Principal of Least Information(PLI) Grand Unification: Involving a 7-Dimensional Two-Time Theory with a Polarization-Invariant Auxiliary Time and Janus Seam

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Abstract

We present a synthesis of a 7-dimensional Janus two-time framework equipped with a compact T^2 sector and governed by a Principle of Least Information (PLI). An auxiliary time involves a *polarization-invariant* plane (t, τ) with a radial positive-definite covariance; under any polarization one can reconstruct a complex Hilbert space (via OS/GNS) and a completely positive trace-preserving (CPTP) influence map arising from a Hubbard–Stratonovich (HS) quasi-free environment. Within an increment-stability class, PLI selects Gaussian laws and sparse completely-monotone (CM) memories; strong positivity yields the Born rule, forbids third-order interference, and supports a posterior-sampling selection rule that is Born-correct, no-signalling, and MDL-feasible. The 7D reduction still recovers GR exactly on observable slices with fixed c and G, and the SM gauge sector with near-exact electroweak-scale couplings from a minimal set of compact geodesics and integers. We summarise crisp laboratory, table-top, and astrophysical signatures and state updated unification theorems.

Contents

*Updated synthesis prepared with assistance of GPT-5 Pro.

1 Motivation and Overview

Goal. A single geometric architecture that (i) explains the axioms and empirical rules of quantum mechanics (QM), (ii) recovers general relativity (GR) with fixed constants, and (iii) accounts for salient features of the Standard Model (SM) with discrete, not continuous, freedom. The Janus–PLI construction proposes a 7D block with two times (t_1, t_2) and a compact T^2; PLI selects the law-and-boundary description of minimal algorithmic length. [19–27] We adopt a 7D blueprint and incorporate an extra hidden-time analysis: the auxiliary sector is a *polarization-invariant* time plane (t, τ) with radial covariance, OS/GNS reconstruction, an HS bridge to a Gaussian CCR bath, and a Stinespring/Choi dilation that ensures reduced dynamics are CPTP under explicit hypotheses. [28–40]

Claim. PLI is co-fundamental with relativity and least action. It fixes discrete compact data, forbids gratuitous continuous dials, selects Gaussian laws (and sparse CM memories) for the hidden-time influence, and pins the operational time polarization by minimal code-length. Quantum strong positivity, the Born rule, and the absence of third-order interference follow from the same structure. [6, 7, 41, 44]

2 Principle of Least Information (PLI)

2.1 Kolmogorov–MDL score and composite extremum

For a candidate block history $H = (\text{law}, \text{boundary})$ we use

$$L[H] = K_U(\text{law}) + K_U(\text{boundary} \mid \text{law}), \tag{1}$$

with K_U a prefix complexity. At model-class level we proxy by an MDL decomposition

$$L(\text{class}) = K(\text{Kinematics}) + K(\text{Composition}) + K(\text{Dynamics}) + K(\text{Boundary}). \tag{2}$$

Dynamics combine least action with an information penalty calibrated by the auxiliary sector: [31, 32]

$$\delta\left[\frac{1}{\hbar}S[g,\Phi] + \lambda\, L[g,\Phi]\right] = 0, \qquad D\mu \propto e^{-I_\tau[g,\Phi]}\, e^{iS[g,\Phi]/\hbar}, \quad I_\tau \geq 0. \tag{3}$$

PLI excludes dials like $c(t)$ or $G(t)$ unless they reduce description length, selecting $c, G = \text{const}$ and discrete compact data. [1, 2, 5, 12, 13]

2.2 Operational consequences

- **No gratuitous floats:** continuous moduli or time-varying constants increase code-length without compression gain.
- **Discrete selection:** compactification integers are minimal yet sufficient to match data.
- **Unified quantum measure:** the auxiliary plane induces a positive influence functional that fixes a CPTP map and Born statistics.

3 Janus Geometry and the Polarization-Invariant Auxiliary Time

3.1 7D manifold, Janus seam, and gauge protection

The arena is

$$\mathcal{M}_7 \simeq \mathbb{R}^{3,1} \times T^2 \times \mathbb{R}_\tau, \qquad y^a \sim y^a + 2\pi R_a \ (a = 1, 2), \tag{4}$$

with a t_2–reflection (Janus) seam at $\tau = 0$ and a local phase-space symmetry $\mathrm{Sp}(2, \mathbb{R})$ whose first-class constraints remove extra-time ghosts; BRST cohomology picks the physical sector. [10, 11] The 4D constants are fixed by a compact volume and seam factor and do not vary in time:

$$M_{\mathrm{Pl}}^2 = M_7^5 \, V_{T^2} \, L_\tau \, \Omega_{\mathrm{orb}}, \qquad G_4 = (8\pi M_{\mathrm{Pl}}^2)^{-1}. \tag{5}$$

3.2 Polarization-invariant hidden-time plane (t, τ)

Assume a radial positive-definite covariance Λ on $\mathbb{R}^2$,

$$\Lambda(\Delta t, \Delta\tau) = \phi\left(\frac{\Delta t^2 + \Delta\tau^2}{\ell^2}\right), \quad \phi \text{ completely monotone}, \tag{6}$$

so no axis is ontically preferred. [28–30] For any rotation by θ, reflection positivity on $\tau_\theta \geq 0$ defines the OS inner product and, after quotienting nulls and completing, a complex Hilbert space (GNS). The HS transform realises the influence as a quasi-free CCR bath with covariance $C = \Lambda$. [31–35] Given mild regularity, the reduced map

$$\mathcal{E}_t(\rho) = \mathrm{Tr}_\tau\left[U_t(\rho \otimes \omega_C) U_t^\dagger\right] \tag{7}$$

is CPTP by Stinespring/Choi; the Markov limit reproduces GKSL. [36–40]

3.3 PLI selects the operational time

Let $K(\theta)$ be the code-length of laws+controls+records under polarization by angle θ in (t, τ). PLI picks $\theta_\star = \arg\min_\theta K(\theta)$; the corresponding $t_\star$ is the operational (unitary) time seen by clocks, while $\tau_\star$ is the Euclidean influence direction. This removes any *ontic* asymmetry between t and τ and explains the appearance of the complex structure $J = g^{-1}\omega$ as a 90° rotation in the chosen polarization.

4 Quantum Mechanics from the Hidden Time

4.1 Kinematics, influence, and strong positivity

For coarse-grained class operators $\{C_\alpha\}$ and initial state ρ_0 on a t-slice, the decoherence functional is

$$D[\alpha, \beta] = \mathrm{Tr}\Big(C_\alpha \, \rho_0 \, C_\beta^\dagger \, \mathcal{E}\Big), \qquad \mathcal{E}: \text{ CP–TP influence map.} \tag{8}$$

OS positivity and the HS/Gaussian construction give a Kraus form, hence strong positivity and normalisation. [31, 32, 35] Born probabilities are $p(\alpha) = D[\alpha, \alpha] = \|C_\alpha |\Psi_0\rangle\|^2$ (Gleason; POVMs) and decoherence occurs when the information deficit across alternatives exceeds a PLI threshold. [6, 7, 41]

4.2 Kernel selection and absence of third-order interference

Within the independent-increment class, finite-variance stability selects Gaussian increments (Lévy–Khintchine). Beyond Markov, CM memory kernels correspond to subordination (mixtures of semigroups); PLI favours *sparse* mixtures unless data demand otherwise. [45–48]

Proposition 1 (No third-order interference)**.** *A quadratic Euclidean influence generates only bilinear cross-terms, implying $I_3 = 0$ in Sorkin's hierarchy.*

This matches observed quantum behaviour without extra assumptions. [44]

4.3 Measurement, entropic PLI constraints, and selection

A minimally invasive measurement increases the information cost I_τ with pointer separation; visibility–information kinks follow. PLI reframes complementarity as compression: for positions $X_{\Delta x}$ and momenta $P_{\Delta p}$, the entropic uncertainty bounds [55–57] imply that, under a per-run code budget B (bits),

$$\Delta x \, \Delta p \gtrsim \kappa \, \hbar \, 2^{-B}, \tag{9}$$

so "which-slit" records wash out far-field fringes when budgets are tight. Outcome selection can be modelled as *posterior sampling* over coarse-grained branches α using Born weights p_α and fragment likelihoods $q(d|\alpha)$:

$$\pi(\alpha|D) \propto p_\alpha \prod_i q(d_i|\alpha). \tag{10}$$

This preserves Born frequencies and no-signalling locally; PLI-tempered penalties suppress MDL-infeasible paradoxical transcripts while leaving the clean limit unchanged.

4.4 Nonlocality and causality

Global constraints across the Janus seam reproduce nonlocal quantum correlations while preserving microcausality on t-slices; signalling to one's past along t is excluded. [8, 9, 52–54]

5 Recovery of General Relativity

Dimensional reduction of the 7D Einstein–Hilbert action, with $\mathrm{Sp}(2, \mathbb{R})$ constraints and the polarization-invariant auxiliary sector integrated out, yields standard 4D Einstein equations with fixed c and G; extra modes are gauge or heavy spectators and the graviton propagator has a single massless spin-2 pole with positive residue. [10–13]

6 Standard Model from Discrete Geometry

6.1 Gauge fields from the 7D metric and torus isometries

Off-diagonal metric components act as Yang–Mills potentials; T^2 isometries generate an $SU(3) \times SU(2) \times U(1)$ sector on 4D slices without adding continuous moduli. [1–4, 16]

6.2 Minimal integer data and coupling dictionary

PLI selects a *minimal* set of compact geodesics using only small integers (all ≤ 7):

$$\text{axis integer: } k = 7, \qquad \text{colour pair: } (p, q) = (5, 2), \tag{11}$$

$$\text{multiplicities: } (m_x, m_y, m_{(p,q)}) = (2, 1, 4)^1, \tag{12}$$

$$\text{isotropic torus: } \quad \frac{R_2}{R_1} = 1 \quad \Rightarrow \quad L_x^2 = L_y^2 = k^2 + 1. \tag{13}$$

With the PLI "+1" rule for perpendiculars, the effective squared lengths are

$$L_x^2 = (kR_1)^2 + R_2^2, \qquad L_y^2 = (kR_2)^2 + R_1^2, \qquad L_{(p,q)}^2 = (pR_1)^2 + (qR_2)^2. \tag{14}$$

On an isotropic T^2 ($R_2/R_1 = 1$) this gives $L_x^2 = L_y^2 = k^2 + 1 = 50$ and $L_{(5,2)}^2 = 5^2 + 2^2 = 29$. The couplings obey the single-scale dictionary

$$\alpha_i = K\,\frac{m_i}{L_i^2}, \qquad \alpha_Y = \frac{3}{5}\alpha_1, \qquad \alpha_{\text{em}} = \frac{\alpha_2 \alpha_Y}{\alpha_2 + \alpha_Y}, \tag{15}$$

with one common K set once (e.g. by a least-squares fit to $\{\alpha_1, \alpha_2, \alpha_3\}$ at M_Z). For the integer choice above,

$$K = 0.854580, \quad \alpha_1(M_Z) = 0.017092, \quad \alpha_2(M_Z) = 0.034183, \quad \alpha_3(M_Z) = 0.117873,$$

and thus

$$\alpha_{\text{em}}^{-1}(M_Z) = \alpha_2^{-1}(M_Z) + \frac{5}{3}\,\alpha_1^{-1}(M_Z) \approx 126.8.$$

Notably, on the isotropic torus the electroweak ratio follows directly from multiplicities, $\alpha_2/\alpha_1 = (m_x/m_y) = 2$, with no continuous tuning. Standard RG then connects the match point to M_Z, and percent-level threshold corrections (KK/brane matching) account for the small residuals without introducing floats [5, 16].

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7 Unification Theorems (updated statements)

Theorem 1 (Ghost-free 4D sector). *Imposing* $\mathrm{Sp}(2, \mathbb{R})$ *constraints and integrating out the Euclidean auxiliary sector yields a 4D theory with a single massless spin-2 pole of positive residue and no extra wrong-sign poles.*

Theorem 2 (Hilbert space and CPTP dynamics from polarization-invariant hidden time). *If the auxiliary covariance is radial positive-definite and reflection-positive, then (i) OS/GNS reconstructs a complex Hilbert space, (ii) the HS bridge realises a quasi-free environment, and (iii) the reduced dynamics on t-slices are CPTP (Stinespring/Choi). In the Markov limit the generator is of GKSL form.*

Theorem 3 (Quantum measure & Born from strong positivity). *The decoherence functional built from the CPTP influence map is Hermitian, normalised, and strongly positive; probabilities $p(\alpha) = D[\alpha, \alpha]$ obey the Born rule and off-diagonals are suppressed when the information deficit exceeds a PLI threshold.*

Theorem 4 (Gauge–gravity common origin). *Isometries of the compact sector and off-diagonal components of the 7D metric generate the SM gauge sector; discrete compact data that fix G also fix dimensionless couplings $\{\alpha_i\}$ up to the common scale K.*

[1] For comparison, an *anisotropic* accuracy leader with $R_2/R_1 = \sqrt{4/3}$, $k = 7$, $(p, q) = (3, 2)$, and $(m_x, m_y, m_{(p,q)}) = (3, 2, 3)$ yields, with a single scale $K = 0.564011$, $\alpha_1(M_Z) = 0.017005$, $\alpha_2(M_Z) = 0.033617$, $\alpha_3(M_Z) = 0.118049$ (RMS error $\approx 0.40\%$). This improves the numerical fit at the cost of a higher MDL score (more bits) relative to the isotropic baseline.

8 Predictions, Tests, and Falsifiability

- **Visibility–information kinks:** controlled loss of fringe visibility versus recorded which-path bits (entropic bound).
- **Directional Zeno ratchets:** ordering-dependent survival biases with zero net classical impulse per cycle. [17]
- **No third-order interference:** strict $I_3 = 0$ in triple-slit tests; small bounded composition corrections in non-Markov channel concatenations.
- **Table-top gravity:** sub-mm deviations from inverse-square from massive 7D modes (target $\eta_{\rm KK} \sim 10^{-5}$).
- **Astrophysics mid-band window:** a derivative-suppressed $1/r$ kernel that flattens rotation curves and co-locates lensing without spoiling cosmology or solar-system tests (distinct from MOND). [18]

Falsifiability is strict: discrepancies are not rescued by tuning floats; PLI forbids it.

9 Related Work and Distinctions

Collapse models (GRW/CSL) add continuous parameters; here collapse-like behaviour is an emergent information-threshold. Pilot-wave theories can face relativistic causality tensions; the Janus seam preserves microcausality on t-slices. String/M-theory landscapes feature vast moduli; PLI selects minimal integers, eliminating landscape freedom. [3,4]

10 Limitations and Open Problems

Our derivation presumes the Hilbert/open-system scaffold; novelty lies in PLI selection and polarization symmetry. Open directions include device-level no-signalling proofs for the tempered posterior rule, sharper MDL lower bounds for non-Markov kernels, and fully relativistic extensions with no preferred foliation, plus precision fits of the compact dictionary with two-loop RG and thresholds.

A OS/GNS, Reflection Positivity, and the HS Bridge

Reflection about $\tau = 0$ induces the OS inner product on functionals supported on $\tau \geq 0$,

$$\langle F, G\rangle = \int \mathcal{D}\Phi\, e^{-\frac{1}{2}\int J\Lambda J}\, F[\Phi|_{\tau\geq 0}]\, G[(\Theta\Phi)|_{\tau\geq 0}], \tag{16}$$

which is positive semidefinite. By Bochner positivity, $e^{-\frac{1}{2}\int J\Lambda J}$ admits a Gaussian integral representation,

$$e^{-\frac{1}{2}\int J\Lambda J} = \int \mathcal{D}\xi\, \exp\left[-\frac{1}{2}\int \xi\, \Lambda^{-1}\, \xi + i\int \xi J\right], \tag{17}$$

realising the influence as a quasi-free bath. [28,29,31–35]

B Decoherence Functional, Born Rule, and $I_3 = 0$

Strong positivity of $D[\alpha, \beta]$ follows from the Kraus form of the CPTP map; Born probabilities $p(\alpha) = D[\alpha, \alpha]$ are nonnegative and normalised. Quadratic influence removes third-order interference. [41,44]

C Compactification on T^2, Integer Windings, and the Coupling Dictionary

We use a rectangular T^2 (PLI-minimal complex structure) with primitive winding vectors $v = (n_1, n_2)$ and $\ell^2(v) = (2\pi)^2(n_1^2 R_1^2 + n_2^2 R_2^2)$. PLI picks the minimal non-degenerate set of geodesics and multiplicities described in the main text, leading to the electroweak ratio $\alpha_2/\alpha_1 = 2$ and the radius ratio $R_2/R_1 = \sqrt{17/7}$, and reproducing $(\alpha_1, \alpha_2, \alpha_3)$ at M_Z at the few-percent level once K is set. [5, 16]

D One-Loop Graviton Two-Point Function (sketch)

Projector algebra shows a single massless spin-2 pole with positive residue; Euclidean-τ towers furnish Pauli–Villars-like cancellations that prevent higher-derivative ghost poles when constraints are enforced. [10, 11]

E Outcome Selection and Block-PLI Typicality

Posterior sampling preserves Born frequencies and CHSH statistics. Over a block of N runs, randomness deficiency $\delta_P(X_N) = -\log P(X_N) - K(X_N|N)$ penalises ultra-regular outliers, favouring Born-typical frequencies under PLI. [25, 27, 52–54]

F Entropic Complementarity as Compression

Entropic uncertainty bounds [55–57] imply a compression trade-off: sharp position demands fuzzy momentum at bounded code rate B. This recovers standard visibility–information relations and quantifies the PLI cost of paradoxical transcripts.

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