deteriorated inventory at $t = t_1$ to T_1
- $I_3(t)$: Undecayed inventory of buffer stock at $t = 0$ to T_1
- δ_1: Degraded inventory at $t = 0$ to t_1
- δ_2: Degraded inventory at $t = t_1$ to T_1
- δ_3: Degraded inventory of buffer stock at $t = 0$ to T_1
- $TP(Q)$: Profit Function per cycle

4. Mathematical Model

This section frames a mathematical model with linear demand rate which may changes because of various reasons. Products with a limited shelf life are a good fit for this model.

Since the model considers a buffer stock, production begins at a rate of ω and a quantity of Q at the beginning when t = 0. Throughout the production cycle, it is anticipated that the production rate will remain constant. The suggested inventory model is displayed in **Fig.1** below.

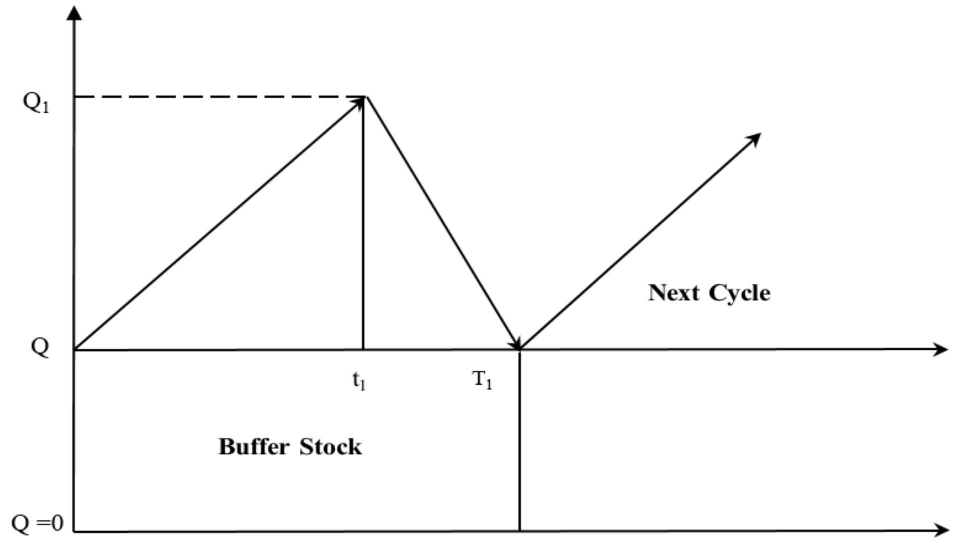

Fig. 1. Graphical representation of the proposed model

At the beginning of each cycle, the inventory increases from $t = 0$ to t_1 at a rate of $\omega - \{\alpha + \beta I(t) + \varphi I(t)\}$.
In this case, the market demand rate is $\alpha + \beta I(t)$, and the deterioration of $\text{I}I(t)$ inventories at any time t is represented by $\varphi I(t)$, where φ denotes the decline in unit inventory during the specified time period. Thus the differential equation can be formed as:

$$I(t + dt) = I(t)dt + \{\omega - \alpha - \beta I(t)\}dt - \varphi I(t)dt$$

$$\lim_{dt\to 0} \frac{I(t+dt)-I(t)}{dt} = \{\omega - \alpha - \beta I(t) - \varphi I(t)$$

$$\frac{d}{dt}I(t) + \varphi I(t) = \omega - \alpha - \beta I(t)$$

In general, we can describe the solution of the differential equation as: $I(t) = \frac{\omega-\alpha}{\varphi+\beta} + A.\exp\{-(\varphi+\beta)t\}$

Using the boundary condition, which states that at $t = 0, I(t) = Q$, we get

$$A = Q - \frac{\omega-\alpha}{\varphi+\beta}$$

Therefore, $I(t) = \frac{\omega-\alpha}{\varphi+\beta} + \left(Q - \frac{\omega-\alpha}{\varphi+\beta}\right).\exp\{-(\varphi+\beta)t\}$ (1)

With respect to the other boundary condition, that is, at $t = t_1, I(t) = Q_1$ and considering up to the first degree of φ we obtain the following equation:

$$Q_1 = \frac{\omega-\alpha}{\varphi+\beta} + \left(Q - \frac{\omega-\alpha}{\varphi+\beta}\right).\exp\{-(\varphi+\beta)t_1\}$$

$$= \frac{\omega-\alpha}{\varphi+\beta} + (Q - \frac{\omega-\alpha}{\varphi+\beta})\{1 - (\varphi+\beta)t_1\}$$

$$= Q + Q(\varphi+\beta)t_1 - (\omega-\alpha)t_1$$

$$Q_1 = Q + \{\omega - \alpha - Q(\varphi+\beta)\}t_1 \tag{2}$$

For simplicity, the undecayed inventory during $t = 0$ to t_1 can be calculated using equation (1) and taking into account up to the second degree of φ.

$$I_1 = \int_0^{t_1} I(t)dt = \int_0^{t_1} \left\{\frac{\omega-\alpha}{\varphi+\beta} + \left(\text{Q} - \frac{\omega-\alpha}{\varphi+\beta}\right).\exp\{-(\varphi+\beta)t\}\right\}dt$$

$$= [\frac{\omega-\alpha}{\varphi+\beta}t + (\text{Q} - \frac{\omega-\alpha}{\varphi+\beta})\ \frac{\exp\{-(\varphi+\beta)t\}}{-(\varphi+\beta)}]_0^{t_1}$$

$$= \frac{\omega-\alpha}{\varphi+\beta} t_1 - (\mathrm{Q} - \frac{\omega-\alpha}{\varphi+\beta})(\frac{1}{\varphi+\beta})\{-(\varphi+\beta)t_1 + \frac{1}{2}(\varphi+\beta)^2 {t_1}^2\}$$

$$I_1 = Qt_1 - \frac{1}{2}Q(\varphi+\beta){t_1}^2 + \frac{1}{2}(\omega-\alpha){t_1}^2 \quad (3)$$

During the same period, the condition of the items was deteriorating due to the decay of the items.as follows:

$$\delta_1 = \int_0^{t_1} \varphi\, I(t)dt = Q\varphi t_1 - \frac{1}{2}Q(\varphi+\beta)\varphi {t_1}^2 + \frac{1}{2}(\omega-\alpha)\varphi {t_1}^2 \quad (4)$$

On the other hand, since no production takes place beyond time t_1, the inventory decreases at a rate of $\alpha + \beta I(t) + \varphi I(t)$ from $t = t_1$ to T_1. The market's demand and the products' slow degradation cause the inventory to decline. Based in this situation, we can derive the differential equation that follows:

$$\frac{d}{dt} I(t) + \varphi I(t) = -\{\alpha + \beta I(t)\}$$

The overall solution to the differential equation is expressed as : $I(t) = -\frac{\alpha}{\varphi+\beta} + B.\exp\{-(\varphi+\beta)t\}$

By applying the boundary condtion, i.e. at $t = T_1, I(t) = Q$

By solving we get $B = \left(Q + \frac{\alpha}{\varphi+\beta}\right).\exp\{-(\varphi+\beta)T_1\}$

Therefore , $I(t) = -\frac{\alpha}{\varphi+\beta} + (Q + \frac{\alpha}{\varphi+\beta})e^{(\varphi+\beta)(T_1-t)}$ (5)

By substituting a different boundary condition, such as, at $t = t_1, I(t) = Q_1$and and considering the first order of φ, we can derive the following equation

$$Q_1 = -\frac{\alpha}{\varphi+\beta} + \left(Q + \frac{\alpha}{\varphi+\beta}\right).\exp\{(\varphi+\beta)(T_1 - t_1)\}$$

$$= -\frac{\alpha}{\varphi+\beta} + (Q + \frac{\alpha}{\varphi+\beta})\{1 + (\varphi+\beta)(T_1 - t_1)\}$$

$$Q_1 = Q + \{\alpha + Q(\varphi+\beta)\}(T_1 - t_1) \quad (6)$$

The un-decayed inventory by $t = t_1$ to T_1 is now obtained by applying equation (5) and taking into account up to the first degree of φ as follows:

$$I_2 = \int_{t_1}^{T_1} I(t)dt = \int_{t_1}^{T_1}\{-\frac{\alpha}{\varphi+\beta} + \left(\mathrm{Q} + \frac{\alpha}{\varphi+\beta}\right).\exp\{(\varphi+\beta)(T_1 - t)\}\}dt$$

$$= -\frac{\alpha}{\varphi+\beta}(T_1 - t_1) - (\mathrm{Q} + \frac{\alpha}{\varphi+\beta})(T_1 - t_1)$$

$$I_2 = Q(T_1 - t_1) \quad (7)$$

Taking into account the deterioration of the items, we determine the decaying items as follows:

$$\delta_2 = \int_{t_1}^{T_1} \varphi I(t)dt$$

$$\delta_2 = Q\varphi(T_1 - t_1) \quad (8)$$

4.1. Buffer Stock Dynamics

The rate of change of inventory due to decay can be designed as a linear decay process, which means that the change in the inventory level at any instant t is proportional to the current inventory level.

Therefore the differential equation can be modeled as:

$$\frac{d}{dt} I(t) = -\psi I(t)$$

$\frac{d}{I(t)} I(t) = -\psi dt$

$ln|I(t)| = -\psi t + C$

$I(t) = \exp\{C - \psi t\} = \exp(C). \exp(-\psi t)$

From the boundary condition we get, at $t = 0, I(t) = Q$

Thus, $Q = \exp(C)$

Therefore, $I(t) = Q. \exp(-\psi t)$ (9)

We now get the un-decayed inventory in buffer stock as follows, taking into account up to the first degree of ,

$I_3 = \int_0^{T_1} I(t)dt \; = Q \int_0^{T_1} \exp(-\psi t)dt$

$= Q[\frac{\exp(-\psi t)}{-\psi}]_0^{T_1} = \frac{Q}{-\psi}[\exp(-\psi T_1) - 1]$

$= \frac{Q}{-\psi}(1 - \psi T_1 - 1)$

$I_3 = QT_1$ (10)

We determine the degrading items at time $t_1 = 0$ to T_1 as follows, taking into account the decay of the items in buffer stock:

$\delta_3 = \int_0^{T_1} \psi I(t)dt$

$\delta_3 = \psi Q T_1$ (11)

From equation (2) and (6), we obtain

$T_1 = \frac{\omega}{\alpha + Q(\varphi + \beta)} t_1$ (12)

Total Cost Function

The following form describes the cost function,

$TC = \frac{1}{T_1}\{C_0 + h_1(I_1 + I_2) + h_2(\delta_1 + \delta_2) + h_3\delta_3 + \gamma P I_1\}$

by using equation (3), (4), (7) and (8)

$$TC = \frac{1}{T_1}[\gamma P Q t_1 + \frac{1}{2}\{(\omega - \alpha)(h_1 + \gamma P + h_2\varphi) - Q(\varphi + \beta)$$
$$(h_1 + \gamma P + h_2\varphi)\}{t_1}^2 + C_0] + Q(h_1 + h_2\varphi + h_3\psi) \quad (13)$$

Total Revenue Function

Total revenue function is as follows:

$TR = \frac{1}{T_1}[P(I_1 + I_2)]$

Using equation (3) and (7)

$TR = \frac{1}{T_1}\left[\frac{1}{2}P\{(\omega - \alpha) - Q(\varphi + \beta)\}t_1^2\right]$ (14)

Total Profit Function

Total profit function can be obtained subtracting equation (13) from (14)

$$TP = TR - TC$$

$$= \frac{1}{T_1}\left[\frac{1}{2}P\{(\omega-\alpha)-Q(\varphi+\beta)\}{t_1}^2 - \gamma Q t_1 - \frac{1}{2}\{(\omega-\alpha)(h_1+\gamma P+h_2\varphi)-Q(\varphi+\beta)(h_1+\gamma P+h_2\varphi)\}t_1^2 - C_0\}\right] + PQ - Q(h_1+h_2\varphi+h_3\psi)$$

$$TP = \frac{-\gamma PQ\{\alpha+Q(\varphi+\beta)\}}{\omega} + \frac{\alpha+Q(\varphi+\beta)}{2\omega}\{(\omega-\alpha)-Q(\varphi+\beta)\}(P-h_1-\gamma P-h_2\varphi)t_1 - \frac{C_0\{\alpha+Q(\varphi+\beta)\}}{\omega t_1} + Q(P-h_1-h_2\varphi-h_3\psi) \tag{15}$$

Now differentiating (15) partially with respect to t_1 and equate to zero, we get

$$\frac{\partial}{\partial t_1}(TP) = \frac{\alpha+Q(\varphi+\beta)}{2\omega}\{(\omega-\alpha)-Q(\varphi+\beta)\}(P-h_1-\gamma P-h_2\varphi) + \frac{C_0\{\alpha+Q(\varphi+\beta)\}}{\omega {t_1}^2} = 0 \tag{16}$$

It is obvious from (16) that the second partial derivatives is less than zero. hence we get,

$$t_1^* = \sqrt{\frac{2C_0}{\{\alpha-\omega+Q(\varphi+\beta)\}(P-h_1-\gamma P-h_2\varphi)}} \tag{17}$$

which ensure the maximum profit for the system.

Substituting the value of t_1^* in equation (12) we obtain T_1^* as:

$$T_1^* = \frac{\omega}{\alpha+Q(\varphi+\beta)}\sqrt{\frac{2C_0}{\{\alpha-\omega+Q(\varphi+\beta)\}(P-\gamma P-h_1-h_2\varphi)}} \tag{18}$$

By using the equation (17), we can have the total profit function in terms of the quantity Q as follows:

$$TP(Q) = \frac{-\gamma PQ\{\alpha+Q(\varphi+\beta)\}}{\omega} + \frac{\alpha+Q(\varphi+\beta)}{2\omega}\{(\omega-\alpha)-Q(\varphi+\beta)\}(P-h_1-\gamma P-h_2\varphi)$$

$$\frac{\sqrt{2C_0}}{\sqrt{\{\alpha-\omega+Q(\varphi+\beta)\}(P-\gamma P-h_1-h_2\varphi)}} + Q(P-h_1-h_2\varphi-h_3\psi)$$

$$-\frac{C_0\{\alpha+Q(\varphi+\beta)\}}{\omega}\frac{\sqrt{\{\alpha-\omega+Q(\varphi+\beta)\}(P-\gamma P-h_1-h_2\varphi)}}{\sqrt{2C_0}}$$

$$TP(Q) = -\frac{1}{\omega}\left[\gamma\alpha PQ + \gamma P(\varphi+\beta)Q^2 - \omega(P-h_1-h_2\varphi-h_3\psi)Q + \{\alpha+Q(\varphi+\beta)\}\sqrt{2C_0\{\alpha-\omega+Q(\varphi+\beta)\}(P-\gamma P-h_1-h_2\varphi)}\right] \tag{19}$$

By differentiating (19) with respect to Q and equate to zero, we get

$$\frac{d}{dQ}\{TP(Q)\} = \alpha\gamma P + 2\gamma(\varphi+\beta)Q - \omega(P-h_1-h_2\varphi-h_3\psi) + (\varphi+\beta)\sqrt{2C_0\{\alpha-\omega+Q(\varphi+\beta)\}(P-\gamma P-h_1-h_2\varphi)} + +C_0(P-\gamma P-h_1-h_2\varphi)(\varphi+\beta)\frac{\alpha+Q(\varphi+\beta)}{\sqrt{2C_0\{\alpha-\omega+Q(\varphi+\beta)\}(P-\gamma P-h_1-h_2\varphi)}} = 0 \tag{20}$$

$\Rightarrow$ $8(\gamma P)^2(\varphi+\beta)^3Q^3+(\varphi+\beta)^2[8(\gamma P)^2(\alpha-\omega)-8\gamma P\{\omega(P-h_1-h_2\varphi-h_3\psi)-\alpha\gamma P\}-9C_0(P-\gamma P-h_1-h_2\varphi)(\varphi+\beta)^2]Q^2+(\varphi+\beta)[2\{\omega(P-h_1-h_2\varphi-h_3\psi)-\alpha\gamma P\}^2-8\gamma P(\alpha-\omega)\{\omega(P-h_1-h_2\varphi-h_3\psi)-\alpha\gamma P\}-6C_0(P-\gamma P-h_1-h_2\varphi)(\varphi+\beta)^2(3\alpha-2\omega)]Q+2(\alpha-\omega)\{\omega(P-h_1-h_2\varphi-h_3\psi)-\alpha\gamma P\}^2-C_0(P-\gamma P-h_1-h_2\varphi)$

$$(\varphi+\beta)^2(3\alpha-2\omega)^2=0 \qquad (21)$$

Equation (21) can be written as cubic form:

$$aQ^3+bQ^2+cQ+d=0 \qquad (22)$$

Where, $a=8(\gamma P)^2(\varphi+\beta)^3$;

$b=(\varphi+\beta)^2[8(\gamma P)^2(\alpha-\omega)-8\gamma P\{\omega(P-h_1-h_2\varphi-h_3\psi)-\alpha\gamma P\}$
$-9C_0(P-\gamma P-h_1-h_2\varphi)(\varphi+\beta)^2]$;

$c=(\varphi+\beta)[2\{\omega(P-h_1-h_2\varphi-h_3\psi)-\alpha\gamma P\}^2-8\gamma P(\alpha-\omega)\{\omega(P-h_1-h_2\varphi-h_3\psi)-\alpha\gamma P\}-6C_0(P-\gamma P-h_1-h_2\varphi)(\varphi+\beta)^2(3\alpha-2\omega)]$;

$d=2(\alpha-\omega)\{\omega(P-h_1-h_2\varphi-h_3\psi)-\alpha\gamma P\}^2$
$-C_0(P-\gamma P-h_1-h_2\varphi)(\varphi+\beta)^2(3\alpha-2\omega)^2$;

Solving equation (22), we obtain Q as follows:

$$Q=\sqrt[3]{-(k/2)+\sqrt{(k/2)^2+(j/3)^3}}+\sqrt[3]{-(k/2)-\sqrt{(k/2)^2+(j/3)^3}}-\frac{b}{3a} \qquad (23)$$

Where $j=\frac{3ac-b^2}{3a^2}$ and $k=\frac{2b^3-9abc+27a^2d}{27a^3}$

Differentiating equation (20) and after some simplification, we can have

$$\frac{d^2}{dQ^2}\{TP(Q)\}=-\frac{1}{\omega}\left[2\gamma(\varphi+\beta)+\frac{C_0(\varphi+\beta)^2(P-\gamma-h_1-h_2\varphi)}{\sqrt{2C_0\{\alpha-\omega+Q(\varphi+\beta)\}(P-\gamma-h_1-h_2\varphi)}}+\frac{\{C_0(\varphi+\beta)(P-\gamma-h_1-h_2\varphi)\}^2\{\alpha-2\omega+Q(\varphi+\beta)\}}{[2C_0\{\alpha-\omega+Q(\varphi+\beta)\}(P-\gamma-h_1-h_2\varphi)]^{\frac{3}{2}}}\right] \qquad (24)$$

The value of equation (24) will be less than zero iff

$$\frac{C_0(\varphi+\beta)^2(P-\gamma-h_1-h_2\varphi)}{\sqrt{2C_0\{\alpha-\omega+Q(\varphi+\beta)\}(P-\gamma-h_1-h_2\varphi)}}+\frac{\{C_0(\varphi+\beta)(P-\gamma-h_1-h_2\varphi)\}^2\{\alpha-2\omega+Q(\varphi+\beta)\}}{[2C_0\{\alpha-\omega+Q(\varphi+\beta)\}(P-\gamma-h_1-h_2\varphi)]^{\frac{3}{2}}}<(1/\omega)[2\gamma(\varphi+\beta)$$

Theorem-01: the inventory system will be stable iff

$$\frac{C_0(\varphi+\beta)^2(P-\gamma-h_1-h_2\varphi)}{\sqrt{2C_0\{\alpha-\omega+Q(\varphi+\beta)\}(P-\gamma-h_1-h_2\varphi)}}+\frac{\{C_0(\varphi+\beta)(P-\gamma-h_1-h_2\varphi)\}^2\{\alpha-2\omega+Q(\varphi+\beta)\}}{[2C_0\{\alpha-\omega+Q(\varphi+\beta)\}(P-\gamma-h_1-h_2\varphi)]^{\frac{3}{2}}}<(1/\omega)[2\gamma(\varphi+\beta)$$

Proof: For maximizing the profit function, equation (24) must be less than zero. Hence

$$-\frac{1}{\omega}\left[2\gamma(\varphi+\beta)+\frac{C_0(\varphi+\beta)^2(P-\gamma-h_1-h_2\varphi)}{\sqrt{2C_0\{\alpha-\omega+Q(\varphi+\beta)\}(P-\gamma-h_1-h_2\varphi)}}+\frac{\{C_0(\varphi+\beta)(P-\gamma-h_1-h_2\varphi)\}^2\{\alpha-2\omega+Q(\varphi+\beta)\}}{[2C_0\{\alpha-\omega+Q(\varphi+\beta)\}(P-\gamma-h_1-h_2\varphi)]^{\frac{3}{2}}}\right]<0$$

$$\Rightarrow \frac{C_0(\varphi+\beta)^2(P-\gamma-h_1-h_2\varphi)}{\sqrt{2C_0\{\alpha-\omega+Q(\varphi+\beta)\}(P-\gamma-h_1-h_2\varphi)}}+\frac{\{C_0(\varphi+\beta)(P-\gamma-h_1-h_2\varphi)\}^2\{\alpha-2\omega+Q(\varphi+\beta)\}}{[2C_0\{\alpha-\omega+Q(\varphi+\beta)\}(P-\gamma-h_1-h_2\varphi)]^{\frac{3}{2}}} < (1/\omega)[2\gamma(\varphi+\beta)$$

5. Computational Analysis

In this section, a numerical example is considered to illustrate this model. We derived numerical illustrations by taking the values of the parameters involve in this model as follows:

$C_0 = 70, \omega = 60, \psi = 0.1, \ \alpha = 7, \beta = 0.5, \varphi = 0.01, \ \gamma = 0.1, P = 30, h_1 = 1.5,$
$h_2 = 1.75, \ h_3 = 0.5$

The optimum order interval $T_1{}^* = 0.443$, optimum time $t_1{}^* = 0.362$, maximum order quantity $Q^* = 355.213$ and total average optimum profit $TP^* = 4637.31$ are the outcomes obtained by using equations (17), (18), and (23) respectively.

Details are shown in the **Table 1** and **Fig. 2**

Q	$t_1{}^*$	$T_1{}^*$	TP^*
353.713	0.359	0.441	4637.23
354.213	0.360	0.441	4637.27
354.713	0.361	0.442	4637.3
355.213	**0.362**	**0.443**	**4637.31**
355.713	0.363	0.444	4637.3
356.213	0.365	0.444	4637.27
356.713	0.366	0.445	4637.23

Table 1. Impact of quantity Q, t_1 and T_1 on TP

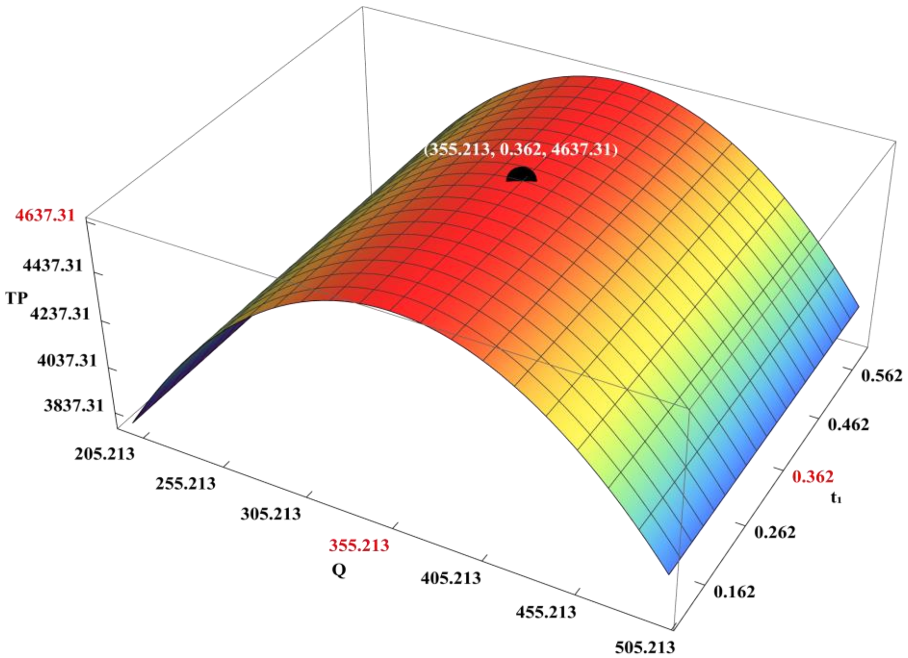

Fig. 2. 3D view of Quantity against Total Profit and t_1

From table 1 and fig. 2, it is observed that as Q increases, the total profit generally rises because higher production levels generate more revenue. This is particularly true when the selling price and other parameters like demand are relatively high. Beyond a certain point, the increase in Q may lead to falling returns on profit. This happens because of increasing costs, such as holding costs, decay rates (and ψ), and other variable costs that escalate with higher production levels. As these costs grow, they begin to offset the additional revenue gained from producing more.

6. Sensitivity Analysis

This section will now show how even slight changes to the parameters $Q, \omega, \alpha, \beta, \varphi, \psi, h_1, h_2,$ and h_3 affect the inventory system or the solution.

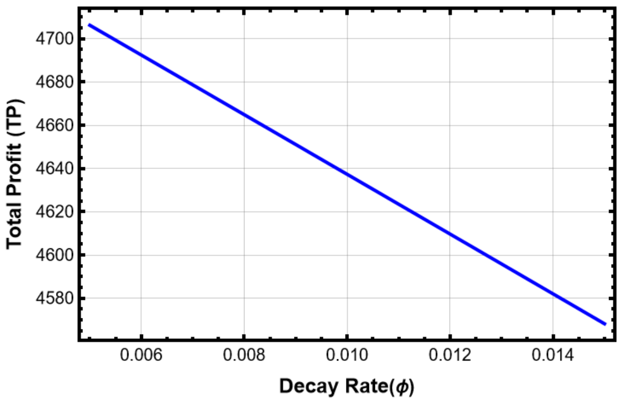

Fig. 3(a). Decay Rate verses Total Profit

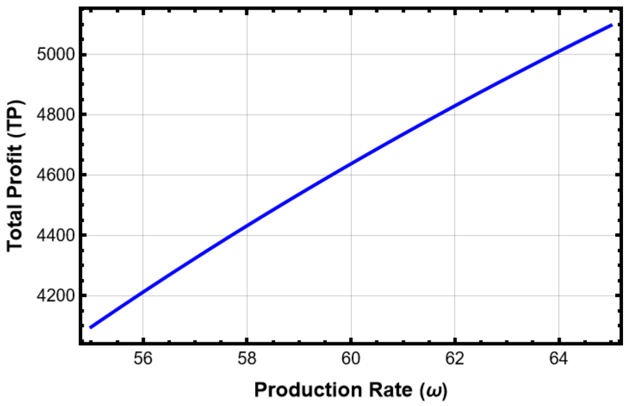

Fig. 3(b). Production Rate verses Total Profit

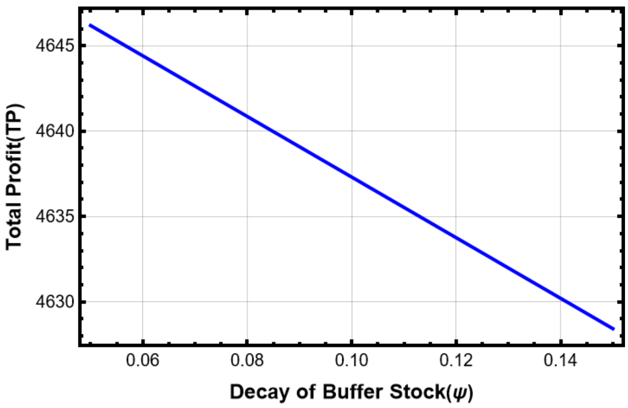

Fig. 3(c). Decay Rate verses Total Profit

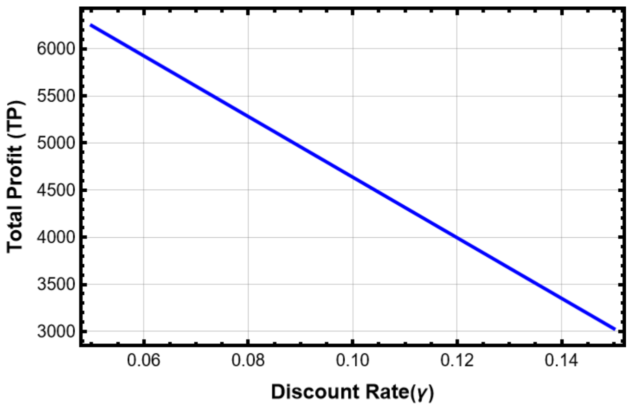

Fig. 3(d). Discount Rate verses Total Profit

Fig. 3(a) indicates that ω (production rate) increases from lower values, the total profit tends to rise. This is because higher production rates allow the business to meet higher demand or produce more efficiently, which boosts revenue. A higher production rate also spreads out fixed costs like setup costs, making each unit produced cheaper which might be true for an optimat amount Q^*. It is observed from the fig.3 (b) that the total profit TP decreases significantly as φ increases. This indicates that a higher decay rate leads to greater losses, reducing overall profitability. It is found in the fig. 3(c), as γ increases, total profit TP gradually decreases. This is because γ representing a form of decay or cost factor, directly reduces the profitability as it increases. A higher γ reduces the remaining profit after accounting for the cost influences in the system. The decrease in TP is consistent with the fact that γ is subtracted in both the linear and quadratic components of the profit equation, which significantly affects the overall outcome. The fig. 03(d) shows the impact of the parameter ψ on total profit (TP), while keeping other variables constant. As ψ increases, it directly influences the term involving the holding cost component $(h_3\psi)$ in the total profit equation. This means that as ψ rises, it increases the holding cost, which reduces the overall profit. Therefore, the Total Profit (TP) tends to decrease as ψ increases. This highlights the sensitivity of profit to changes in holding costs.

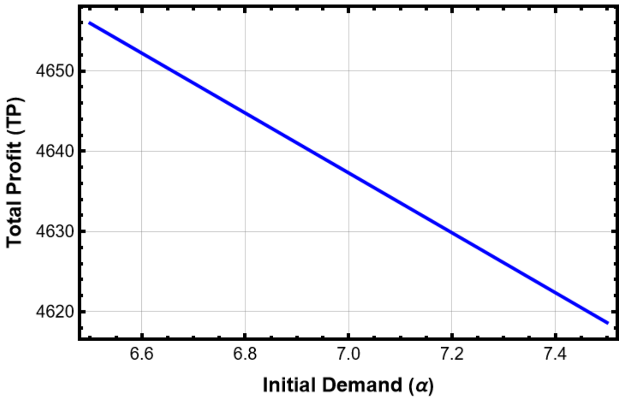

Fig. 4(a). Initial Demand verses Total Profit

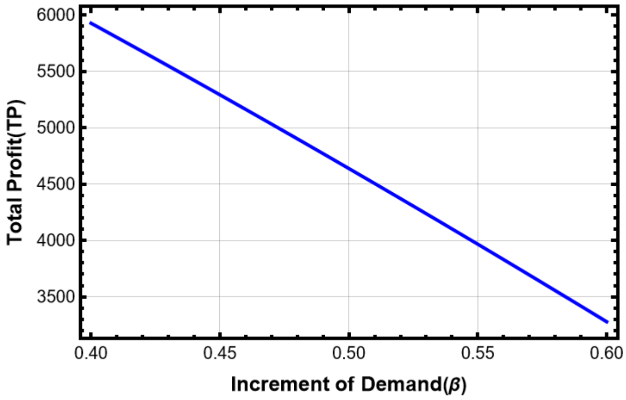

Fig. 4(b). Increment of Demand verses Total Profi

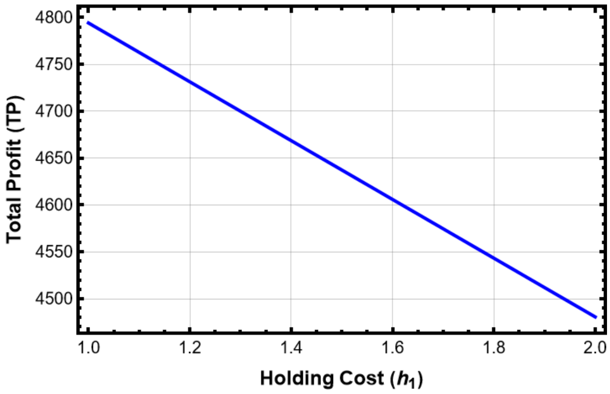

Fig. 4(c). Initial Demand verses Total Profit

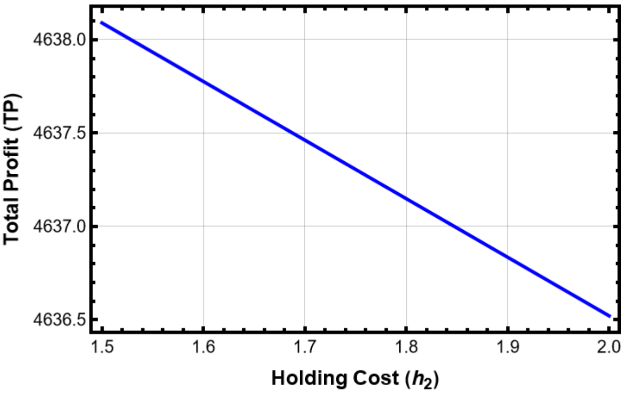

Fig. 4(d). Initial Demand verses Total Profit

Fig. 4(a) informs that when α is lower (e.g., 0.25 or 0.5), the Total Profit tends to be higher. This is because a smaller α reduces the negative impact of the term related to a in the profit equation, allowing other positive terms, such as the revenue component, to dominate. It is found from fig. 4(b) that The impact of the parameter β on Total Profit (TP) is observed through its influence on the cost structure related to inventory holding and decay rates. As β increases, representing a higher rate of decay or deterioration, the total profit TP tends to decrease. This happens because a higher value of β increases the decay-related costs, thus reducing the profit margin. The relationship is logical because a higher deterioration rate leads to more losses in stock, requiring either more replenishment or higher holding costs, both of which reduce profitability.

In contrast, when b is lower, the decay rate is minimal, which results in lower associated costs, and consequently, higher total profit TP. This shows that controlling the decay rate is critical for maintaining profitability, especially in systems where the product deterioration rate plays a significant role in cost calculations.

Fig. 4(c) demonstrates that as holding cost h_1 for undecayed inventory increase, it reduces the net profit, as more resources are spent on storage and management. h_1's impact on profit also indirectly affects pricing and discounting strategies. If h_1 is high, companies may need to accelerate sales (potentially through discounts) to avoid high holding costs, impacting revenue. Lower h_1 allows more flexibility in timing inventory turnover without eroding profit.

We observed from fig. 4(d) that an increase in h_2 raises the holding cost for deteriorated items, thus increasing the total cost associated with carrying decaying inventory over time.. As h_2 rises, the resulting increase in total costs diminishes the overall profit. Since h_2 directly influences the portion of the inventory that deteriorates, higher h_2 values result in a sharper decline in profit margins, especially in scenarios with high decay rates (φ). Conversely, a lower h_2 mitigates this effect, allowing more of the revenue generated from sales to contribute positively to the total profit.

7. Conclusion:

The model outlined in this study offers a comprehensive approach to managing a perishable production inventory system, particularly for items with level-dependent demand. Through the integration of a buffer stock that restocks at the conclusion of each cycle and by maintaining an elevated production rate (ω) beyond demand, the model guarantees an ongoing supply while avoiding deficiencies. The assumptions pertain to the essential dynamics of inventory degradation, demand-related exhaustion, and decline, which are crucial for handling perishable objects. Key features of the model include the incorporation of various holding costs (h_1, h_2, h_3) for non-decayed and degraded inventory, along with a discount rate (γ) in the early inventory phases, supply a comprehensive cost framework that enables enhanced decision-making. This allows companies to modify carrying expenses based regarding stock status, enhancing profitability. Moreover, establishing distinct stages for inventory $I_1$and I_2 for items that are undeteriorated and δ_1,δ_2 for those that are deteriorated guarantees precise cost assessments throughout the inventory lifecycle.
The model's use of infinite replenishment at the end of each cycle, combined with a fixed setup cost (C_0), provides stability and allows for flexible cycle planning, enhancing responsiveness to demand fluctuations. Overall, this framework is a practical tool for industries dealing with perishable goods, ensuring cost-effective inventory levels, minimizing wastage through deterioration, and optimizing profits through strategic discounting.

Declarations:

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Authors' Contributions

This work was carried out in collaboration between both authors. Author MEI conceptualized the study and critically reviewed the manuscript, providing guidance and feedback throughout the research process. Author MAAM contributed in model formulation, mathematical derivations, numerical analysis, optimization and wrote the first draft of the manuscript. Both authors read and approved the final manuscript.

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