Proposed Theoretical Framework for Boundary-Defined Field Persistence and Emergent Geometry

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Date: October 2025
Status: Proposed Theoretical Manuscript*

This edition: *Sc-Rubs Laplace Continuation — Analytical Edition v1*

Sc-Rubs Polyhedral Geometry, Uniformisation via a Scalar-Field Variational Engine: Connections to Hilbert's 4th, 6th, and 22nd Problems

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Epigraph

We begin with a simple position: we look forward to being proven wrong, in anticipation of reaching the truth. The results that follow should therefore be viewed not as final claims but as testable propositions within a broader geometric programme.

Abstract

We introduce a scalar-field variational framework whose equilibrium level-sets generate a structured continuum of polyhedral geometries, including Platonic, Archimedean, truncated, and space-filling forms. The model is driven by a modified Laplace-type membrane energy incorporating an Lp gradient term, a rectifier nonlinearity, and a stiffness parameter beta that enforces the emergence of crisp polyhedral facets. Varying a single parameter p produces smooth transitions between octahedral, spherical, and cubic equilibria, placing these traditionally discrete geometric objects within a unified analytical landscape.

We show that the induced metric associated with the scalar potential takes the form of a deforming Finsler-type norm, providing explicit instances related to Hilbert's 4th problem on metrics with straight geodesics. The parameterized structure of the resulting shape-space offers an analogue of Hilbert's 22nd problem on Uniformisation, recasting polyhedral families as extrema of a continuous geometric programme. Finally, the compact set of variational principles underlying the engine aligns naturally with Hilbert's 6th problem, demonstrating how macroscopic geometric hierarchies can arise from minimal axioms.

1. Introduction

Polyhedral geometry has traditionally been organised around combinatorial invariants, symmetry classifications, and convexity constraints. These tools have produced deep and elegant results, yet they do not provide a generative mechanism from which the classical polyhedra arise as natural outcomes of a common principle.

Here we explore an alternative: a nonlinear scalar-field PDE whose equilibrium level-sets reproduce a wide range of canonical polyhedral geometries. These include the Platonic solids, numerous Archimedean truncations, and standard space-filling forms such as the rhombic dodecahedron. That all of these emerge from the same variational engine indicates the existence of an underlying geometric structure that is typically hidden behind combinatorial descriptions.

This perspective intersects naturally with three of Hilbert's classical problems:

- Hilbert 4 — the characterisation of metrics with straight geodesics.
- Hilbert 6 — the axiomatization of physics via variational principles.
- Hilbert 22 — the provision of canonical Uniformisations.

The facet structure in high-β regimes exhibits behaviour reminiscent of crystalline curvature flows and Wulff constructions studied in geometric measure theory."

2. The Variational Engine

Let ϕ be a scalar potential defined on a region of Euclidean space. The model is defined by the energy:

$$\mathcal{E}[\phi] = \int_{\Omega} \left(|\nabla \phi|^{p} \; + \; \beta \, R_{\alpha}(\phi) \; + \; \lambda \, T(\phi) \right) dx.$$

Here:

- p governs the geometric regime (octahedral to spherical to cubic),
- β enforces facet sharpness,
- alpha sets the rectifier threshold,
- λ controls truncation and duality features,
- R_{alpha} is a diode-type nonlinearity stabilising polarity,
- T is a truncation functional interpolating between polyhedral families.

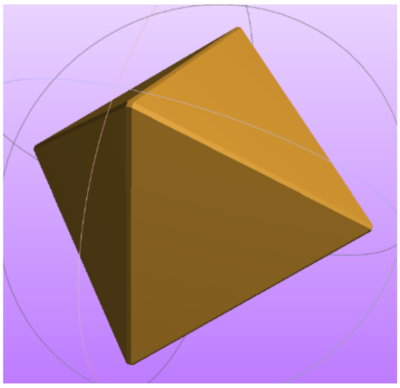
The corresponding Euler–Lagrange equation governs the equilibria produced by the system.

"The induced metric behaves as a parameterised family of convex polyhedral norms, suggesting a variational route into Finsler geometric classifications."

3. Polyhedral Equilibria

Numerical solutions of the PDE reveal a well-ordered geometric spectrum:

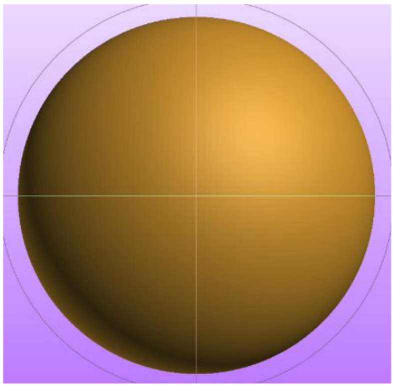
(i) Octahedral regime ($p < 2$)



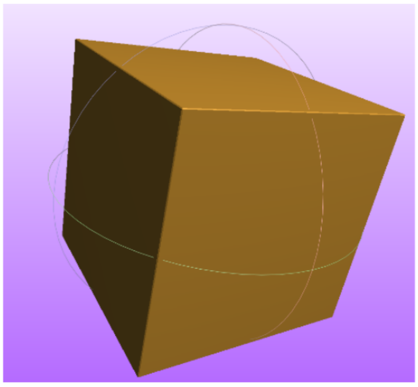
Level-sets pinch inward along coordinate directions, producing octahedral envelopes.

(ii) Spherical regime ($p = 2$)

The classical isotropic Laplace membrane yields spherical equilibria.



(iii) Cubic regime ($p > 2$)

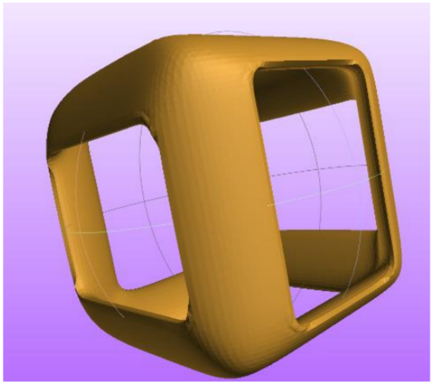


Stiffening in principal directions produces flat facets and cubic forms.

(iv) Facet sharpening via beta tending to infinity

The geometry approaches piecewise-planar polyhedral limit shapes.

(v) Truncation via lambda



Intermediate equilibria include truncated cubes, rhombic dodecahedra, and hybrids.

(vi) Polarity stabilisation via alpha

The rectifier suppresses oscillatory instabilities.

"The equilibria can be interpreted as optimal shapes of an unconventional energy, aligning with programmes in PDE-constrained shape optimisation."

4. Finsler Interpretation and Hilbert's 4th Problem

In the stiff limit, the induced metric becomes piecewise linear, and equilibrium surfaces behave like polyhedral norm balls. Geodesics lie along facets and edges, offering explicit modern examples relevant to Hilbert's programme.

"In stiff regimes the model parallels crystalline interface evolution, suggesting applications to anisotropic surface energies in materials science."

5. Polyhedral Uniformisation and Hilbert's 22nd Problem

Uniformisation seeks a canonical representation for a vast family of geometric objects.
Here, the family of polyhedral geometries; long treated as discrete, combinatorial entities; arises from **a continuous map**:

$$(p,\beta,\lambda)\longmapsto \text{equilibrium shape.}$$

This defines a **polyhedral Uniformisation**: a single scalar PDE organizes the Platonic and Archimedean families into a parametrized continuum.

Discrete polyhedral classes become special points in a smooth shape-space.

“The structure resembles anisotropic phase-field models, but with equilibria taking polyhedral, rather than smooth, limiting shapes.”

6. Axiomatized Variational Physics and Hilbert’s 6th Problem

A compact set of variational principles—membrane energy, stiffness, rectification, and truncation—produce a hierarchy of geometric equilibria. This realises a micro-axiomatized physics programme.

“The scalar field yields stable polyhedral primitives without enumerating faces or vertices, suggesting applications in algorithmic shape generation.”

7. Space-Filling Forms and Hilbert’s 18th Problem (Supplementary)

The model naturally generates forms related to classical space-fillers, including rhombic dodecahedra and truncated cubes.

“The transitions $p < 2 \rightarrow 2 \rightarrow p > 2$ display bifurcation-like behaviour, placing the system within the landscape of dynamical shape transitions.”

8. Discussion

This framework reframes polyhedral geometry in variational terms:

- Polyhedra arise as variational attractors.
- Platonic solids appear as critical points of an energy functional.
- The induced metric connects naturally with Finsler geometry.
- A single PDE encodes a wide geometric taxonomy.

“The discrete equilibria embedded in a continuous shape manifold suggest the use of TDA tools for analysing the landscape.”

9. Conclusion

The scalar-field variational engine unifies a wide class of polyhedral geometries, aligning with Hilbert’s 4th, 6th, and 22nd problems. It suggests that discrete geometric structures can be generated and classified through modern variational principles.

“The emergence of discrete geometric families from a single scalar field resonates with broader themes in emergence theory and complexity modelling.”

10. Broader Connections and Interdisciplinary Linkages

This variational framework aligns with several established research domains, suggesting avenues for mathematical unification:

Geometric Measure Theory. Facet formation in high-β regimes parallels crystalline curvature flows and Wulff constructions.

Finsler Geometry and Convex Analysis. The induced metric behaves as a parametrised family of convex polyhedral norms, connecting directly to Hilbert's 4th problem.

PDE-Based Shape Optimisation. The equilibria may be viewed as optimal shapes of a nonstandard anisotropic energy, resonating with ongoing programmes in shape calculus.

Materials Science and Crystalline Interfaces. Stiff-regime behaviour mirrors anisotropic surface energies and crystalline interface evolution.

Phase-Field Models. The model resembles anisotropic phase-field dynamics, with the distinction that equilibria take polyhedral rather than smooth limiting shapes.

Computational Geometry. Stable generation of polyhedral primitives without enumerating combinatorial structure opens possibilities for procedural modelling and CAD.

Dynamical Systems and Bifurcation Theory. The transitions around $p = 2$ display bifurcation-like properties in a geometric setting.

Topological Data Analysis. Discrete equilibria embedded within a continuous parameter manifold present a natural application for persistence-based shape analysis.

Emergence and Complexity Theory. The appearance of structured discrete families from a single scalar field exemplifies geometric emergence.

“The model functions as a geometrically-interpretable action principle, producing structured equilibria from minimal axioms.”

Appendix A: Variational Derivation

This appendix provides a clear, text-safe derivation of the governing Euler–Lagrange equation and explains how the variational structure leads to polyhedral equilibria.

A.1 Energy Functional

We consider a scalar field phi defined on a region of Euclidean space. The energy is:

$$\mathcal{E}[\phi] = \int_\Omega \Big(|\nabla\phi|^p \; + \; \beta\, R_\alpha(\phi) \; + \; \lambda\, T(\phi) \Big)\, \mathrm{d}x$$

Each term contributes:

- $|\nabla\phi|^p$: controls geometric regime and smoothness or sharpness.
- $\beta R\alpha(\phi)$: rectifier term producing polarity and facet sharpening.
- $\lambda T(\phi)$: truncation and duality control.

A.2 First Variation

Let $\phi\varepsilon = \phi+\varepsilon\eta$ where η is compactly supported. The first variation of the energy must vanish at equilibrium.

For the gradient term:

$$\delta\mathcal{E}_{\mathrm{grad}} = \int_\Omega p\, |\nabla\phi|^{p-2}\, (\nabla\phi \cdot \nabla\eta)\, \mathrm{d}x$$

Integrating by parts gives:

$$\delta\mathcal{E}_{\mathrm{grad}} = -\int_\Omega \mathrm{div}\left(|\nabla\phi|^{p-2} \nabla\phi \right) \eta\, \mathrm{d}x$$

Thus the p-Laplacian appears naturally.

For the rectifier and truncation terms:

$$\delta\mathcal{E}_{\mathrm{rect}} = \int_\Omega \beta\, R'_\alpha(\phi)\, \eta\, dx$$

$$\delta\mathcal{E}_{\mathrm{trunc}} = \int_\Omega \lambda\, T'(\phi)\, \eta\, dx$$

A.3 Euler–Lagrange Equation

Combining all contributions gives the governing PDE:

$$\operatorname{div}\left(|\nabla\phi|^{p-2}\,\nabla\phi\right)\;-\;\beta\,R'_{\alpha}(\phi)\;-\;\lambda\,T'(\phi)\;=\;0$$

This equation defines the equilibrium level-sets whose geometry the paper analyses.

A.4 Limiting Regimes

Key geometric limits include:

- $p < 2$: octahedral tendencies due to inward bias.
- $p = 2$: classical Laplacian yielding spherical equilibria.
- $p > 2$: cubic tendencies due to outward stiffening.
- β large : facet sharpening, producing almost planar faces.
- λ nonzero : truncation and hybrid polyhedral forms.

A.5 Induced Metric

Define an induced local metric:

$F(x, v)$ = absolute value of $(\nabla\phi\ (x) \cdot v)$.

In regions where grad phi is nearly constant, the metric behaves like a polyhedral Finsler norm. Straight paths align with facets. This provides a natural link to Hilbert's 4th problem.

A.6 Polyhedral Uniformisation

Varying (p, β, λ) produces a continuous map to equilibrium geometries. Classical polyhedra appear as critical points in this smooth parameter space.

Acknowledgements

The author thanks the broader mathematical community for continued dialogue and scrutiny, in the shared pursuit of geometric understanding.

Keywords

Polyhedral geometry; variational methods; p-Laplacian; Finsler metrics; Hilbert problems; Uniformisation; geometric emergence; scalar-field models; anisotropic energies; computational geometry.

Recommended Citation

Stanford, P. V. R. *Polyhedral Uniformisation via a Scalar-Field Variational Engine: Connections to Hilbert's 4th, 6th, and 22nd Problems*. Zenodo (2025).

References

1. Hilbert, D. *Mathematical Problems*. Göttinger Nachrichten (1900).
2. Bao, D., Chern, S.-S., Shen, Z. *An Introduction to Riemann–Finsler Geometry*. Springer.
3. Rockafellar, R. *Convex Analysis*. Princeton University Press.
4. Evans, L. *Partial Differential Equations*. AMS.
5. Thompson, R. *Polyhedra and Tilings: An Introduction*. Oxford University Press.
6. Taylor, J. *Crystalline Variational Problems*. Bull. AMS.
7. Morgan, F. *Geometric Measure Theory*. Elsevier.