An Examination of Algebraic Structures Derived from the Integers by Sequential Adjunction of Elements with Prescribed Identity and Absorbing Properties

Abstract

We conduct a detailed investigation into two algebraic structures constructed sequentially starting from the ring of integers $\mathbb{Z}$. The first structure, $\mathcal{S}$, is the globalization of $\mathbb{Z}$, obtained by adjoining an element $\mathcal{T}$ defined as the additive identity and multiplicative absorber. We provide an exhaustive verification that $\mathcal{S}$ is a commutative unital standard semiring. Its algebraic properties are analyzed, establishing it as a Principal Ideal Semiring (PIS) with Krull dimension 2. We classify its ideals, all of which are subtractive, its prime spectrum, and its congruences. A detailed analysis of factorization reveals that while $\mathcal{S}$ is an integral semidomain, it is not a Unique Factorization Domain (UFD), due to the existence of an element ($0_{\mathbb{Z}}$) that is prime but reducible, and which lacks a factorization into irreducibles. The ideal zeta function is computed, $\zeta_{\mathcal{S}}(s) = \zeta(s)$. The action of the unit group $U(\mathcal{S}) \cong \mathbb{Z}/2\mathbb{Z}$ yields a set of fixed points (singlets) $\mathcal{A} = \{0_{\mathbb{Z}}, \mathcal{T}\}$, isomorphic to the Boolean semiring $\mathbb{B}$.

The second structure, $\mathcal{S}'$, is constructed by adjoining an element Ω to $\mathcal{S}$, defined as an absorbing element for both addition and multiplication. We establish that $\mathcal{S}'$ is a commutative unital hemiring, but not a standard semiring, as the additive identity ($\mathcal{T}$) differs from the unique multiplicative absorber (Ω). We characterize its ideal structure, proving it is a PIS, but demonstrating that no proper ideals are subtractive due to the presence of an additive absorber. We determine its spectrum, showing its Krull dimension is 3. We analyze the singlets of $\mathcal{S}'$, which form an idempotent sub-semiring $\mathcal{A}' = \{0_{\mathbb{Z}}, \mathcal{T}, \Omega\}$, isomorphic to the extended Boolean semiring $\mathbb{B}_{\text{ext}}$. These constructions are generalized to rings of integers $\mathcal{O}_K$ in algebraic number fields, where we establish that the class semigroup of $S(\mathcal{O}_K)$ is isomorphic to the class group $\mathrm{Cl}(K)$. We examine the topological properties of the spectra, demonstrating how these sequential adjunctions systematically increase the Krull dimension by introducing new generic points.

Contents

1 Set-Theoretic Foundations and the Construction of the Integers

We operate within the axiomatic framework of Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC) [7]. We commence by recapitulating the construction of the standard number systems, ensuring precision in the foundational definitions upon which subsequent algebraic structures are erected.

1.1 The Natural Numbers ω

We employ the von Neumann construction of the natural numbers, which is predicated upon the Axiom of Infinity.

Definition 1.1. A set S is termed *inductive* if it satisfies the following two conditions:

1. The empty set $\emptyset$ is an element of S.
2. For every $x \in S$, the successor of x, defined as $S(x) = x \cup \{x\}$, is also an element of S.

The Axiom of Infinity asserts the existence of at least one inductive set.

Definition 1.2. The set of natural numbers, denoted by ω (or $\mathbb{N}_0$), is defined as the intersection of all inductive sets:

$$\omega := \bigcap \{S \mid S \text{ is inductive}\}. \tag{1.1}$$

Lemma 1.3. *The set ω is an inductive set, and it is the smallest such set with respect to the inclusion relation; that is, if S' is an inductive set, then $\omega \subseteq S'$.*

Proof. Let $\mathcal{I}$ denote the collection of all inductive sets. By the Axiom of Infinity, $\mathcal{I}$ is non-empty. We must verify that $\omega = \bigcap \mathcal{I}$ satisfies the definition of an inductive set.

(i) Verification of $\emptyset \in \omega$. By the definition of an inductive set (Condition 1), $\emptyset \in S$ for every $S \in \mathcal{I}$. Therefore, $\emptyset$ is an element of the intersection of all elements of $\mathcal{I}$. Consequently, $\emptyset \in \omega$.

(ii) Verification of closure under the successor operation. Let $x \in \omega$. By the definition of intersection, $x \in S$ for every $S \in \mathcal{I}$. Since each $S \in \mathcal{I}$ possesses the property of being inductive (Condition 2), it follows that the successor $S(x) = x \cup \{x\}$ is an element of S for every $S \in \mathcal{I}$. Consequently, $S(x)$ is an element of the intersection $\bigcap \mathcal{I}$. Therefore, $S(x) \in \omega$.

The two conditions being satisfied, ω is an inductive set.

To demonstrate the minimality property, let S' be an arbitrary inductive set. By definition, $S' \in \mathcal{I}$. By the properties of set intersection, $\bigcap \mathcal{I} \subseteq S'$. Therefore, $\omega \subseteq S'$. □

Notation 1.4. We denote the elements of ω using standard numerals: $0 := \emptyset$, $1 := S(0) = \{0\}$, $2 := S(1) = \{0, 1\}$, and so forth.

Theorem 1.5 (Principle of Mathematical Induction)**.** *Let A be a subset of ω. If $0 \in A$ and for all $n \in \omega$, the condition $n \in A$ implies $S(n) \in A$, then $A = \omega$.*

Proof. The hypotheses imposed upon the set A precisely state that A is an inductive set. By the minimality established in Lemma 1.3, we have the inclusion $\omega \subseteq A$. Since $A \subseteq \omega$ by hypothesis, we conclude by the Axiom of Extensionality that $A = \omega$. □

The binary operations of addition and multiplication on ω are established via the Recursion Theorem.

Definition 1.6 (Operations on ω)**.** For $m \in \omega$:

1. Addition ($+$): Defined recursively by the equations $m + 0 := m$ and $m + S(n) := S(m + n)$.
2. Multiplication ($\times$): Defined recursively by the equations $m \times 0 := 0$ and $m \times S(n) := (m \times n) + m$.

Proposition 1.7. *The structure $(\omega, +, \times, 0, 1)$ is a commutative unital standard semiring (Definition 2.5) satisfying the cancellation laws:*

1. *Additive cancellation: for all $m, n, l \in \omega$, the equality $m + l = n + l$ implies $m = n$.*

2. *Multiplicative cancellation (for $l \neq 0$): for all $m, n, l \in \omega$, the equality $m \times l = n \times l$ implies $m = n$.*

Proof. The verification of the semiring axioms (associativity, commutativity, distributivity, identities, and the absorbing property of 0) relies on successive applications of the Principle of Mathematical Induction (Theorem 1.5). We provide verification of the cancellation laws.

1. Additive cancellation law. We proceed by induction on the variable l. Base Case ($l = 0$): Assume $m + 0 = n + 0$. By the definition of addition (Definition 1.6.1), $m + 0 = m$ and $n + 0 = n$. Thus $m = n$.

Inductive Step: Assume the cancellation law holds for a specific $l \in \omega$. We consider the successor $S(l)$. Suppose $m+S(l) = n+S(l)$. By Definition 1.6.1, this equality is equivalent to $S(m+l) = S(n+l)$.

We utilize the injectivity of the successor function S. This property is derived from the definition of ordinals in ZFC. Specifically, if $S(x) = S(y)$, then $x \cup \{x\} = y \cup \{y\}$. By properties related to the well-founded nature of the membership relation on the ordinals ω, this equality implies $x = y$.

By the injectivity of S, we deduce $m + l = n + l$. By the inductive hypothesis, $m = n$. By the Principle of Mathematical Induction, the additive cancellation law holds for all $l \in \omega$.

2. Multiplicative cancellation law. This relies on the property that ω has no zero divisors (if $m \times l = 0$, then $m = 0$ or $l = 0$), which is also established by induction. Assuming this property, suppose $m \times l = n \times l$ and $l \neq 0$. We utilize the standard total order relation on ω, defined by $n \leq m$ if and only if there exists $k \in \omega$ such that $m = n + k$. Assume without loss of generality $m \geq n$, so $m = n + k$. The equality becomes $(n + k) \times l = n \times l$. By distributivity, $(n \times l) + (k \times l) = n \times l$. We write the right side as $(n \times l) + 0$. By the additive cancellation law (Part 1), we conclude $k \times l = 0$. Since $l \neq 0$, by the absence of zero divisors, we must have $k = 0$. Thus $m = n + 0 = n$. ☐

1.2 The Ring of Integers $\mathbb{Z}$

We construct the ring of integers $\mathbb{Z}$ from the semiring ω utilizing the method of Grothendieck group completion.

Definition 1.8. Define the relation $\sim$ on the Cartesian product $\omega \times \omega$ by the condition $(a, b) \sim (c, d)$ if and only if $a + d = b + c$ in ω.

Lemma 1.9. *The relation $\sim$ is an equivalence relation on $\omega \times \omega$.*

Proof. We verify the defining properties of an equivalence relation: reflexivity, symmetry, and transitivity. These properties rely on the algebraic structure of $(\omega, +)$ established in Proposition 1.7.

1. Reflexivity: For any $(a, b) \in \omega \times \omega$. We require $(a, b) \sim (a, b)$, which necessitates $a + b = b + a$. This holds by the commutativity of addition in ω.

2. Symmetry: Assume $(a, b) \sim (c, d)$. This means $a + d = b + c$. We require $(c, d) \sim (a, b)$, which necessitates $c + b = d + a$. By the commutativity of addition in ω, $b + c = c + b$ and $a + d = d + a$. Thus $c + b = d + a$.

3. Transitivity: Assume $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. We have the equalities in ω:

$$a + d = b + c \tag{1.2}$$

$$c + f = d + e \tag{1.3}$$

We wish to demonstrate that $(a, b) \sim (e, f)$, i.e., $a + f = b + e$. We sum the two equations (1.2) and (1.3): $(a + d) + (c + f) = (b + c) + (d + e)$. Applying the associativity and commutativity of addition in ω, we rearrange the terms: $(a + f) + (d + c) = (b + e) + (c + d)$. Since $d + c = c + d$, we apply the additive cancellation law (Proposition 1.7.1) to cancel the term $(c+d)$ from both sides of the equality. This yields the desired result $a + f = b + e$. Therefore, $\sim$ is transitive. ☐

Definition 1.10. The set of integers, $\mathbb{Z}$, is defined as the set of equivalence classes $\mathbb{Z} := (\omega \times \omega)/ \sim$. We denote the equivalence class containing the pair (a, b) by $[a, b]$.

Definition 1.11 (Operations on $\mathbb{Z}$)**.** Addition ($+_{\mathbb{Z}}$) and multiplication ($\times_{\mathbb{Z}}$) on $\mathbb{Z}$ are defined as follows:

$$[a, b] +_{\mathbb{Z}} [c, d] := [a + c, b + d], \tag{1.4}$$
$$[a, b] \times_{\mathbb{Z}} [c, d] := [ac + bd, ad + bc]. \tag{1.5}$$

(We use juxtaposition xy to denote $x \times y$ in ω).

Lemma 1.12. *The operations $+_{\mathbb{Z}}$ and $\times_{\mathbb{Z}}$ are well-defined on the set of equivalence classes $\mathbb{Z}$.*

Proof. We must demonstrate that the definitions are independent of the choice of representatives for the equivalence classes. Let $[a, b] = [a', b']$ and $[c, d] = [c', d']$. This implies the following equalities in ω:

$$a + b' = b + a' \tag{1.6}$$
$$c + d' = d + c' \tag{1.7}$$

1. Well-definedness of $+_{\mathbb{Z}}$. We must show $[a+c, b+d] = [a'+c', b'+d']$. This requires verification of the condition $(a + c) + (b' + d') = (b + d) + (a' + c')$. Using associativity and commutativity in ω: LHS $= (a + c) + (b' + d') = (a + b') + (c + d')$. RHS $= (b + d) + (a' + c') = (b + a') + (d + c')$. By equations (1.6) and (1.7), and commutativity, LHS $=$ RHS. Thus $+_{\mathbb{Z}}$ is well-defined.

2. Well-definedness of $\times_{\mathbb{Z}}$. We must show $[ac+bd, ad+bc] = [a'c'+b'd', a'd'+b'c']$. It suffices, by the symmetry inherent in the definition (commutativity of $\times_{\mathbb{Z}}$ follows immediately from the commutativity of $\times$ in ω), to verify independence with respect to the first argument, assuming $[a, b] = [a', b']$. We require $[ac + bd, ad + bc] = [a'c + b'd, a'd + b'c]$. This requires verification of the condition $(ac + bd) + (a'd + b'c) = (ad + bc) + (a'c + b'd)$. LHS: $ac + bd + a'd + b'c$. Rearranging terms using properties of ω and applying distributivity: $c(a + b') + d(b + a')$. RHS: $ad + bc + a'c + b'd$. Rearranging terms: $c(b + a') + d(a + b')$. By equation (1.6), $a + b' = b + a'$. Thus LHS $=$ RHS. $\times_{\mathbb{Z}}$ is well-defined. □

Theorem 1.13. *The structure $(\mathbb{Z}, +_{\mathbb{Z}}, \times_{\mathbb{Z}})$ is a commutative ring with unity. Furthermore, it is an integral domain of characteristic 0, and a Principal Ideal Domain (PID).*

Proof. The verification of the axioms of a commutative ring with unity relies on the established properties of the semiring $(\omega, +, \times)$, as detailed in standard algebra texts [9].

Part I: $(\mathbb{Z}, +_{\mathbb{Z}})$ is an abelian group. Associativity and commutativity follow directly from the corresponding properties in ω. The additive identity is $0_{\mathbb{Z}} = [0, 0]$. Verification: $[a, b] +_{\mathbb{Z}} [0, 0] = [a + 0, b+0] = [a, b]$. The additive inverse of $[a, b]$ is $-[a, b] = [b, a]$. Verification: $[a, b]+_{\mathbb{Z}}[b, a] = [a+b, b+a]$. We check $[a + b, b + a] = [0, 0]$. This requires $(a + b) + 0 = (b + a) + 0$, which holds by commutativity in ω.

Part II: $(\mathbb{Z}, \times_{\mathbb{Z}})$ is a commutative monoid. Associativity and commutativity follow from the corresponding properties and distributivity in ω. The multiplicative identity is $1_{\mathbb{Z}} = [1, 0]$. Verification: $[a, b] \times_{\mathbb{Z}} [1, 0] = [a(1) + b(0), a(0) + b(1)] = [a, b]$.

Part III: Distributivity. Verification follows by expanding the definitions and applying distributivity in ω.

Part IV: Integral Domain. We verify the absence of zero divisors. Suppose $[a, b] \times_{\mathbb{Z}} [c, d] = 0_{\mathbb{Z}}$. This means $[ac + bd, ad + bc] = [0, 0]$, so $ac + bd = ad + bc$. Assume $[a, b] \neq 0_{\mathbb{Z}}$ (i.e., $a \neq b$). Without loss of generality, assume $a > b$ (using the standard order on ω). Then $a = b + k$ for some $k \in \omega, k \neq 0$. Substituting: $(b + k)c + bd = (b + k)d + bc$. Expanding using distributivity in ω: $bc+kc+bd = bd+kd+bc$. By the additive cancellation law (Proposition 1.7.1), we cancel $bc+bd$ from both sides, yielding $kc = kd$. Since $k \neq 0$, we utilize the multiplicative cancellation law (Proposition 1.7.2) to conclude $c = d$. Thus $[c, d] = [c, c] = 0_{\mathbb{Z}}$.

Part V: Characteristic 0. The characteristic is the smallest positive integer n such that $n \cdot 1_{\mathbb{Z}} = 0_{\mathbb{Z}}$. $n \cdot 1_{\mathbb{Z}} = [n, 0]$. The condition $[n, 0] = [0, 0]$ implies $n + 0 = 0 + 0$, so $n = 0$. Since no such positive n exists, the characteristic is 0.

Part VI: PID. It is a standard result in ring theory that $\mathbb{Z}$ is a Euclidean Domain (with the Euclidean function being the absolute value), and every Euclidean Domain is a PID. □

We identify $n \in \omega$ with the equivalence class $[n, 0] \in \mathbb{Z}$. This defines an injective homomorphism $\iota : \omega \to \mathbb{Z}$. We henceforth utilize standard notation for the elements and operations of $\mathbb{Z}$.

2 Algebraic Preliminaries: Semirings and Related Concepts

We establish the precise definitions for the algebraic structures central to this investigation, adhering to conventions such as those presented in [5].

2.1 Definitions of Semirings and Hemirings

Definition 2.1. A *semiring* is an algebraic structure $(R, +, \times)$ consisting of a non-empty set R equipped with two binary operations, addition $(+)$ and multiplication $(\times)$, satisfying the following axioms:

(S1) $(R, +)$ is a commutative semigroup (addition is associative and commutative).

(S2) $(R, \times)$ is a semigroup (multiplication is associative).

(S3) Multiplication distributes over addition from the left and the right: $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$.

Definition 2.2. Let $(M, *)$ be a semigroup.

1. An element $e \in M$ is an *identity element* if $a * e = e * a = a$ for all $a \in M$.
2. An element $z \in M$ is an *absorbing element* (or *annihilator*) if $a * z = z * a = z$ for all $a \in M$.

Lemma 2.3. *In a semigroup, an identity element, if it exists, is unique. An absorbing element, if it exists, is unique.*

Proof. Uniqueness of identity: Let e_1, e_2 be identities. $e_1 = e_1 * e_2$ (since e_2 is a right identity) $= e_2$ (since e_1 is a left identity). Uniqueness of absorber: Let z_1, z_2 be absorbers. $z_1 = z_1 * z_2$ (since z_2 is a right absorber) $= z_2$ (since z_1 is a left absorber). $\square$

Definition 2.4. A semiring $(R, +, \times)$ is further characterized as follows:

1. *Commutative* if $(R, \times)$ is commutative.
2. *Unital* if $(R, \times)$ possesses an identity element 1_R.
3. A *hemiring* if $(R, +)$ possesses an identity element 0_R.

A crucial distinction concerns the relationship between the additive identity and the multiplicative absorber.

Definition 2.5. A semiring R is called a *standard semiring* (often referred to as a *semiring with zero*) if it is a hemiring and its unique additive identity 0_R is also the unique multiplicative absorbing element. That is, $a + 0_R = a$ and $a \times 0_R = 0_R \times a = 0_R$ for all $a \in R$.

We shall analyze structures that are hemirings but may or may not satisfy the conditions of a standard semiring.

Definition 2.6. Let R be a commutative unital hemiring with additive identity 0_R.

1. R is *zerosumfree* if $a + b = 0_R$ implies $a = 0_R$ and $b = 0_R$.
2. R is an *integral semidomain relative to* 0_R if $1_R \neq 0_R$ and it has no zero divisors relative to 0_R (i.e., if $a \times b = 0_R$, then $a = 0_R$ or $b = 0_R$).

If R is a standard semiring, we simply refer to it as an integral semidomain.

Definition 2.7. Let R be a commutative unital semiring possessing a multiplicative absorber z_R. R is a *z-integral semidomain* if $1_R \neq z_R$ and it has no z-divisors (i.e., if $a \times b = z_R$, then $a = z_R$ or $b = z_R$).

Example 2.8 (The Boolean Semiring)**.** The Boolean semiring $\mathbb{B} = (\{0, 1\}, \vee, \wedge)$ is a structure where addition is logical OR ($1 \vee 1 = 1$) and multiplication is logical AND. It is a standard commutative unital semiring. It is characterized by additive idempotence: $x + x = x$.

2.2 Ideals and Congruences

Definition 2.9. Let R be a commutative semiring. An *ideal* I of R is a non-empty subset $I \subseteq R$ such that $I + I \subseteq I$ (closure under addition) and $R \times I \subseteq I$ (absorption under multiplication by R).

Lemma 2.10. *If a commutative semiring R possesses a multiplicative absorbing element z_R, then $z_R \in I$ for any ideal I.*

Proof. Let I be an ideal. Since I is non-empty by definition, let $x \in I$. By the absorption property of ideals, $z_R \times x \in I$. By the definition of an absorbing element (Definition 2.2), $z_R \times x = z_R$. Thus $z_R \in I$. □

Definition 2.11. A commutative unital semiring R is a *Principal Ideal Semiring* (PIS) if every ideal is principal. The principal ideal generated by a is $(a)_R = Ra = \{ra \mid r \in R\}$.

Definition 2.12. An ideal I of a semiring R is called *subtractive* (or a *k-ideal*) if for all $a, b \in R$, whenever $a \in I$ and $a + b \in I$, it follows that $b \in I$.

The presence of an additive absorbing element imposes severe restrictions on the existence of subtractive ideals.

Lemma 2.13. *Let R be a semiring. If an ideal I contains an element y such that y is an additive absorber for R (i.e., $y + b = y$ for all $b \in R$), then I is subtractive if and only if $I = R$.*

Proof. ($\Longleftarrow$) If $I = R$, it is trivially subtractive, as the conclusion $b \in R$ is always satisfied.

($\Longrightarrow$) Assume I is subtractive and $y \in I$. We show $I = R$. Let $b \in R$ be an arbitrary element. Let $a = y$. We have $a \in I$. Consider the sum $a + b = y + b$. By the property of the additive absorber y, $y + b = y$. Thus $a + b \in I$. By the definition of a subtractive ideal, since $a \in I$ and $a + b \in I$, we must conclude $b \in I$. Since b was arbitrary, $I = R$. □

Definition 2.14. A *congruence* ρ on a semiring R is an equivalence relation on R compatible with the operations: if $a\rho b$ and $c\rho d$, then $(a + c)\rho(b + d)$ and $(a \times c)\rho(b \times d)$. The set of congruences is $\mathrm{Cong}(R)$.

Definition 2.15. Let I be an ideal of a commutative hemiring R. The *Bourne relation* ρ_I associated with I is defined by $a\rho_I b$ if and only if there exist $x, y \in I$ such that $a + x = b + y$. If I is subtractive, ρ_I is a congruence, and the quotient semiring is $R/I := R/\rho_I$.

2.3 The Spectrum and Dimension

Definition 2.16. Let R be a commutative unital semiring.

1. An ideal P is *prime* if P is proper ($P \neq R$) and if $a \times b \in P$ implies $a \in P$ or $b \in P$. The set of prime ideals is $\mathrm{Spec}(R)$.

2. An ideal M is *maximal* if M is proper and there is no ideal I such that $M \subsetneq I \subsetneq R$. The set of maximal ideals is $\mathrm{MaxSpec}(R)$.

Definition 2.17. The *Krull dimension* of R, $\mathrm{Kdim}(R)$, is the supremum of the lengths n of chains of distinct prime ideals $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$.

2.4 Factorization Theory in Integral Semidomains

We define concepts related to factorization.

Definition 2.18. Let R be an integral semidomain (relative to 0_R).

1. Divisibility: $a|b$ if there exists $c \in R$ such that $b = ac$.
2. Associates: a and b are associates if $a|b$ and $b|a$.
3. Irreducible element: A non-zero ($x \neq 0_R$), non-unit element x is *irreducible* if $x = ab$ implies a or b is a unit.
4. Prime element: A non-zero, non-unit element p is *prime* if $p|ab$ implies $p|a$ or $p|b$.
5. Unique Factorization Domain (UFD): R is a UFD if every non-zero, non-unit element can be written as a product of irreducible elements (Existence), and this factorization is unique up to order and associates (Uniqueness).

2.5 Group Actions and Symmetries

Definition 2.19. Let M be an algebraic structure. A *G-action* on M by automorphisms is a group homomorphism $\rho : G \to \mathrm{Aut}(M)$. An element $x \in M$ is a *fixed point* (or *singlet*) if $\rho(g)(x) = x$ for all $g \in G$. The set of fixed points is denoted M^G.

We recall the inherent symmetries of $\mathbb{Z}$.

Lemma 2.20. *The group of units of the ring $\mathbb{Z}$ is $U(\mathbb{Z}) = \{1_{\mathbb{Z}}, -1_{\mathbb{Z}}\} \cong \mathbb{Z}/2\mathbb{Z}$.*

Theorem 2.21. *The only fixed point (singlet) of the canonical action of $U(\mathbb{Z})$ on the additive group $(\mathbb{Z}, +)$ by multiplication is $0_{\mathbb{Z}}$.*

Proof. A fixed point n must satisfy $g \times n = n$ for all $g \in U(\mathbb{Z})$. We consider the action of $g = -1_{\mathbb{Z}}$. We require $(-1_{\mathbb{Z}}) \times n = n$, which means $-n = n$. Adding n yields $0_{\mathbb{Z}} = 2n$. Since $\mathbb{Z}$ has characteristic 0 (Theorem 1.13), $n = 0_{\mathbb{Z}}$. $\square$

3 The Globalization of $\mathbb{Z}$: Construction and Verification of $\mathcal{S}$

We construct the structure $\mathcal{S}$ by adjoining an element $\mathcal{T}$ to the ring of integers $\mathbb{Z}$, defined such that $\mathcal{T}$ functions simultaneously as the additive identity and the multiplicative absorber. This corresponds to the globalization construction $G(\mathbb{Z})$ applied to $\mathbb{Z}$ [5, Example I.1.10].

3.1 Definition of $\mathcal{S}$ and Operations

Construction 3.1. Let $\mathbb{Z}$ be the ring of integers. Let $\mathcal{T}$ be a formal element such that $\mathcal{T} \notin \mathbb{Z}$. We define the set $\mathcal{S} := \mathbb{Z} \cup \{\mathcal{T}\}$.

Definition 3.2 (Addition on $\mathcal{S}$)**.** We define the operation $+ : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$. For $a, b \in \mathcal{S}$:

$$a + b := \begin{cases} a +_{\mathbb{Z}} b & \text{if } a \in \mathbb{Z}, b \in \mathbb{Z} \quad \text{(Case A1)} \\ a & \text{if } a \in \mathbb{Z}, b = \mathcal{T} \quad \text{(Case A2)} \\ b & \text{if } a = \mathcal{T}, b \in \mathbb{Z} \quad \text{(Case A3)} \\ \mathcal{T} & \text{if } a = \mathcal{T}, b = \mathcal{T} \quad \text{(Case A4)} \end{cases} \tag{3.1}$$

Definition 3.3 (Multiplication on $\mathcal{S}$)**.** We define the operation $\times : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$. For $a, b \in \mathcal{S}$:

$$a \times b := \begin{cases} a \times_{\mathbb{Z}} b & \text{if } a \in \mathbb{Z}, b \in \mathbb{Z} \quad \text{(Case M1)} \\ \mathcal{T} & \text{if } a \in \mathbb{Z}, b = \mathcal{T} \quad \text{(Case M2)} \\ \mathcal{T} & \text{if } a = \mathcal{T}, b \in \mathbb{Z} \quad \text{(Case M3)} \\ \mathcal{T} & \text{if } a = \mathcal{T}, b = \mathcal{T} \quad \text{(Case M4)} \end{cases} \tag{3.2}$$

3.2 Verification of the Semiring Structure

We provide a complete demonstration that $(\mathcal{S}, +, \times)$ satisfies the axioms of a standard commutative unital semiring.

Theorem 3.4. *The structure $(\mathcal{S}, +, \times)$ is a standard commutative unital semiring. The additive identity is $0_{\mathcal{S}} = \mathcal{T}$. The multiplicative identity is $1_{\mathcal{S}} = 1_{\mathbb{Z}}$.*

Proof. We systematically verify the axioms detailed in Definitions 2.1 and 2.5 through an examination of all possible cases.

Part I: $(\mathcal{S}, +)$ is a commutative monoid.

I.1. Commutativity. We must verify $a + b = b + a$ for all $a, b \in \mathcal{S}$. Case I.1.1: $a \in \mathbb{Z}, b \in \mathbb{Z}$. $a + b = a +_{\mathbb{Z}} b$. $b + a = b +_{\mathbb{Z}} a$. By Theorem 1.13, $+_{\mathbb{Z}}$ is commutative, so equality holds. Case I.1.2: $a \in \mathbb{Z}, b = \mathcal{T}$. $a + b = a$ (Case A2). $b + a = a$ (Case A3). Equality holds. Case I.1.3: $a = \mathcal{T}, b \in \mathbb{Z}$. Symmetric to Case I.1.2. Case I.1.4: $a = \mathcal{T}, b = \mathcal{T}$. $a + b = \mathcal{T}$ (Case A4). $b + a = \mathcal{T}$. Equality holds.

I.2. Associativity. We must verify $(a+b)+c = a+(b+c)$ for all $a, b, c \in \mathcal{S}$. We examine the eight possibilities based on the membership of a, b, c in $\mathbb{Z}$ or $\{\mathcal{T}\}$.

Case I.2.1: $a \in \mathbb{Z}, b \in \mathbb{Z}, c \in \mathbb{Z}$. LHS: $(a + b) + c = (a +_{\mathbb{Z}} b) +_{\mathbb{Z}} c$. RHS: $a + (b + c) = a +_{\mathbb{Z}} (b +_{\mathbb{Z}} c)$. Equality holds by associativity of $+_{\mathbb{Z}}$ (Theorem 1.13).

Case I.2.2: $a \in \mathbb{Z}, b \in \mathbb{Z}, c = \mathcal{T}$. LHS: $(a + b) + c = (a +_{\mathbb{Z}} b) + \mathcal{T}$. Since $a +_{\mathbb{Z}} b \in \mathbb{Z}$, by Case A2, this equals $a +_{\mathbb{Z}} b$. RHS: $a + (b + c) = a + (b + \mathcal{T})$. By Case A2, $b + \mathcal{T} = b$. RHS $= a + b = a +_{\mathbb{Z}} b$.

Case I.2.3: $a \in \mathbb{Z}, b = \mathcal{T}, c \in \mathbb{Z}$. LHS: $(a + b) + c = (a + \mathcal{T}) + c$. By Case A2, $a + \mathcal{T} = a$. LHS $= a + c = a +_{\mathbb{Z}} c$. RHS: $a + (b + c) = a + (\mathcal{T} + c)$. By Case A3, $\mathcal{T} + c = c$. RHS $= a + c = a +_{\mathbb{Z}} c$.

Case I.2.4: $a = \mathcal{T}, b \in \mathbb{Z}, c \in \mathbb{Z}$. LHS: $(a + b) + c = (\mathcal{T} + b) + c$. By Case A3, $\mathcal{T} + b = b$. LHS $= b + c = b +_{\mathbb{Z}} c$. RHS: $a + (b + c) = \mathcal{T} + (b +_{\mathbb{Z}} c)$. Since $b +_{\mathbb{Z}} c \in \mathbb{Z}$, by Case A3, this equals $b +_{\mathbb{Z}} c$.

Case I.2.5: $a \in \mathbb{Z}, b = \mathcal{T}, c = \mathcal{T}$. LHS: $(a + \mathcal{T}) + \mathcal{T} = a + \mathcal{T} = a$. RHS: $a + (\mathcal{T} + \mathcal{T}) = a + \mathcal{T} = a$.

Case I.2.6: $a = \mathcal{T}, b \in \mathbb{Z}, c = \mathcal{T}$. LHS: $(\mathcal{T} + b) + \mathcal{T} = b + \mathcal{T} = b$. RHS: $\mathcal{T} + (b + \mathcal{T}) = \mathcal{T} + b = b$.

Case I.2.7: $a = \mathcal{T}, b = \mathcal{T}, c \in \mathbb{Z}$. LHS: $(\mathcal{T} + \mathcal{T}) + c = \mathcal{T} + c = c$. RHS: $\mathcal{T} + (\mathcal{T} + c) = \mathcal{T} + c = c$.

Case I.2.8: $a = b = c = \mathcal{T}$. LHS: $(\mathcal{T} + \mathcal{T}) + \mathcal{T} = \mathcal{T}$. RHS: $\mathcal{T} + (\mathcal{T} + \mathcal{T}) = \mathcal{T}$.

I.3. Additive Identity. By inspection of Definition 3.2 (Cases A2, A3, A4), the element $\mathcal{T}$ satisfies $a + \mathcal{T} = a$ and $\mathcal{T} + a = a$ for all $a \in \mathcal{S}$. By Lemma 2.3, the identity is unique. Thus $0_{\mathcal{S}} = \mathcal{T}$. $\mathcal{S}$ is a hemiring.

Part II: $(\mathcal{S}, \times)$ is a commutative monoid.

II.1. Commutativity. We must verify $a \times b = b \times a$. Case II.1.1: $a, b \in \mathbb{Z}$. Equality holds by commutativity of $\times_{\mathbb{Z}}$ (Theorem 1.13). Case II.1.2: $a \in \mathbb{Z}, b = \mathcal{T}$. $a \times b = \mathcal{T}$ (Case M2). $b \times a = \mathcal{T}$ (Case M3). Case II.1.3: $a = \mathcal{T}, b = \mathcal{T}$. $a \times b = \mathcal{T}$.

II.2. Associativity. We must verify $(a \times b) \times c = a \times (b \times c)$.

Case II.2.1: $a, b, c \in \mathbb{Z}$. Equality holds by inheritance from the associativity of $\times_{\mathbb{Z}}$.

Case II.2.2: At least one element is $\mathcal{T}$. We observe from Definition 3.3 (Cases M2, M3, M4) that $\mathcal{T}$ is a multiplicative absorbing element (Definition 2.2). If $a = \mathcal{T}$. LHS: $(\mathcal{T} \times b) \times c = \mathcal{T} \times c = \mathcal{T}$. RHS: $\mathcal{T} \times (b \times c) = \mathcal{T}$. If $b = \mathcal{T}$. LHS: $(a \times \mathcal{T}) \times c = \mathcal{T} \times c = \mathcal{T}$. RHS: $a \times (\mathcal{T} \times c) = a \times \mathcal{T} = \mathcal{T}$. If $c = \mathcal{T}$. LHS: $(a \times b) \times \mathcal{T} = \mathcal{T}$. RHS: $a \times (b \times \mathcal{T}) = a \times \mathcal{T} = \mathcal{T}$.

II.3. Multiplicative Identity. We determine the multiplicative identity $1_{\mathcal{S}}$. We claim $1_{\mathbb{Z}}$ is the identity. Let $a \in \mathcal{S}$. If $a \in \mathbb{Z}$. $1_{\mathbb{Z}} \times a = 1_{\mathbb{Z}} \times_{\mathbb{Z}} a = a$ (Case M1). If $a = \mathcal{T}$. $1_{\mathbb{Z}} \times \mathcal{T} = \mathcal{T}$ (Case M2). Thus $1_{\mathcal{S}} = 1_{\mathbb{Z}}$.

Part III: Distributivity. We must verify $a \times (b + c) = (a \times b) + (a \times c)$. By commutativity (Part II.1), this suffices.

Case III.1: $a, b, c \in \mathbb{Z}$. Equality holds by distributivity in the ring $\mathbb{Z}$.

Case III.2: $a \in \mathbb{Z}$. Subcase III.2.a: $b \in \mathbb{Z}, c = \mathcal{T}$. LHS: $a \times (b + \mathcal{T}) = a \times b$. RHS: $(a \times b) + (a \times \mathcal{T}) = (a \times b) + \mathcal{T}$. Since $a \times b \in \mathbb{Z}$, this equals $a \times b$. Subcase III.2.b: $b = \mathcal{T}, c \in \mathbb{Z}$. Symmetric. Subcase III.2.c: $b = \mathcal{T}, c = \mathcal{T}$. LHS: $a \times (\mathcal{T} + \mathcal{T}) = a \times \mathcal{T} = \mathcal{T}$. RHS: $(a \times \mathcal{T}) + (a \times \mathcal{T}) = \mathcal{T} + \mathcal{T} = \mathcal{T}$.

Case III.3: $a = \mathcal{T}$. LHS: $\mathcal{T} \times (b + c) = \mathcal{T}$. RHS: $(\mathcal{T} \times b) + (\mathcal{T} \times c) = \mathcal{T} + \mathcal{T} = \mathcal{T}$.

Part IV: Standard Semiring (Annihilation by Zero). The additive identity is $0_{\mathcal{S}} = \mathcal{T}$. The multiplicative absorbing property $a \times 0_{\mathcal{S}} = 0_{\mathcal{S}}$ was verified in Part II.2.2.

All axioms are satisfied. □

4 Algebraic Properties of the Semiring $\mathcal{S}$

4.1 Basic Element Properties

Proposition 4.1. *The additive idempotents of $\mathcal{S}$ ($x + x = x$) are $\{0_\mathbb{Z}, \mathcal{T}\}$. The multiplicative idempotents of $\mathcal{S}$ ($x \times x = x$) are $\{0_\mathbb{Z}, 1_\mathbb{Z}, \mathcal{T}\}$.*

Proof. 1. Additive idempotents. Case 1: $x \in \mathbb{Z}$. $x + x = 2x$. We require $2x = x$. In $\mathbb{Z}$, this implies $x = 0_\mathbb{Z}$. Case 2: $x = \mathcal{T}$. $\mathcal{T} + \mathcal{T} = \mathcal{T}$.

2. Multiplicative idempotents. Case 1: $x \in \mathbb{Z}$. $x \times x = x^2$. We require $x^2 = x$. In the integral domain $\mathbb{Z}$, this implies $x = 0_\mathbb{Z}$ or $x = 1_\mathbb{Z}$. Case 2: $x = \mathcal{T}$. $\mathcal{T} \times \mathcal{T} = \mathcal{T}$. □

Proposition 4.2. *The group of units of $\mathcal{S}$ is $U(\mathcal{S}) = \{1_\mathbb{Z}, -1_\mathbb{Z}\}$.*

Proof. We seek $x, y \in \mathcal{S}$ such that $x \times y = 1_\mathcal{S} = 1_\mathbb{Z}$. If $x = \mathcal{T}$ or $y = \mathcal{T}$, then $x \times y = \mathcal{T}$. Since $1_\mathbb{Z} \neq \mathcal{T}$ (by Construction 3.1), neither x nor y can be $\mathcal{T}$. Thus $x, y \in \mathbb{Z}$. The condition becomes $x \times_\mathbb{Z} y = 1_\mathbb{Z}$. This implies $x \in U(\mathbb{Z})$. By Lemma 2.20, $U(\mathbb{Z}) = \{1_\mathbb{Z}, -1_\mathbb{Z}\}$. □

Proposition 4.3. *The semiring $\mathcal{S}$ is an integral semidomain.*

Proof. We verify the conditions of Definition 2.6. $\mathcal{S}$ is commutative and unital. $1_\mathbb{Z} \neq \mathcal{T}$. We check for zero divisors. We analyze the equation $a \times b = 0_\mathcal{S} = \mathcal{T}$. Case 1: $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$. $a \times b = a \times_\mathbb{Z} b$. Since $a, b \in \mathbb{Z}$, the product $a \times_\mathbb{Z} b \in \mathbb{Z}$. Thus $a \times b \neq \mathcal{T}$. Therefore, the condition $a \times b = \mathcal{T}$ implies that at least one of a or b must not be in $\mathbb{Z}$. This means $a = \mathcal{T}$ or $b = \mathcal{T}$. $\mathcal{S}$ has no zero divisors. □

Proposition 4.4. *The semiring $\mathcal{S}$ is zerosumfree.*

Proof. We analyze the condition $a + b = 0_\mathcal{S} = \mathcal{T}$. Case 1: $a, b \in \mathbb{Z}$. $a + b = a +_\mathbb{Z} b \in \mathbb{Z}$. Thus $a + b \neq \mathcal{T}$. Case 2: $a \in \mathbb{Z}, b = \mathcal{T}$. $a + b = a$. We require $a = \mathcal{T}$. This contradicts $a \in \mathbb{Z}$. Case 3: $a = \mathcal{T}, b \in \mathbb{Z}$. $a + b = b$. We require $b = \mathcal{T}$. Contradiction. Case 4: $a = \mathcal{T}, b = \mathcal{T}$. $a + b = \mathcal{T}$. The only solution to $a + b = \mathcal{T}$ is $a = \mathcal{T}$ and $b = \mathcal{T}$. Thus $\mathcal{S}$ is zerosumfree. □

4.2 The Ideal Structure of $\mathcal{S}$

We characterize the ideals of $\mathcal{S}$, establishing a correspondence with the ideals of $\mathbb{Z}$.

Theorem 4.5. *The ideals of $\mathcal{S}$ are precisely the following sets:*

1. *The zero ideal $I_0 = \{\mathcal{T}\}$.*
2. *The sets of the form $I_J = J \cup \{\mathcal{T}\}$, where J is a (non-empty) ideal of the ring $\mathbb{Z}$.*

Proof. Part I: Characterization of an arbitrary ideal. Let I be an ideal of $\mathcal{S}$. Since $\mathcal{T}$ is the multiplicative absorber in $\mathcal{S}$ (Theorem 3.4), by Lemma 2.10, $\mathcal{T} \in I$. Define $J = I \cap \mathbb{Z}$. Then $I = (I \cap \mathbb{Z}) \cup (I \cap \{\mathcal{T}\}) = J \cup \{\mathcal{T}\}$.

Case 1: $J = \emptyset$. Then $I = \emptyset \cup \{\mathcal{T}\} = \{\mathcal{T}\} = I_0$.

Case 2: $J \neq \emptyset$. We demonstrate that J is an ideal of the ring $\mathbb{Z}$. (i) Closure under $+_\mathbb{Z}$. Let $a, b \in J$. Since $J \subset I$, $a, b \in I$. Since I is an ideal of $\mathcal{S}$, $a + b \in I$. Since $a, b \in \mathbb{Z}$, $a + b = a +_\mathbb{Z} b$ (Case A1), and $a +_\mathbb{Z} b \in \mathbb{Z}$. Thus $a +_\mathbb{Z} b \in I \cap \mathbb{Z} = J$. (ii) Absorption under $\times_\mathbb{Z}$. Let $r \in \mathbb{Z}$ and $a \in J$. $r \in \mathcal{S}$ and $a \in I$. Thus $r \times a \in I$. Since $r, a \in \mathbb{Z}$, $r \times a = r \times_\mathbb{Z} a$ (Case M1), and $r \times_\mathbb{Z} a \in \mathbb{Z}$. Thus $r \times_\mathbb{Z} a \in I \cap \mathbb{Z} = J$. A non-empty subset $J \subseteq \mathbb{Z}$ satisfying these conditions is an ideal of $\mathbb{Z}$.

Part II: Verification that the forms define ideals. 1. $I_0 = \{\mathcal{T}\}$. Additive closure: $\mathcal{T} + \mathcal{T} = \mathcal{T}$. Absorption: $r \times \mathcal{T} = \mathcal{T}$. I_0 is an ideal.

2. Let J be an ideal of $\mathbb{Z}$. (Ideals of $\mathbb{Z}$ are non-empty, as $0_\mathbb{Z} \in J$). Let $I_J = J \cup \{\mathcal{T}\}$. (i) Additive closure. Let $a, b \in I_J$. If $a, b \in J$, $a + b \in J \subset I_J$. If $a \in J, b = \mathcal{T}$, $a + \mathcal{T} = a \in J \subset I_J$. If $a = \mathcal{T}, b = \mathcal{T}$, $a + b = \mathcal{T} \in I_J$.

(ii) Absorption by $\mathcal{S}$. Let $r \in \mathcal{S}, a \in I_J$. If $r \in \mathbb{Z}$. If $a \in J$, $r \times a = r \times_\mathbb{Z} a \in J$. If $a = \mathcal{T}$, $r \times a = \mathcal{T}$. If $r = \mathcal{T}$. $r \times a = \mathcal{T}$. I_J is an ideal of $\mathcal{S}$. □

Theorem 4.6. *The semiring $\mathcal{S}$ is a Principal Ideal Semiring (PIS).*

Proof. We utilize the characterization in Theorem 4.5 and the fact that $\mathbb{Z}$ is a PID (Theorem 1.13).

1. The zero ideal I_0. It is generated by $\mathcal{T}$. $(\mathcal{T})_{\mathcal{S}} = \mathcal{S} \times \mathcal{T} = \{\mathcal{T}\} = I_0$.

2. Ideals I_J. $J = (n)_{\mathbb{Z}} = n\mathbb{Z}$ for some $n \geq 0$. We compute the principal ideal generated by n in $\mathcal{S}$. $(n)_{\mathcal{S}} = \{r \times n \mid r \in \mathcal{S}\}$. If $r \in \mathbb{Z}$: $r \times n = r \times_{\mathbb{Z}} n$. The collection is $n\mathbb{Z} = J$. If $r = \mathcal{T}$: $\mathcal{T} \times n = \mathcal{T}$. Thus, $(n)_{\mathcal{S}} = J \cup \{\mathcal{T}\} = I_J$.

Since every ideal is principal, $\mathcal{S}$ is a PIS. □

Corollary 4.7. *The semiring $\mathcal{S}$ is Noetherian.*

Proof. We must show that $\mathcal{S}$ satisfies the Ascending Chain Condition (ACC) on ideals. Let $I_1 \subseteq I_2 \subseteq \dots$ be an ascending chain of ideals in $\mathcal{S}$. By Theorem 4.5, each I_k corresponds to a subset $J_k = I_k \cap \mathbb{Z}$. The inclusion $I_k \subseteq I_{k+1}$ implies $J_k \subseteq J_{k+1}$. If all $I_k = I_0$, the chain stabilizes. If $I_N \neq I_0$ for some N, then for $k \geq N$, $I_k = I_{J_k}$ where J_k is an ideal of $\mathbb{Z}$. Thus we have an ascending chain of ideals $J_N \subseteq J_{N+1} \subseteq \dots$ in $\mathbb{Z}$. Since $\mathbb{Z}$ is Noetherian, the chain J_k must stabilize; there exists $M \geq N$ such that $J_k = J_M$ for all $k \geq M$. Consequently, $I_k = J_k \cup \{\mathcal{T}\} = J_M \cup \{\mathcal{T}\} = I_M$ for all $k \geq M$. The chain in $\mathcal{S}$ stabilizes. □

4.3 Subtractive Ideals in $\mathcal{S}$

Theorem 4.8. *Every ideal of the semiring $\mathcal{S}$ is subtractive.*

Proof. Let I be an ideal of $\mathcal{S}$. We must show that if $a \in I$ and $a + b \in I$, then $b \in I$ (Definition 2.12).

Case 1: $I = I_0 = \{\mathcal{T}\}$. If $a \in I_0$, then $a = \mathcal{T}$. If $a + b \in I_0$, then $\mathcal{T} + b = \mathcal{T}$. By the identity property of $\mathcal{T}$, $\mathcal{T} + b = b$. Thus $b = \mathcal{T} \in I_0$.

Case 2: $I = I_J = J \cup \{\mathcal{T}\}$. Let $a \in I_J$ and $a + b \in I_J$.

Subcase 2a: $a = \mathcal{T}$. Then $a + b = \mathcal{T} + b = b$. Since $a + b \in I_J$, we have $b \in I_J$.

Subcase 2b: $a \in J$. (Note $J \subset \mathbb{Z}$). We examine the possibilities for $b \in \mathbb{Z} \cup \{\mathcal{T}\}$.

(i) $b = \mathcal{T}$. Then $b \in I_J$.

(ii) $b \in \mathbb{Z}$. The addition is $a + b = a +_{\mathbb{Z}} b$. Since $a, b \in \mathbb{Z}$, $a +_{\mathbb{Z}} b \in \mathbb{Z}$. We are given $a + b \in I_J$. Since $a + b \in \mathbb{Z}$, we must have $a + b \in I_J \cap \mathbb{Z} = J$. We have $a \in J$ and $a +_{\mathbb{Z}} b \in J$. Since J is an ideal of the ring $\mathbb{Z}$, it is an additive subgroup of $(\mathbb{Z}, +_{\mathbb{Z}})$. Therefore, the difference $(a +_{\mathbb{Z}} b) -_{\mathbb{Z}} a$ must belong to J. $(a +_{\mathbb{Z}} b) -_{\mathbb{Z}} a = b$. Thus $b \in J$. Since $J \subset I_J$, $b \in I_J$.

In all cases, $b \in I$. Thus every ideal I is subtractive. □

5 Factorization and Divisibility in $\mathcal{S}$

We examine concepts of divisibility and factorization within the integral semidomain $\mathcal{S}$. This analysis reveals distinctions between the factorization theory of $\mathcal{S}$ and that of standard integral domains (rings), particularly concerning the element $0_{\mathbb{Z}}$.

5.1 Divisibility and Associates

We analyze the divisibility relation (Definition 2.18.1) in $\mathcal{S}$. Recall $0_{\mathcal{S}} = \mathcal{T}$.

Lemma 5.1. *Let $a, b \in \mathcal{S}$.*

1. *$\mathcal{T} | a$ if and only if $a = \mathcal{T}$.*
2. *$a | \mathcal{T}$ for all $a \in \mathcal{S}$.*
3. *If $a, b \in \mathbb{Z}$ and $a \neq 0_{\mathbb{Z}}$. Then $a|_{\mathcal{S}} b$ if and only if $a|_{\mathbb{Z}} b$.*
4. *$0_{\mathbb{Z}} | a$ if and only if $a \in \{0_{\mathbb{Z}}, \mathcal{T}\}$.*
5. *If $a \in \mathbb{Z} \setminus \{0_{\mathbb{Z}}\}$, then $a | 0_{\mathbb{Z}}$.*

Proof. 1. If $\mathcal{T}|a$, then $a = \mathcal{T} \times c$. By the absorbing property of $\mathcal{T}$, $a = \mathcal{T}$. Conversely, $\mathcal{T}|\mathcal{T}$ since $\mathcal{T} = \mathcal{T} \times \mathcal{T}$.

2. $\mathcal{T} = a \times \mathcal{T}$ by the absorbing property of $\mathcal{T}$. Thus $a|\mathcal{T}$.

3. Let $a, b \in \mathbb{Z}, a \neq 0_{\mathbb{Z}}$. ($\Longleftarrow$) If $a|_{\mathbb{Z}}b$, $b = ac$ for $c \in \mathbb{Z}$. Since $\mathbb{Z} \subset \mathcal{S}$, $a|_{\mathcal{S}}b$. ($\Longrightarrow$) If $a|_{\mathcal{S}}b$, $b = ac$ for $c \in \mathcal{S}$. If $c = \mathcal{T}$, $b = a\mathcal{T} = \mathcal{T}$ (Case M2). This contradicts $b \in \mathbb{Z}$. So $c \in \mathbb{Z}$. $b = a \times_{\mathbb{Z}} c$. Thus $a|_{\mathbb{Z}}b$.

4. $0_{\mathbb{Z}}|a$ means $a = 0_{\mathbb{Z}} \times c$ for some $c \in \mathcal{S}$. If $c \in \mathbb{Z}$. $a = 0_{\mathbb{Z}} \times_{\mathbb{Z}} c = 0_{\mathbb{Z}}$ (Case M1). If $c = \mathcal{T}$. $a = 0_{\mathbb{Z}} \times \mathcal{T} = \mathcal{T}$ (Case M2). Thus the set of elements divisible by $0_{\mathbb{Z}}$ is $\{0_{\mathbb{Z}}, \mathcal{T}\}$.

5. Let $a \in \mathbb{Z} \setminus \{0_{\mathbb{Z}}\}$. $0_{\mathbb{Z}} = a \times_{\mathbb{Z}} 0_{\mathbb{Z}}$ (Case M1). Thus $a|0_{\mathbb{Z}}$ in $\mathcal{S}$. □

We analyze the associates (Definition 2.18.2).

Proposition 5.2. *Elements $a, b \in \mathcal{S}$ are associates if and only if $a = ub$ for some unit $u \in U(\mathcal{S}) = \{\pm 1_{\mathbb{Z}}\}$.*

Proof. ($\Longleftarrow$) If $a = ub$ where u is a unit, then $b|a$ and $b = u^{-1}a$, so $a|b$.

($\Longrightarrow$) Assume $a|b$ and $b|a$. $b = ac_1, a = bc_2$. $a = ac_1c_2$.

Case 1: $a = \mathcal{T}$. By Lemma 5.1.1, $a|b \implies \mathcal{T}|b \implies b = \mathcal{T}$. So $a = b = \mathcal{T}$. $a = 1_{\mathbb{Z}}b$.

Case 2: $a \in \mathbb{Z}, a \neq 0_{\mathbb{Z}}$. We show b must also be in $\mathbb{Z} \setminus \{0_{\mathbb{Z}}\}$. If $b = \mathcal{T}$. $a|\mathcal{T}$ holds. $\mathcal{T}|a$ implies $a = \mathcal{T}$, contradiction. If $b = 0_{\mathbb{Z}}$. $a|0_{\mathbb{Z}}$ holds (Lemma 5.1.5). $0_{\mathbb{Z}}|a$ implies $a \in \{0_{\mathbb{Z}}, \mathcal{T}\}$ (Lemma 5.1.4), contradiction. So $b \in \mathbb{Z} \setminus \{0_{\mathbb{Z}}\}$. Since $a, b \in \mathbb{Z} \setminus \{0_{\mathbb{Z}}\}$, $a|_{\mathcal{S}}b \iff a|_{\mathbb{Z}}b$ (Lemma 5.1.3). Thus a, b are associates in $\mathbb{Z}$. $a = \pm b$.

Case 3: $a = 0_{\mathbb{Z}}$. We require $0_{\mathbb{Z}}|b$ and $b|0_{\mathbb{Z}}$. $0_{\mathbb{Z}}|b$ implies $b \in \{0_{\mathbb{Z}}, \mathcal{T}\}$. If $b = \mathcal{T}$. We check if $\mathcal{T}|0_{\mathbb{Z}}$. This requires $0_{\mathbb{Z}} = \mathcal{T} \times c = \mathcal{T}$. Contradiction. If $b = 0_{\mathbb{Z}}$. $0_{\mathbb{Z}}|0_{\mathbb{Z}}$ holds. Thus $0_{\mathbb{Z}}$ is associated only with itself. $0_{\mathbb{Z}} = 1_{\mathbb{Z}} \times 0_{\mathbb{Z}}$. The characterization holds in all cases. □

5.2 Irreducibles and Primes

We analyze the concepts of irreducible and prime elements in $\mathcal{S}$. The set of non-zero ($\neq \mathcal{T}$), non-unit elements in $\mathcal{S}$ is $\mathbb{Z} \setminus \{1_{\mathbb{Z}}, -1_{\mathbb{Z}}\}$.

Proposition 5.3. *The irreducible elements of $\mathcal{S}$ are precisely the elements associated with prime numbers in $\mathbb{Z}$ (i.e., elements $\pm p$, where p is a prime number). The element $0_{\mathbb{Z}}$ is reducible.*

Proof. Let x be a non-zero, non-unit element.

Case 1: $x = 0_{\mathbb{Z}}$. We examine if $0_{\mathbb{Z}}$ is irreducible. We seek a factorization $0_{\mathbb{Z}} = ab$ where a, b are not units. Consider $a = 2$ and $b = 0_{\mathbb{Z}}$. Both a and b are not units in $\mathcal{S}$. We verify the product: $2 \times 0_{\mathbb{Z}}$. Since both are in $\mathbb{Z}$, the product is $2 \times_{\mathbb{Z}} 0_{\mathbb{Z}} = 0_{\mathbb{Z}}$ (Case M1). Thus $0_{\mathbb{Z}}$ admits a factorization into non-units. Therefore, $0_{\mathbb{Z}}$ is reducible.

Case 2: $x \in \mathbb{Z} \setminus \{0_{\mathbb{Z}}, 1_{\mathbb{Z}}, -1_{\mathbb{Z}}\}$. Suppose $x = ab$ in $\mathcal{S}$. Since $x \in \mathbb{Z}$ and $x \neq \mathcal{T}$, we must have $a, b \in \mathbb{Z}$ (otherwise the product would be $\mathcal{T}$). The factorization $x = ab$ in $\mathcal{S}$ is equivalent to the factorization $x = a \times_{\mathbb{Z}} b$ in $\mathbb{Z}$. The units of $\mathcal{S}$ are exactly the units of $\mathbb{Z}$. Thus x is irreducible in $\mathcal{S}$ if and only if it is irreducible in $\mathbb{Z}$. The irreducibles in $\mathbb{Z}$ are precisely $\pm p$ where p is a prime number. □

Proposition 5.4. *The prime elements of $\mathcal{S}$ are precisely the irreducible elements ($\pm p$) and the element $0_{\mathbb{Z}}$.*

Proof. We check the definition of prime (Definition 2.18.4).

1. Irreducibles $x = \pm p$. We show x is prime. Suppose $x|ab$ in $\mathcal{S}$. Case 1a: $a, b \in \mathbb{Z}$. If $ab \neq 0_{\mathbb{Z}}$. Then $x|_{\mathcal{S}}ab$ is equivalent to $x|_{\mathbb{Z}}ab$ (Lemma 5.1.3). Since x is prime in $\mathbb{Z}$, $x|_{\mathbb{Z}}a$ or $x|_{\mathbb{Z}}b$. Thus $x|_{\mathcal{S}}a$ or $x|_{\mathcal{S}}b$. If $ab = 0_{\mathbb{Z}}$. Then $a = 0_{\mathbb{Z}}$ or $b = 0_{\mathbb{Z}}$. By Lemma 5.1.5, $x|0_{\mathbb{Z}}$. Thus $x|a$ or $x|b$. Case 1b: $a = \mathcal{T}$ or $b = \mathcal{T}$. Then $ab = \mathcal{T}$. $x|\mathcal{T}$ holds (Lemma 5.1.2). Also $x|a$ or $x|b$. Thus $\pm p$ are prime in $\mathcal{S}$.

2. Reducible element $x = 0_{\mathbb{Z}}$. We show $0_{\mathbb{Z}}$ is prime. Suppose $0_{\mathbb{Z}}|ab$. By Lemma 5.1.4, this means $ab \in \{0_{\mathbb{Z}}, \mathcal{T}\}$.

Case 2a: $ab = 0_{\mathbb{Z}}$. This implies $a, b \in \mathbb{Z}$ and $a \times_{\mathbb{Z}} b = 0_{\mathbb{Z}}$. Since $\mathbb{Z}$ is an integral domain, $a = 0_{\mathbb{Z}}$ or $b = 0_{\mathbb{Z}}$. If $a = 0_{\mathbb{Z}}$, then $0_{\mathbb{Z}}|a$. If $b = 0_{\mathbb{Z}}$, then $0_{\mathbb{Z}}|b$.

Case 2b: $ab = \mathcal{T}$. By Proposition 4.3, this implies $a = \mathcal{T}$ or $b = \mathcal{T}$. If $a = \mathcal{T}$. By Lemma 5.1.4, $0_{\mathbb{Z}}|\mathcal{T}$. So $0_{\mathbb{Z}}|a$. Similarly if $b = \mathcal{T}$. Thus $0_{\mathbb{Z}}$ is a prime element in $\mathcal{S}$. □

Theorem 5.5. *In the integral semidomain $\mathcal{S}$, there exist prime elements that are not irreducible.*

Proof. The element $0_{\mathbb{Z}}$ is prime by Proposition 5.4. The element $0_{\mathbb{Z}}$ is reducible by Proposition 5.3. □

This behavior contrasts with that of integral domains (rings), where every prime element is irreducible. The distinction arises because the cancellation law does not hold universally for the element $0_{\mathbb{Z}}$ in $\mathcal{S}$ (e.g., $0_{\mathbb{Z}} \times 1_{\mathbb{Z}} = 0_{\mathbb{Z}} \times 2_{\mathbb{Z}}$ but $1_{\mathbb{Z}} \neq 2_{\mathbb{Z}}$).

5.3 Unique Factorization Domains (UFDs)

Theorem 5.6. *The semiring $\mathcal{S}$ is not a Unique Factorization Domain (UFD).*

Proof. We demonstrate that the existence condition (Definition 2.18.5) fails in $\mathcal{S}$. Consider the element $0_{\mathbb{Z}}$. It is non-zero ($0_{\mathbb{Z}} \neq \mathcal{T}$) and non-unit ($0_{\mathbb{Z}} \neq \pm 1_{\mathbb{Z}}$). We examine whether $0_{\mathbb{Z}}$ can be expressed as a product of irreducible elements. Suppose, for the sake of contradiction, that $0_{\mathbb{Z}}$ possesses such a factorization: $0_{\mathbb{Z}} = x_1 x_2 \ldots x_k$, where x_i are irreducible elements of $\mathcal{S}$. By Proposition 5.3, the irreducible elements are $x_i = \pm p_i$ for prime numbers p_i. In particular, $x_i \in \mathbb{Z}$ and $x_i \neq 0_{\mathbb{Z}}$. The product $P = x_1 \ldots x_k$ in $\mathcal{S}$ is the product in $\mathbb{Z}$: $P = x_1 \times_{\mathbb{Z}} \cdots \times_{\mathbb{Z}} x_k$. Since $\mathbb{Z}$ is an integral domain and $x_i \neq 0_{\mathbb{Z}}$, their product is non-zero: $P \neq 0_{\mathbb{Z}}$. This contradicts the assumption that the product equals $0_{\mathbb{Z}}$. Therefore, $0_{\mathbb{Z}}$ cannot be factored into a product of irreducible elements. The existence condition for a UFD is violated. □

6 The Spectrum and Topology of $\mathcal{S}$

6.1 The Spectrum of Prime Ideals

We determine the set of prime ideals $\mathrm{Spec}(\mathcal{S})$ and analyze the poset structure under inclusion to determine the Krull dimension.

Theorem 6.1. *The spectrum* $\mathrm{Spec}(\mathcal{S})$ *consists precisely of the following ideals:*

1. *The zero ideal $P_{\mathcal{T}} = \{\mathcal{T}\}$.*
2. *The ideal $P_{(0)} = \{0_{\mathbb{Z}}, \mathcal{T}\}$.*
3. *The ideals $P_{(p)} = p\mathbb{Z} \cup \{\mathcal{T}\}$, where p is a prime number in $\mathbb{Z}$.*

Proof. We utilize the characterization of ideals from Theorem 4.5.

1. The zero ideal $I_0 = \{\mathcal{T}\}$. I_0 is proper since $1_{\mathbb{Z}} \notin I_0$. Let $a \times b \in I_0$, so $a \times b = \mathcal{T}$. By Proposition 4.3, this implies $a = \mathcal{T}$ or $b = \mathcal{T}$. Thus I_0 is prime. We denote it $P_{\mathcal{T}}$.

2. Ideals of the form $I_J = J \cup \{\mathcal{T}\}$. I_J is proper if and only if J is a proper ideal of $\mathbb{Z}$. We establish the equivalence: I_J is prime in $\mathcal{S}$ if and only if J is prime in $\mathbb{Z}$.

($\Longrightarrow$) Assume I_J is prime in $\mathcal{S}$. Let $a, b \in \mathbb{Z}$ such that $a \times_{\mathbb{Z}} b \in J$. Then $a \times b \in I_J$. Since I_J is prime, $a \in I_J$ or $b \in I_J$. Since $a, b \in \mathbb{Z}$, this means $a \in J$ or $b \in J$. Thus J is prime in $\mathbb{Z}$.

($\Longleftarrow$) Assume J is prime in $\mathbb{Z}$. Let $a, b \in \mathcal{S}$ such that $a \times b \in I_J$. If $a = \mathcal{T}$ or $b = \mathcal{T}$. Then $a \in I_J$ or $b \in I_J$. If $a, b \in \mathbb{Z}$. Then $a \times b = a \times_{\mathbb{Z}} b$. Since $a \times b \in \mathbb{Z}$, $a \times b \in I_J$ implies $a \times b \in J$. Since J is prime in $\mathbb{Z}$, $a \in J$ or $b \in J$. Thus $a \in I_J$ or $b \in I_J$.

The prime ideals of $\mathbb{Z}$ are the zero ideal $(0) = \{0_{\mathbb{Z}}\}$ and the principal ideals $(p) = p\mathbb{Z}$. If $J = (0)$, $I_J = P_{(0)}$. If $J = (p)$, $I_J = P_{(p)}$. □

Theorem 6.2. *The maximal spectrum* $\mathrm{MaxSpec}(\mathcal{S})$ *consists of the ideals $P_{(p)}$. The minimal spectrum* $\mathrm{MinSpec}(\mathcal{S})$ *consists solely of $P_{\mathcal{T}}$.*

Proof. We analyze the inclusions among the prime ideals identified in Theorem 6.1. (i) $P_{\mathcal{T}} \subsetneq P_{(0)}$. $0_{\mathbb{Z}} \in P_{(0)}$ and $0_{\mathbb{Z}} \notin P_{\mathcal{T}}$. (ii) $P_{(0)} \subsetneq P_{(p)}$. Since $0_{\mathbb{Z}} \in p\mathbb{Z}$, $P_{(0)} \subseteq P_{(p)}$. The inclusion is strict because $p \in P_{(p)}$ and $p \notin P_{(0)}$ (as $p \neq 0_{\mathbb{Z}}$). (iii) $P_{(p)} \subseteq P_{(q)}$ if and only if $p\mathbb{Z} \subseteq q\mathbb{Z}$. This occurs iff $q|p$. Since p, q are prime, $p = q$.

The ideals $P_{(p)}$ are the maximal elements. $P_{\mathcal{T}}$ is the unique minimum element, as it is contained in every ideal. □

Theorem 6.3. *The Krull dimension of the semiring $\mathcal{S}$ is 2.*

Proof. Based on the inclusion analysis in the proof of Theorem 6.2, the maximal chains of distinct prime ideals are of the form:

$$P_{\mathcal{T}} \subsetneq P_{(0)} \subsetneq P_{(p)}. \tag{6.1}$$

The length of this chain is $n = 2$. Therefore, $\mathrm{Kdim}(\mathcal{S}) = 2$. □

6.2 The Zariski Topology on $\mathrm{Spec}(\mathcal{S})$

Definition 6.4. The Zariski topology on $\mathrm{Spec}(R)$ is defined by the collection of closed sets $\mathcal{V} = \{V(I) \mid I \text{ is an ideal of } R\}$, where $V(I) = \{P \in \mathrm{Spec}(R) \mid I \subseteq P\}$.

We characterize the closed sets in $\mathrm{Spec}(\mathcal{S})$.

Lemma 6.5. *Let I be an ideal of $\mathcal{S}$. The closed sets $V(I)$ are characterized as follows:*

1. *If $I = P_{\mathcal{T}}$ (the zero ideal), then $V(I) = \mathrm{Spec}(\mathcal{S})$.*
2. *If $I = I_J$ where $J = (n)_{\mathbb{Z}}$, $n \geq 0$.*
   (a) *If $n = 0$, $I = P_{(0)}$. $V(I) = \{P_{(0)}\} \cup \{P_{(p)} \mid p \text{ prime}\}$.*
   (b) *If $n = 1$, $I = \mathcal{S}$. $V(I) = \emptyset$.*
   (c) *If $n > 1$. $V(I) = \{P_{(p)} \mid p \text{ is a prime divisor of } n\}$.*

Proof. We analyze the condition $I \subseteq P$.
1. $I = P_{\mathcal{T}}$. $P_{\mathcal{T}}$ is contained in every ideal. $V(P_{\mathcal{T}}) = \mathrm{Spec}(\mathcal{S})$.
2. $I = I_J$. $I_J \subseteq P$ is equivalent to $J \subseteq P$.
2a. $J = (0)$. $I = P_{(0)}$. $P_{(0)} \subseteq P$. This excludes $P = P_{\mathcal{T}}$ (since $0_{\mathbb{Z}} \notin P_{\mathcal{T}}$). It includes $P = P_{(0)}$ and $P = P_{(p)}$.
2b. $J = (1)$. $I = \mathcal{S}$. $V(\mathcal{S}) = \emptyset$ because prime ideals must be proper.
2c. $J = (n), n > 1$. $J \subseteq P_{\mathcal{T}}$ is impossible as J is non-empty ($0_{\mathbb{Z}} \in J$). $J \subseteq P_{(0)}$ implies $(n) \subseteq (0)$, so $n = 0$, impossible. $J \subseteq P_{(p)}$ implies $(n) \subseteq (p)$, equivalent to $p|n$. □

Theorem 6.6. *The topological space $\mathrm{Spec}(\mathcal{S})$ exhibits the following properties:*

1. *The closed points are the maximal ideals $P_{(p)}$.*
2. *The point $P_{\mathcal{T}}$ is the unique generic point. Consequently, $\mathrm{Spec}(\mathcal{S})$ is irreducible.*
3. *$\mathrm{Spec}(\mathcal{S})$ is a Noetherian topological space.*
4. *$\mathrm{Spec}(\mathcal{S})$ is T_0 but not T_1.*

Proof. 1. A point P is closed if $\{P\}$ is a closed set. This occurs if and only if P is maximal. By Theorem 6.2, these are the $P_{(p)}$.
2. The closure of $\{P_{\mathcal{T}}\}$ is $V(P_{\mathcal{T}}) = \mathrm{Spec}(\mathcal{S})$ (Lemma 6.5.1). A point whose closure is the entire space is a generic point. It is unique as $P_{\mathcal{T}}$ is the unique minimal prime ideal. The existence of a generic point implies irreducibility.
3. Follows since $\mathcal{S}$ is Noetherian (Corollary 4.7).
4. Since $P_{\mathcal{T}}$ and $P_{(0)}$ are not closed, the space is not T_1. It is T_0 (Kolmogorov) because the specialization order (inclusion) distinguishes the points. □

7 Congruences and Quotient Structures of $\mathcal{S}$

We provide a complete classification of the congruences on $\mathcal{S}$ and analyze the resulting quotient semirings.

7.1 Classification of Congruences

Let $\rho \in \text{Cong}(\mathcal{S})$. The kernel of ρ is $\text{Ker}(\rho) = [\mathcal{T}]_\rho$.

Lemma 7.1. *If $\rho \in \text{Cong}(\mathcal{S})$, the kernel $I = [\mathcal{T}]_\rho$ is an ideal of $\mathcal{S}$.*

Proof. I is non-empty ($\mathcal{T} \in I$). 1. Additive closure. Let $a, b \in I$. $a\rho\mathcal{T}, b\rho\mathcal{T}$. By compatibility, $(a + b)\rho(\mathcal{T} + \mathcal{T})$. $\mathcal{T} + \mathcal{T} = \mathcal{T}$. So $(a + b)\rho\mathcal{T}$. $a + b \in I$. 2. Absorption. Let $r \in \mathcal{S}, a \in I$. $a\rho\mathcal{T}$. By compatibility, $(r \times a)\rho(r \times \mathcal{T})$. $r \times \mathcal{T} = \mathcal{T}$. So $(r \times a)\rho\mathcal{T}$. $r \times a \in I$. □

Let $\rho_\mathbb{Z} = \rho \cap (\mathbb{Z} \times \mathbb{Z})$ be the restriction of ρ to $\mathbb{Z}$. $\rho_\mathbb{Z}$ is a congruence on the ring $\mathbb{Z}$, corresponding uniquely to an ideal $(n)_\mathbb{Z}, n \geq 0$.

Lemma 7.2. *Let $\rho \in \text{Cong}(\mathcal{S})$. Let $I = [\mathcal{T}]_\rho$ and $J = I \cap \mathbb{Z}$. Let $\rho_\mathbb{Z}$ correspond to $(n)_\mathbb{Z}$.*

1. *If $I = \{\mathcal{T}\}$ (i.e., $J = \emptyset$), any $n \geq 0$ defines a valid congruence ρ_n.*
2. *If $I \neq \{\mathcal{T}\}$ (i.e., $J = (m)_\mathbb{Z} \neq \emptyset$). Then we must have $(n) = (m)$.*

Proof. 1. If $I = \{\mathcal{T}\}$. The relation ρ_n is defined by: $x\rho_n y$ iff ($x, y \in \mathbb{Z}$ and $x \equiv y(n)$) or ($x = y = \mathcal{T}$). We verify compatibility. Let $a\rho_n b$ and $c\rho_n d$. If $a, b, c, d \in \mathbb{Z}$. $a \equiv b(n), c \equiv d(n)$. Then $a+c \equiv b+d(n)$ and $ac \equiv bd(n)$. Compatibility holds. If $a = \mathcal{T}$, then $b = \mathcal{T}$. Addition: $(\mathcal{T} + c)\rho_n(\mathcal{T} + d)$. Simplifies to $c\rho_n d$. Holds by assumption. Multiplication: $(\mathcal{T} \times c)\rho_n(\mathcal{T} \times d)$. Simplifies to $\mathcal{T}\rho_n\mathcal{T}$. Holds.

2. If $I \neq \{\mathcal{T}\}$. $J \neq \emptyset$. Since J is an ideal of $\mathbb{Z}$, $0_\mathbb{Z} \in J$. Thus $0_\mathbb{Z} \in I$, so $0_\mathbb{Z}\rho\mathcal{T}$.

We show $(n) = (m)$. (i) $(n) \subseteq (m)$. Let $x \in (n)$. $x\rho_\mathbb{Z}0_\mathbb{Z}$. So $x\rho0_\mathbb{Z}$. Since $0_\mathbb{Z}\rho\mathcal{T}$, by transitivity, $x\rho\mathcal{T}$. So $x \in I$. Since $x \in \mathbb{Z}$, $x \in J = (m)$.

(ii) $(m) \subseteq (n)$. Let $x \in (m)$. $x \in J$, so $x\rho\mathcal{T}$. Since $0_\mathbb{Z}\rho\mathcal{T}$, by symmetry and transitivity, $x\rho0_\mathbb{Z}$. Since $x, 0_\mathbb{Z} \in \mathbb{Z}$, $x\rho_\mathbb{Z}0_\mathbb{Z}$, so $x \in (n)$. □

Theorem 7.3. *The congruences on $\mathcal{S}$ are completely classified into the following types:*

1. *Type A (Trivial Kernel): Congruences ρ_n ($n \geq 0$), defined in Lemma 7.2.1. $\ker(\rho_n) = \{\mathcal{T}\}$.*
2. *Type B (Zero Kernel): The congruence ρ_B (case $n = m = 0$). $I = \{0_\mathbb{Z}, \mathcal{T}\}$. Defined by $x\rho_B y$ iff $x = y$ or $x, y \in \{0_\mathbb{Z}, \mathcal{T}\}$. This is the Rees congruence associated with the ideal $I_{(0)}$.*
3. *Type C (Principal Kernel): Congruences ρ'_n ($n \geq 1$) (case $n = m \geq 1$). $I = (n) \cup \{\mathcal{T}\}$. Defined by the partition consisting of the kernel I and the distinct classes $C_k = \{x \in \mathbb{Z} \mid x \equiv k(n)\}$ for $k = 1, \ldots, n - 1$.*

Proof. The necessity follows from Lemma 7.2. Sufficiency requires verification that Type B and Type C relations are indeed congruences.

Type B Verification (ρ_B). This is the Rees congruence modulo $I_{(0)}$. Compatibility is standard for Rees congruences. We verify explicitly. Let $a\rho_B b, c\rho_B d$. If $a, b \in I_{(0)}, c = d$. Addition: $a + c, b + c$. If $c \in I_{(0)}$, sums are in $I_{(0)}$. If $c \notin I_{(0)}$ ($c \in \mathbb{Z}, c \neq 0$). Consider $a = 0_\mathbb{Z}, b = \mathcal{T}$. $a + c = c$. $b + c = c$. $c\rho_B c$. Multiplication: ac, bc. Since $I_{(0)}$ is an ideal, $ac, bc \in I_{(0)}$.

Type C Verification (ρ'_n). Let A, B be classes. Let $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Addition: If $A = C_j, B = C_k$. $a_i + b_i \equiv j + k(n)$. They belong to the same class (C_{j+k} or $I_{(n)}$). If $A = I_{(n)}, B = C_k$. If $a_i \in (n)$, $a_i + b_i \equiv k(n) \in C_k$. If $a_i = \mathcal{T}$, $a_i + b_i = b_i \in C_k$. The sum is always in C_k. Multiplication: Products belong to the same class (either C_{jk} or $I_{(n)}$ if one factor is in $I_{(n)}$ due to the ideal property). □

7.2 Quotient Structures

Theorem 7.4. *The quotient semirings of $\mathcal{S}$ are characterized up to isomorphism as follows:*

1. *$\mathcal{S}/\rho_n \cong S(\mathbb{Z}/n\mathbb{Z})$ (The globalization of the ring $\mathbb{Z}/n\mathbb{Z}$). ($n \geq 0$).*
2. *$\mathcal{S}/\rho_B \cong \mathbb{Z}$.*

3. $\mathcal{S}/\rho'_n \cong \mathbb{Z}/n\mathbb{Z}$*.* *(*$n \geq 1$*).*

Proof. 1. $\mathcal{S}/\rho_n$. The classes are $[\mathcal{T}]$ (zero element) and the classes $[k]_n = \{x \in \mathbb{Z} \mid x \equiv k(n)\}$. The operations are $[j]_n + [k]_n = [j+k]_n$, $[j]_n \times [k]_n = [jk]_n$, $[k]_n + [\mathcal{T}] = [k]_n$, $[k]_n \times [\mathcal{T}] = [\mathcal{T}]$. This is precisely the definition of $S(\mathbb{Z}/n\mathbb{Z})$.

2. $\mathcal{S}/\rho_B$. The classes are $I_{(0)} = \{0_{\mathbb{Z}}, \mathcal{T}\}$ (zero element) and $\{k\}$ for $k \in \mathbb{Z} \setminus \{0\}$. Let $Q_B = \mathcal{S}/\rho_B$. Define $\phi : \mathbb{Z} \to Q_B$ by $\phi(0) = I_{(0)}$ and $\phi(k) = \{k\}$ for $k \neq 0$. ϕ is a bijection. Homomorphism verification: $\phi(a+b) = \phi(a) + \phi(b)$. If $a, b, a+b \neq 0$. LHS $= \{a+b\}$. RHS $= \{a\} + \{b\} = \{a+b\}$. If $a+b = 0$. LHS $= I_{(0)}$. RHS $= \{a\} + \{-a\} = \{0\} = I_{(0)}$. $\phi(ab) = \phi(a)\phi(b)$. Verified similarly. Thus $Q_B \cong \mathbb{Z}$.

3. $\mathcal{S}/\rho'_n$. The classes are $I_{(n)}$ (zero element) and C_k ($k = 1, \ldots, n-1$). Let $Q_C = \mathcal{S}/\rho'_n$. Define $\phi : \mathbb{Z}/n\mathbb{Z} \to Q_C$ by $\phi(\bar{0}) = I_{(n)}$ and $\phi(\bar{k}) = C_k$ for $k \neq 0$. This is a bijective homomorphism, as the operations on the classes C_k mirror the operations in $\mathbb{Z}/n\mathbb{Z}$, with $I_{(n)}$ serving as the zero element. $\square$

8 Number Theoretic Aspects of $\mathcal{S}$

8.1 Ideal Zeta Function of $\mathcal{S}$

We define and compute the ideal zeta function associated with $\mathcal{S}$. Since all ideals are subtractive (Theorem 4.8), the norm is defined via the Bourne congruence (Definition 2.15).

Definition 8.1. Let R be a commutative unital hemiring. The ideal zeta function of R is $\zeta_R(s) = \sum_I (N(I))^{-s}$, where the sum is over all subtractive ideals I of finite norm $N(I) = |R/I|$.

Theorem 8.2. *The ideal zeta function of the semiring* $\mathcal{S}$ *is identical to the Riemann zeta function* $\zeta(s)$*.*

Proof. We determine the norms of the ideals I_0 and $I_{(n)}$ for $n \geq 0$. We analyze the associated Bourne congruences and the resulting quotients, which correspond to specific types identified in Theorem 7.3 and analyzed in Theorem 7.4.

1. $I_0 = \{\mathcal{T}\}$. The Bourne congruence associated with I_0 is the identity relation ρ_0 (Type A, $n = 0$). The quotient is $\mathcal{S}/I_0 \cong \mathcal{S}$. The norm $N(I_0)$ is infinite.

2. $I_{(0)} = \{0_{\mathbb{Z}}, \mathcal{T}\}$. The Bourne congruence is the Type B congruence ρ_B. The quotient is $\mathcal{S}/\rho_B \cong \mathbb{Z}$. The norm $N(I_{(0)})$ is infinite.

3. $I_{(n)}, n \geq 1$. The Bourne congruence is the Type C congruence ρ'_n. The quotient is $\mathcal{S}/\rho'_n \cong \mathbb{Z}/n\mathbb{Z}$. The norm is $N(I_{(n)}) = |\mathbb{Z}/n\mathbb{Z}| = n$. This is finite.

We sum over the ideals of finite norm:

$$\zeta_{\mathcal{S}}(s) = \sum_{n=1}^{\infty} (N(I_{(n)}))^{-s} = \sum_{n=1}^{\infty} n^{-s}. \tag{8.1}$$

This series is the definition of the Riemann zeta function $\zeta(s)$. $\square$

8.2 Diophantine Equations over $\mathcal{S}$

We examine solutions to polynomial equations in $\mathcal{S}$.

Theorem 8.3 (Pythagorean Triples in $\mathcal{S}$)**.** *The solutions* $(X, Y, Z) \in \mathcal{S}^3$ *to the Diophantine equation* $X^2 + Y^2 = Z^2$ *are characterized as follows:*

1. *Integer solutions:* $(x, y, z) \in \mathbb{Z}^3$ *such that* $x^2 +_{\mathbb{Z}} y^2 = z^2$*.*

2. *Solutions involving* $\mathcal{T}$*:*

   - $(x, \mathcal{T}, z)$ *where* $x, z \in \mathbb{Z}$ *and* $x^2 = z^2$ *(i.e.,* $z = \pm x$*).*
   - $(\mathcal{T}, y, z)$ *where* $y, z \in \mathbb{Z}$ *and* $y^2 = z^2$ *(i.e.,* $z = \pm y$*).*

Proof. We analyze the equation $X^2 + Y^2 = Z^2$ by cases based on the membership of X, Y, Z in $\mathbb{Z}$ or $\{\mathcal{T}\}$. Note that $A^2 = A \times A$.

Case 1: $X, Y, Z \in \mathbb{Z}$. The operations are those of $\mathbb{Z}$ (Cases A1, M1). This yields the standard integer solutions (Type 1).

Case 2: At least one variable is $\mathcal{T}$.

Subcase 2a: $Z = \mathcal{T}$. $X^2 + Y^2 = \mathcal{T}$. Since $\mathcal{S}$ is zerosumfree (Proposition 4.4), this implies $X^2 = \mathcal{T}$ and $Y^2 = \mathcal{T}$. We analyze $A^2 = \mathcal{T}$. If $A \in \mathbb{Z}$, $A^2 \in \mathbb{Z}$, so $A^2 \neq \mathcal{T}$. Thus $A = \mathcal{T}$. The only solution is $(X, Y, Z) = (\mathcal{T}, \mathcal{T}, \mathcal{T})$.

Subcase 2b: $X = \mathcal{T}$. $Y, Z \in \mathcal{S}$. $\mathcal{T}^2 + Y^2 = Z^2$. $\mathcal{T} + Y^2 = Z^2$. Since $\mathcal{T}$ is the additive identity, $Y^2 = Z^2$. We analyze $A^2 = B^2$ in $\mathcal{S}$. If $A, B \in \mathbb{Z}$. $A^2 = B^2$ in $\mathbb{Z}$, so $A = \pm B$. If $A = \mathcal{T}$, then $B^2 = \mathcal{T}$, so $B = \mathcal{T}$. So $Y^2 = Z^2$ implies $(Y, Z \in \mathbb{Z}, Z = \pm Y)$ or $Y = Z = \mathcal{T}$. This yields solutions $(\mathcal{T}, y, \pm y)$ for $y \in \mathbb{Z}$.

Subcase 2c: $Y = \mathcal{T}$. Symmetric to Subcase 2b. Solutions $(x, \mathcal{T}, \pm x)$ for $x \in \mathbb{Z}$.

We consolidate the findings. The solutions are the integer triples, and the degenerate cases $(\mathcal{T}, y, \pm y)$ and $(x, \mathcal{T}, \pm x)$. Note that $(\mathcal{T}, \mathcal{T}, \mathcal{T})$ is covered when $x = 0_{\mathbb{Z}}$ or $y = 0_{\mathbb{Z}}$ in these cases. ☐

9 Symmetry Analysis and the Characterization of Singlets in $\mathcal{S}$

We analyze the symmetries of $\mathcal{S}$ arising from its group of units.

9.1 The Canonical $\mathbb{Z}/2\mathbb{Z}$ Action on $\mathcal{S}$

By Proposition 4.2, $U(\mathcal{S}) = \{1_{\mathbb{Z}}, -1_{\mathbb{Z}}\} \cong \mathbb{Z}/2\mathbb{Z}$.

Definition 9.1. The canonical $\mathbb{Z}/2\mathbb{Z}$ action on $\mathcal{S}$ is the action $\Psi : U(\mathcal{S}) \times \mathcal{S} \to \mathcal{S}$ defined by multiplication: $\Psi(g, s) := g \times s$.

Theorem 9.2. *The action Ψ is an action by automorphisms of the additive monoid $(\mathcal{S}, +)$.*

Proof. We must show that the map $\Sigma_g(s) = g \times s$ is in $\mathrm{Aut}((\mathcal{S}, +))$ for each $g \in U(\mathcal{S})$, and that $g \mapsto \Sigma_g$ is a group homomorphism (Definition 2.19).

1. Σ_g is an additive homomorphism. $\Sigma_g(a + b) = g(a + b)$. By distributivity in $\mathcal{S}$ (Theorem 3.4), this equals $ga + gb = \Sigma_g(a) + \Sigma_g(b)$.
2. Σ_g is bijective. The inverse is $\Sigma_{g^{-1}}$. $\Sigma_{g^{-1}}(\Sigma_g(s)) = g^{-1}(gs) = (g^{-1}g)s$. Since $g \in \mathbb{Z}$, $g^{-1}g = 1_{\mathbb{Z}}$. Thus $\Sigma_{g^{-1}}(\Sigma_g(s)) = 1_{\mathbb{Z}}s = s$.
3. The map $g \mapsto \Sigma_g$ is a group homomorphism. $\Sigma_{g_1 g_2}(s) = (g_1 g_2)s = g_1(g_2 s) = \Sigma_{g_1}(\Sigma_{g_2}(s))$. ☐

9.2 Characterization of Singlets

We determine the fixed points (singlets) of the action Ψ. An element s is a singlet if and only if it is fixed by the generator $-1_{\mathbb{Z}}$, i.e., $(-1_{\mathbb{Z}}) \times s = s$.

Theorem 9.3. *The set of singlets in $\mathcal{S}$ under the canonical $\mathbb{Z}/2\mathbb{Z}$ action Ψ is $\mathcal{A} = \{0_{\mathbb{Z}}, \mathcal{T}\}$.*

Proof. We solve the equation $(-1_{\mathbb{Z}}) \times s = s$.

Case 1: $s = n \in \mathbb{Z}$. $(-1_{\mathbb{Z}}) \times n = (-1_{\mathbb{Z}}) \times_{\mathbb{Z}} n = -n$. We require $-n = n$. This implies $2n = 0_{\mathbb{Z}}$. Since $\mathrm{char}(\mathbb{Z}) = 0$, $n = 0_{\mathbb{Z}}$.

Case 2: $s = \mathcal{T}$. $(-1_{\mathbb{Z}}) \times \mathcal{T}$. By Definition 3.3 (Case M2), this equals $\mathcal{T}$. So $\mathcal{T}$ is a singlet.

The set of singlets is $\{0_{\mathbb{Z}}, \mathcal{T}\}$. ☐

9.3 The Algebraic Structure of the Set of Singlets $\mathcal{A}$

Theorem 9.4. *The subset $\mathcal{A} = \{0_{\mathbb{Z}}, \mathcal{T}\} \subset \mathcal{S}$, equipped with the inherited operations $(+, \times)$, forms a sub-semiring of $\mathcal{S}$. This sub-semiring is isomorphic to the Boolean semiring $\mathbb{B}$.*

Proof. We verify closure under the operations.

Addition ($+$): $0_{\mathbb{Z}} + 0_{\mathbb{Z}} = 0_{\mathbb{Z}}$. $0_{\mathbb{Z}} + \mathcal{T} = 0_{\mathbb{Z}}$ (Case A2). $\mathcal{T} + 0_{\mathbb{Z}} = 0_{\mathbb{Z}}$ (Case A3). $\mathcal{T} + \mathcal{T} = \mathcal{T}$.

Multiplication ($\times$): $0_{\mathbb{Z}} \times 0_{\mathbb{Z}} = 0_{\mathbb{Z}}$. $0_{\mathbb{Z}} \times \mathcal{T} = \mathcal{T}$ (Case M2). $\mathcal{T} \times 0_{\mathbb{Z}} = \mathcal{T}$ (Case M3). $\mathcal{T} \times \mathcal{T} = \mathcal{T}$.

We identify the identities within $\mathcal{A}$. Additive Identity ($0_{\mathcal{A}}$): $\mathcal{T}$. Multiplicative Identity ($1_{\mathcal{A}}$): $0_{\mathbb{Z}}$.

The structure $(\mathcal{A}, +, \times)$ is a commutative unital standard semiring.

We establish the isomorphism with $\mathbb{B} = (\{0, 1\}, \vee, \wedge)$. Define the map $\psi : \mathcal{A} \to \mathbb{B}$ by mapping the identities: $\psi(0_{\mathcal{A}}) = \psi(\mathcal{T}) = 0$ and $\psi(1_{\mathcal{A}}) = \psi(0_{\mathbb{Z}}) = 1$. The comparison of the operation tables confirms that ψ is an isomorphism. For example, $\psi(0_{\mathbb{Z}} + \mathcal{T}) = \psi(0_{\mathbb{Z}}) = 1$, and $\psi(0_{\mathbb{Z}}) \vee \psi(\mathcal{T}) = 1 \vee 0 = 1$. Also $\psi(0_{\mathbb{Z}} \times \mathcal{T}) = \psi(\mathcal{T}) = 0$, and $\psi(0_{\mathbb{Z}}) \wedge \psi(\mathcal{T}) = 1 \wedge 0 = 0$. □

10 Adjunction of a Universal Absorbing Element: The Hemiring $\mathcal{S}'$

We investigate the structure obtained by adjoining an element Ω to $\mathcal{S}$ that is defined to be absorbing for both addition and multiplication. This is the absorber adjunction construction $A(\mathcal{S})$.

10.1 Construction and Verification of $\mathcal{S}'$

Construction 10.1. Let $\mathcal{S} = \mathbb{Z} \cup \{\mathcal{T}\}$. Let Ω be a formal element such that $\Omega \notin \mathcal{S}$. We define the set $\mathcal{S}' := \mathcal{S} \cup \{\Omega\} = \mathbb{Z} \cup \{\mathcal{T}, \Omega\}$.

Definition 10.2 (Operations on $\mathcal{S}'$)**.** We define operations $+_{\mathcal{S}'}$ and $\times_{\mathcal{S}'}$ on $\mathcal{S}'$ that extend the operations on $\mathcal{S}$ and enforce the absorbing properties of Ω. For $a, b \in \mathcal{S}'$:

Addition ($+_{\mathcal{S}'}$):

$$a +_{\mathcal{S}'} b := \begin{cases} \Omega & \text{if } a = \Omega \text{ or } b = \Omega \\ a + b & \text{if } a, b \in \mathcal{S} \end{cases} \tag{10.1}$$

Multiplication ($\times_{\mathcal{S}'}$):

$$a \times_{\mathcal{S}'} b := \begin{cases} \Omega & \text{if } a = \Omega \text{ or } b = \Omega \\ a \times b & \text{if } a, b \in \mathcal{S} \end{cases} \tag{10.2}$$

We henceforth omit the subscripts on the operations.

Theorem 10.3. *The structure $(\mathcal{S}', +, \times)$ is a commutative unital hemiring. It is not a standard semiring.*

Proof. Part I: Verification of Semiring Axioms (Definition 2.1).

I.1. Commutativity. $a * b = b * a$ ($*$ is $+$ or $\times$). If $a = \Omega$ or $b = \Omega$, both sides equal Ω. If $a, b \in \mathcal{S}$, commutativity is inherited from $\mathcal{S}$.

I.2. Associativity. $(a * b) * c = a * (b * c)$. If $a, b, c \in \mathcal{S}$, associativity is inherited. If at least one element is Ω. Suppose $a = \Omega$. LHS: $(\Omega * b) * c$. By (10.1) or (10.2), $\Omega * b = \Omega$. LHS $= \Omega * c = \Omega$. RHS: $\Omega * (b * c) = \Omega$. Equality holds. This establishes that Ω is the absorbing element for both operations.

I.3. Distributivity. $a \times (b + c) = (a \times b) + (a \times c)$. Case I.3.1: $a, b, c \in \mathcal{S}$. Distributivity is inherited from $\mathcal{S}$. Case I.3.2: At least one element is Ω. If $a = \Omega$. LHS $= \Omega \times (b + c) = \Omega$. RHS $= (\Omega \times b) + (\Omega \times c) = \Omega + \Omega = \Omega$. If $b = \Omega$. LHS $= a \times (\Omega + c)$. Since Ω is the additive absorber, $\Omega + c = \Omega$. LHS $= a \times \Omega = \Omega$. RHS $= (a \times \Omega) + (a \times c) = \Omega + (a \times c)$. Since Ω is the additive absorber, RHS $= \Omega$. If $c = \Omega$. Symmetric to $b = \Omega$.

Part II: Identities (Hemiring and Unital properties). Additive Identity ($0_{\mathcal{S}'}$). We claim the identity is $\mathcal{T}$. Let $a \in \mathcal{S}'$. We verify $a + \mathcal{T} = a$. If $a \in \mathcal{S}$. $a + \mathcal{T} = a +_{\mathcal{S}} \mathcal{T} = a$. If $a = \Omega$. $\Omega + \mathcal{T}$. By (10.1) (since the first argument is Ω), the sum is Ω. Thus $0_{\mathcal{S}'} = \mathcal{T}$. $\mathcal{S}'$ is a hemiring.

Multiplicative Identity ($1_{\mathcal{S}'}$). We claim the identity is $1_{\mathbb{Z}}$. $1_{\mathbb{Z}} \times a = a$ for $a \in \mathcal{S}$. $1_{\mathbb{Z}} \times \Omega = \Omega$ by (10.2). Thus $1_{\mathcal{S}'} = 1_{\mathbb{Z}}$.

Part III: Standard Semiring Condition. The additive identity is $\mathcal{T}$. The unique multiplicative absorber is Ω. Since $\mathcal{T} \neq \Omega$ by construction (Construction 10.1), $\mathcal{S}'$ is not a standard semiring (Definition 2.5). The axiom $a \times 0_R = 0_R$ fails for $a = \Omega$: $\Omega \times \mathcal{T} = \Omega \neq \mathcal{T}$. □

10.2 Algebraic Properties of $\mathcal{S}'$

Proposition 10.4. *The hemiring $\mathcal{S}'$ is zerosumfree. It is an integral semidomain relative to $\mathcal{T}$. It is also a z-integral semidomain relative to its multiplicative absorber Ω.*

Proof. 1. Zerosumfree. We analyze $a+b=0_{\mathcal{S}'}=\mathcal{T}$. If $a=\Omega$ or $b=\Omega$, $a+b=\Omega$. We require $\Omega=\mathcal{T}$, which is false. Thus $a,b\in\mathcal{S}$. $a+b=\mathcal{T}$ in $\mathcal{S}$. Since $\mathcal{S}$ is zerosumfree (Proposition 4.4), $a=\mathcal{T}$ and $b=\mathcal{T}$.

2. Integral semidomain relative to $\mathcal{T}$. We analyze $a\times b=\mathcal{T}$. If $a=\Omega$ or $b=\Omega$, $a\times b=\Omega$. We require $\Omega=\mathcal{T}$, false. Thus $a,b\in\mathcal{S}$. $a\times b=\mathcal{T}$ in $\mathcal{S}$. Since $\mathcal{S}$ is an integral semidomain (Proposition 4.3), $a=\mathcal{T}$ or $b=\mathcal{T}$.

3. z-integral semidomain relative to Ω. We analyze $a\times b=\Omega$. By definition (10.2), this occurs if and only if $a=\Omega$ or $b=\Omega$. (If $a,b\in\mathcal{S}$, $a\times b\in\mathcal{S}$, so $a\times b\neq\Omega$). □

Proposition 10.5. *The group of units of $\mathcal{S}'$ is $U(\mathcal{S}')=\{1_{\mathbb{Z}},-1_{\mathbb{Z}}\}$.*

Proof. We seek $x,y\in\mathcal{S}'$ such that $x\times y=1_{\mathcal{S}'}=1_{\mathbb{Z}}$. If $x=\Omega$ or $y=\Omega$, $x\times y=\Omega\neq 1_{\mathbb{Z}}$. Thus $x,y\in\mathcal{S}$. The equation reduces to the corresponding equation in $\mathcal{S}$. By Proposition 4.2, $U(\mathcal{S}')=U(\mathcal{S})$. □

10.3 The Ideal Structure of $\mathcal{S}'$

Theorem 10.6. *The ideals of $\mathcal{S}'$ are precisely the following sets:*

1. *The minimal ideal $I_\Omega=\{\Omega\}$.*
2. *The sets of the form $I'=I\cup\{\Omega\}$, where I is an ideal of the semiring $\mathcal{S}$.*

Proof. Part I: Characterization of an arbitrary ideal. Let I' be an ideal of $\mathcal{S}'$. The multiplicative absorber is Ω. By Lemma 2.10, $\Omega\in I'$. Define $I=I'\cap\mathcal{S}$. Then $I'=I\cup\{\Omega\}$.

Case 1: $I=\emptyset$. $I'=\{\Omega\}=I_\Omega$.

Case 2: $I\neq\emptyset$. We demonstrate that I is an ideal of $\mathcal{S}$. (i) Additive closure. $a,b\in I \implies a,b\in I'$. $a+b\in I'$. Since $a,b\in\mathcal{S}$, $a+b\in\mathcal{S}$ by (10.1). Thus $a+b\in I$. (ii) Absorption by $\mathcal{S}$. $r\in\mathcal{S}, a\in I$. $r\times a\in I'$. Since $r,a\in\mathcal{S}$, $r\times a\in\mathcal{S}$ by (10.2). Thus $r\times a\in I$.

Part II: Verification that the forms define ideals. 1. $I_\Omega=\{\Omega\}$. Closure: $\Omega+\Omega=\Omega$. Absorption: $r\times\Omega=\Omega$.

2. Let I be an ideal of $\mathcal{S}$. $I'=I\cup\{\Omega\}$. (i) Additive closure. If $a,b\in I$, $a+b\in I$. If one is Ω, the sum is Ω (additive absorption). (ii) Absorption by $\mathcal{S}'$. Let $r\in\mathcal{S}', a\in I'$. If $r\in\mathcal{S}$. If $a\in I$, $r\times a\in I$. If $a=\Omega$, $r\times a=\Omega$. If $r=\Omega$. $r\times a=\Omega$. □

Theorem 10.7. *The semiring $\mathcal{S}'$ is a Principal Ideal Semiring (PIS).*

Proof. We demonstrate that every ideal characterized in Theorem 10.6 is principal.

1. $I_\Omega=(\Omega)_{\mathcal{S}'}$.

2. Ideals $I'=I\cup\{\Omega\}$. Since $\mathcal{S}$ is a PIS (Theorem 4.6), $I=(a)_{\mathcal{S}}$ for some $a\in\mathcal{S}$. We compute $(a)_{\mathcal{S}'}=\{r\times a\mid r\in\mathcal{S}'\}$. If $r\in\mathcal{S}$: The collection is $\{r\times a\mid r\in\mathcal{S}\}=(a)_{\mathcal{S}}=I$. If $r=\Omega$: $r\times a=\Omega\times a=\Omega$. Thus, $(a)_{\mathcal{S}'}=I\cup\{\Omega\}=I'$. □

10.4 Subtractive Ideals in $\mathcal{S}'$

Theorem 10.8. *The only subtractive ideal of $\mathcal{S}'$ is the improper ideal $\mathcal{S}'$.*

Proof. The semiring $\mathcal{S}'$ possesses an additive absorbing element, Ω (Theorem 10.3). Furthermore, Ω is the multiplicative absorber, so $\Omega\in I$ for any ideal I. By Lemma 2.13, if an ideal I contains an additive absorber, it is subtractive if and only if $I=R$. Therefore, the only subtractive ideal is $\mathcal{S}'$. □

10.5 The Spectrum of $\mathcal{S}'$

Theorem 10.9. *The spectrum* $\operatorname{Spec}(\mathcal{S}')$ *consists of the ideal* I_Ω *together with the ideals* $P' = P \cup \{\Omega\}$*, where* $P \in \operatorname{Spec}(\mathcal{S})$*.*

Proof. We analyze the ideals from Theorem 10.6 for primality.

1. $I_\Omega = \{\Omega\}$. Proper ($1_\mathbb{Z} \notin I_\Omega$). Let $ab \in I_\Omega$, so $ab = \Omega$. By Proposition 10.4 (z-integral semidomain property), $ab = \Omega$ implies $a = \Omega$ or $b = \Omega$. Thus I_Ω is prime.

2. Ideals $I' = I \cup \{\Omega\}$. I' is proper iff I is proper in $\mathcal{S}$. We establish the equivalence: I' is prime in $\mathcal{S}'$ if and only if I is prime in $\mathcal{S}$.

($\implies$) Assume I' is prime. Let $a, b \in \mathcal{S}$ such that $ab \in I$. Then $ab \in I'$. So $a \in I'$ or $b \in I'$. Since $a, b \in \mathcal{S}$, $a \in I$ or $b \in I$.

($\impliedby$) Assume I is prime in $\mathcal{S}$. Let $ab \in I'$. If $a = \Omega$ or $b = \Omega$, we are done. Assume $a, b \in \mathcal{S}$. Then $ab \in \mathcal{S}$. $ab \in I' \cap \mathcal{S} = I$. Since I is prime, $a \in I$ or $b \in I$. Thus $a \in I'$ or $b \in I'$. $\square$

Corollary 10.10. *Explicitly, the prime ideals of* $\mathcal{S}'$ *are:* $Q_\Omega = \{\Omega\}$, $Q_\mathcal{T} = \{\mathcal{T}, \Omega\}$, $Q_{(0)} = \{0_\mathbb{Z}, \mathcal{T}, \Omega\}$*, and* $Q_{(p)} = p\mathbb{Z} \cup \{\mathcal{T}, \Omega\}$*.*

Theorem 10.11. *The Krull dimension of the semiring* $\mathcal{S}'$ *is 3.*

Proof. Let $\Psi : \operatorname{Spec}(\mathcal{S}) \to \operatorname{Spec}(\mathcal{S}') \setminus \{I_\Omega\}$ be the map $\Psi(P) = P \cup \{\Omega\}$. By Theorem 10.9, this is an order-preserving bijection.

The ideal I_Ω is the unique minimal element of $\operatorname{Spec}(\mathcal{S}')$, as it is contained in every other ideal.

A maximal chain in $\operatorname{Spec}(\mathcal{S}')$ is obtained by taking a maximal chain in $\operatorname{Spec}(\mathcal{S})$, applying Ψ, and prepending I_Ω. A maximal chain in $\operatorname{Spec}(\mathcal{S})$ has length 2 (Theorem 6.3), e.g., $P_\mathcal{T} \subsetneq P_{(0)} \subsetneq P_{(p)}$. The corresponding maximal chain in $\mathcal{S}'$ is (using the notation of Corollary 10.10):

$$Q_\Omega \subsetneq Q_\mathcal{T} \subsetneq Q_{(0)} \subsetneq Q_{(p)}. \tag{10.3}$$

We verify the inclusions are strict. $Q_\Omega \subsetneq Q_\mathcal{T}$ because $\mathcal{T} \neq \Omega$. $Q_\mathcal{T} \subsetneq Q_{(0)}$ because $0_\mathbb{Z} \notin \{\mathcal{T}, \Omega\}$. $Q_{(0)} \subsetneq Q_{(p)}$ because $p \notin \{0_\mathbb{Z}, \mathcal{T}, \Omega\}$.

The length of this chain is 3. Therefore, $\operatorname{Kdim}(\mathcal{S}') = 3$. $\square$

11 Symmetry Analysis and Singlets of $\mathcal{S}'$

11.1 The Canonical $\mathbb{Z}/2\mathbb{Z}$ Action on $\mathcal{S}'$

$U(\mathcal{S}') = \{\pm 1_\mathbb{Z}\}$ (Proposition 10.5). The canonical $\mathbb{Z}/2\mathbb{Z}$ action is $\Psi' : U(\mathcal{S}') \times \mathcal{S}' \to \mathcal{S}'$, $\Psi'(g, s) = g \times s$.

Theorem 11.1. *The action* Ψ' *is an action by automorphisms of the additive semigroup* $(\mathcal{S}', +)$*.*

Proof. The verification is analogous to Theorem 9.2, utilizing distributivity in $\mathcal{S}'$ (Theorem 10.3) and the invertibility of the units. $\square$

11.2 Characterization of Singlets

Theorem 11.2. *The set of singlets in* $\mathcal{S}'$ *under the canonical* $\mathbb{Z}/2\mathbb{Z}$ *action* Ψ' *is* $\mathcal{A}' = \{0_\mathbb{Z}, \mathcal{T}, \Omega\}$*.*

Proof. We solve the equation $(-1_\mathbb{Z}) \times s = s$.

Case 1: $s \in \mathcal{S}$. The equation reduces to the corresponding equation in $\mathcal{S}$. By Theorem 9.3, the solutions are $s \in \{0_\mathbb{Z}, \mathcal{T}\}$.

Case 2: $s = \Omega$. We compute $(-1_\mathbb{Z}) \times \Omega$. By Definition 10.2 (10.2), since the second argument is Ω, the product is Ω. Thus $(-1_\mathbb{Z}) \times \Omega = \Omega$. Ω is a singlet. $\square$

11.3 The Structure of the Singlets $\mathcal{A}'$

Definition 11.3. The extended Boolean semiring $\mathbb{B}_{\text{ext}}$ is the semiring obtained by adjoining a universal absorber ∞ to the Boolean semiring $\mathbb{B} = \{0, 1\}$. $\mathbb{B}_{\text{ext}} = \{0, 1, \infty\}$. Operations extend $\mathbb{B}$ such that ∞ absorbs everything (additively and multiplicatively).

Theorem 11.4. *The subset $\mathcal{A}'$ forms a commutative unital sub-hemiring of $\mathcal{S}'$. It is isomorphic to the extended Boolean semiring $\mathbb{B}_{\text{ext}}$. It is an idempotent semiring ($x + x = x$).*

Proof. We verify closure by examining the operation tables restricted to $\mathcal{A}' = \{0_{\mathbb{Z}}, \mathcal{T}, \Omega\}$.

Addition ($+$):

$+$	$0_{\mathbb{Z}}$	$\mathcal{T}$	Ω
$0_{\mathbb{Z}}$	$0_{\mathbb{Z}}$	$0_{\mathbb{Z}}$	Ω
$\mathcal{T}$	$0_{\mathbb{Z}}$	$\mathcal{T}$	Ω
Ω	Ω	Ω	Ω

The entries are derived as follows: Operations within $\{0_{\mathbb{Z}}, \mathcal{T}\}$ follow $\mathcal{S}$ (Theorem 9.4). Any sum involving Ω results in Ω (by (10.1)).

Multiplication ($\times$):

$\times$	$0_{\mathbb{Z}}$	$\mathcal{T}$	Ω
$0_{\mathbb{Z}}$	$0_{\mathbb{Z}}$	$\mathcal{T}$	Ω
$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	Ω
Ω	Ω	Ω	Ω

Operations within $\{0_{\mathbb{Z}}, \mathcal{T}\}$ follow $\mathcal{S}$. Any product involving Ω results in Ω (by (10.2)).

The set $\mathcal{A}'$ is closed. Identities in $\mathcal{A}'$. Additive Identity $0_{\mathcal{A}'} = \mathcal{T}$. Multiplicative Identity $1_{\mathcal{A}'} = 0_{\mathbb{Z}}$.

Isomorphism to $\mathbb{B}_{\text{ext}} = \{0, 1, \infty\}$. Define $\psi' : \mathcal{A}' \to \mathbb{B}_{\text{ext}}$ by $\psi'(\mathcal{T}) = 0, \psi'(0_{\mathbb{Z}}) = 1, \psi'(\Omega) = \infty$. A comparison of the operation tables confirms this is an isomorphism.

Idempotency: $0_{\mathbb{Z}} + 0_{\mathbb{Z}} = 0_{\mathbb{Z}}$. $\mathcal{T} + \mathcal{T} = \mathcal{T}$. $\Omega + \Omega = \Omega$. ☐

12 Generalization to Algebraic Number Fields

We generalize the constructions to the ring of integers $\mathcal{O}_K$ of an algebraic number field K. $\mathcal{O}_K$ is a Dedekind domain [10].

12.1 The Globalization $S(\mathcal{O}_K)$

Definition 12.1. Define $S(\mathcal{O}_K) = \mathcal{O}_K \cup \{\mathcal{T}_K\}$ (the globalization $S(\mathcal{O}_K)$), such that $\mathcal{T}_K$ is the additive identity and multiplicative absorber.

Theorem 12.2. *$(S(\mathcal{O}_K), +, \times)$ is a standard commutative unital semiring, integral semidomain, and zerosumfree. The ideals are $\{\mathcal{T}_K\}$ and $I_J = J \cup \{\mathcal{T}_K\}$, where J is an ideal of $\mathcal{O}_K$. All ideals are subtractive.*

Proof. The verification of the axioms and basic properties generalizes the analysis for $\mathcal{S}$ (Theorems 3.4, 4.3, 4.4, 4.5, 4.8), relying on the ring structure of $\mathcal{O}_K$. The proof that ideals are subtractive relies crucially on the fact that J is an additive subgroup of $\mathcal{O}_K$. ☐

12.2 Ideal Theory and the Class Group

Theorem 12.3. *$S(\mathcal{O}_K)$ is a Principal Ideal Semiring (PIS) if and only if $\mathcal{O}_K$ is a Principal Ideal Domain (PID) (i.e., the class number $h_K = 1$).*

Proof. ($\Longrightarrow$) Assume $S(\mathcal{O}_K)$ is a PIS. Let J be a non-zero ideal of $\mathcal{O}_K$. $I_J = J \cup \{\mathcal{T}_K\}$ is an ideal of $S(\mathcal{O}_K)$. $I_J = (a)_{S(\mathcal{O}_K)}$ for some $a \in S(\mathcal{O}_K)$. $a \neq \mathcal{T}_K$, so $a \in \mathcal{O}_K$. $(a)_{S(\mathcal{O}_K)} = S(\mathcal{O}_K) \times a = (\mathcal{O}_K \times a) \cup (\{\mathcal{T}_K\} \times a) = (a)_{\mathcal{O}_K} \cup \{\mathcal{T}_K\}$. The equality $I_J = (a)_{S(\mathcal{O}_K)}$ implies $J = (a)_{\mathcal{O}_K}$. Thus $\mathcal{O}_K$ is a PID.

($\Longleftarrow$) Assume $\mathcal{O}_K$ is a PID. Let I be an ideal of $S(\mathcal{O}_K)$. If $I = \{\mathcal{T}_K\}$, it is principal. If $I = I_J$, then $J = (a)_{\mathcal{O}_K}$. As shown above, $I_J = (a)_{S(\mathcal{O}_K)}$. □

We analyze the class semigroup when $h_K \geq 1$.

Definition 12.4. Let R be an integral semidomain. Let $\mathcal{I}^*(R)$ be the set of non-zero ideals of R. The class semigroup $\mathrm{Cl}(R)$ is $\mathcal{I}^*(R)/\sim$, where $I_1 \sim I_2$ if there exist $a, b \in R \setminus \{0_R\}$ such that $aI_1 = bI_2$.

In $S(\mathcal{O}_K)$, $0_{S(\mathcal{O}_K)} = \mathcal{T}_K$. The set of non-zero elements is $R \setminus \{0_R\} = \mathcal{O}_K$.

Theorem 12.5. *The class semigroup* $\mathrm{Cl}(S(\mathcal{O}_K))$ *is isomorphic to the ideal class group* $\mathrm{Cl}(K)$ *of the number field* K.

Proof. We establish an isomorphism $\Phi : \mathrm{Cl}(K) \to \mathrm{Cl}(S(\mathcal{O}_K))$. We identify $\mathrm{Cl}(K)$ with the equivalence classes of integral ideals of $\mathcal{O}_K$.

Let $[J] \in \mathrm{Cl}(K)$. Define $\Phi([J]) = [I_J]_\sim$.

1. Well-defined. If $[J_1] = [J_2]$, then $\alpha J_1 = \beta J_2$ for $\alpha, \beta \in \mathcal{O}_K \setminus \{0\}$. We verify $\alpha I_{J_1} = \beta I_{J_2}$ in $S(\mathcal{O}_K)$. $\alpha I_{J_1} = \alpha(J_1 \cup \{\mathcal{T}_K\}) = (\alpha J_1) \cup (\alpha \times \mathcal{T}_K)$. Since $\alpha \in \mathcal{O}_K$, $\alpha \times \mathcal{T}_K = \mathcal{T}_K$. $\alpha I_{J_1} = (\alpha J_1) \cup \{\mathcal{T}_K\}$. Similarly, $\beta I_{J_2} = (\beta J_2) \cup \{\mathcal{T}_K\}$. Since $\alpha J_1 = \beta J_2$, the equality holds.

2. Homomorphism. We verify the multiplication of ideals: $I_{J_1} I_{J_2} = I_{J_1 J_2}$. $I_{J_1} I_{J_2} = (J_1 \cup \{\mathcal{T}_K\})(J_2 \cup \{\mathcal{T}_K\}) = (J_1 J_2) \cup (J_1 \mathcal{T}_K) \cup (\mathcal{T}_K J_2) \cup (\mathcal{T}_K \mathcal{T}_K)$. Since $\mathcal{T}_K$ is absorbing, this simplifies to $(J_1 J_2) \cup \{\mathcal{T}_K\} = I_{J_1 J_2}$. Thus $\Phi([J_1][J_2]) = [I_{J_1 J_2}]_\sim = \Phi([J_1])\Phi([J_2])$.

3. Injectivity. Suppose $\Phi([J_1]) = \Phi([J_2])$. $I_{J_1} \sim I_{J_2}$. There exist $a, b \in \mathcal{O}_K \setminus \{0\}$ such that $aI_{J_1} = bI_{J_2}$. $(aJ_1) \cup \{\mathcal{T}_K\} = (bJ_2) \cup \{\mathcal{T}_K\}$. Intersecting both sides with $\mathcal{O}_K$ yields $aJ_1 = bJ_2$. Thus $[J_1] = [J_2]$ in $\mathrm{Cl}(K)$.

4. Surjectivity. Let $[I] \in \mathrm{Cl}(S(\mathcal{O}_K))$. I is a non-zero ideal of $S(\mathcal{O}_K)$. By Theorem 12.2, $I = I_J$ for some non-zero ideal J of $\mathcal{O}_K$. $[I] = \Phi([J])$. □

12.3 Spectrum and Dimension

Theorem 12.6. *The Krull dimension of* $S(\mathcal{O}_K)$ *is 2. The spectrum* $\mathrm{Spec}(S(\mathcal{O}_K))$ *consists of* $P_{\mathcal{T}_K} = \{\mathcal{T}_K\}$, $P_{(0)} = \{0_{\mathcal{O}_K}, \mathcal{T}_K\}$, *and* $P_{\mathfrak{p}} = \mathfrak{p} \cup \{\mathcal{T}_K\}$ *where* $\mathfrak{p}$ *is a non-zero prime ideal of* $\mathcal{O}_K$.

Proof. The characterization of prime ideals follows the methodology of Theorem 6.1: I_J is prime in $S(\mathcal{O}_K)$ iff J is prime in $\mathcal{O}_K$. The prime ideals of the Dedekind domain $\mathcal{O}_K$ are (0) and the non-zero prime (maximal) ideals $\mathfrak{p}$. A maximal chain in $S(\mathcal{O}_K)$ is $P_{\mathcal{T}_K} \subsetneq P_{(0)} \subsetneq P_{\mathfrak{p}}$. The length is 2. □

12.4 Symmetry Actions in $S(\mathcal{O}_K)$

The group of units $U(\mathcal{O}_K)$ acts on $S(\mathcal{O}_K)$ by multiplication: $\Psi_K(g, s) = g \times s$.

Theorem 12.7. *The set of singlets in* $S(\mathcal{O}_K)$ *under the action of* $U(\mathcal{O}_K)$ *is* $A_K = \{0_{\mathcal{O}_K}, \mathcal{T}_K\}$. A_K *forms a sub-semiring isomorphic to the Boolean semiring* $\mathbb{B}$.

Proof. We seek s such that $g \times s = s$ for all $g \in U(\mathcal{O}_K)$. If $s = x \in \mathcal{O}_K$. $gx = x$, so $(g-1)x = 0$. Since the characteristic of K is 0, $-1_{\mathcal{O}_K} \in U(\mathcal{O}_K)$ and $-1_{\mathcal{O}_K} \neq 1_{\mathcal{O}_K}$. Taking $g = -1_{\mathcal{O}_K}$ yields $(-2)x = 0$. Since $\mathcal{O}_K$ is an integral domain, $x = 0_{\mathcal{O}_K}$. If $s = \mathcal{T}_K$. $g \times \mathcal{T}_K = \mathcal{T}_K$. The isomorphism to $\mathbb{B}$ follows as in Theorem 9.4, with $0_{A_K} = \mathcal{T}_K$ and $1_{A_K} = 0_{\mathcal{O}_K}$. □

12.5 The Extended Construction $S'(\mathcal{O}_K)$

Definition 12.8. Define $S'(\mathcal{O}_K) = S(\mathcal{O}_K) \cup \{\Omega_K\}$, where Ω_K is the universal absorbing element (absorber adjunction to $S(\mathcal{O}_K)$).

Theorem 12.9. *$(S'(\mathcal{O}_K), +, \times)$ is a commutative unital hemiring (not a standard semiring). It is zerosumfree.* $\mathrm{Kdim}(S'(\mathcal{O}_K)) = 3$. *$S'(\mathcal{O}_K)$ is a PIS if and only if $\mathcal{O}_K$ is a PID. No proper ideal of $S'(\mathcal{O}_K)$ is subtractive. The singlets under $U(\mathcal{O}_K)$ are $\{0_{\mathcal{O}_K}, \mathcal{T}_K, \Omega_K\}$, isomorphic to $\mathbb{B}_{\mathrm{ext}}$.*

Proof. These properties generalize the results established for $\mathcal{S}'$. The ideal structure follows Theorem 10.6. The PIS property follows by combining Theorem 10.7 and Theorem 12.3. The non-subtractive property follows from Lemma 2.13 due to the additive absorber Ω_K. The Krull dimension increases by 1 from $\mathrm{Kdim}(S(\mathcal{O}_K)) = 2$ due to the adjunction of the generic point $\{\Omega_K\}$, following the logic of Theorem 10.11. The singlet analysis follows Theorem 11.2. □

13 Topological and Categorical Interpretations

13.1 The Spectral Sequence of Adjunctions

We analyze the topological relationships between the spectra induced by the sequence of constructions (illustrated here for $\mathbb{Z}$, but applicable generally to $\mathcal{O}_K$):

$$\mathbb{Z} \hookrightarrow \mathcal{S} \hookrightarrow \mathcal{S}'. \tag{13.1}$$

Theorem 13.1. *The spectra exhibit the following relationships, characterized by the successive addition of generic points corresponding to the adjoined elements:*

1. $\mathrm{Spec}(\mathcal{S})$ *is homeomorphic to the space obtained by augmenting* $\mathrm{Spec}(\mathbb{Z})$ *with a new generic point* $P_{\mathcal{T}}$.
2. $\mathrm{Spec}(\mathcal{S}')$ *is homeomorphic to the space obtained by augmenting* $\mathrm{Spec}(\mathcal{S})$ *with a new generic point* I_Ω.

Proof. 1. Let $X = \mathrm{Spec}(\mathbb{Z})$ and $X_S = \mathrm{Spec}(\mathcal{S})$. Define the map $\pi_{\mathcal{S}} : X_S \setminus \{P_{\mathcal{T}}\} \to X$ by $\pi_{\mathcal{S}}(P) = P \cap \mathbb{Z}$. As established in the proof of Theorem 6.1, this map is an order-preserving bijection. Since the Zariski topology on these Noetherian spaces is determined by the specialization order (the inclusions), $\pi_{\mathcal{S}}$ is a homeomorphism between the subspaces. $P_{\mathcal{T}}$ is the unique generic point of X_S (Theorem 6.2). Thus X_S is X augmented by a generic point.

2. Let $X_{S'} = \mathrm{Spec}(\mathcal{S}')$. Define $\pi_{\mathcal{S}'} : X_{S'} \setminus \{I_\Omega\} \to X_S$ by $\pi_{\mathcal{S}'}(P') = P' \cap \mathcal{S}$. As established in the proof of Theorem 10.9, this map is an order-preserving bijection. Thus, it is a homeomorphism between the subspaces. I_Ω is the unique minimal prime ideal of $X_{S'}$, hence it is the unique generic point. Thus $X_{S'}$ is X_S augmented by a generic point. □

The spectral structure illustrates a successive augmentation of the geometric space, increasing the Krull dimension at each step: $\mathrm{Kdim}(\mathbb{Z}) = 1 \to \mathrm{Kdim}(\mathcal{S}) = 2 \to \mathrm{Kdim}(\mathcal{S}') = 3$.

13.2 Categorical Perspective

Let **Ring** be the category of commutative unital rings. Let $\mathbf{SRing}_0$ be the category of standard commutative unital semirings. Let **SRing** be the category of commutative unital hemirings.

Definition 13.2. The globalization construction defines a functor $G : \mathbf{Ring} \to \mathbf{SRing}_0$. $G(R) = R \cup \{\mathcal{T}_R\}$. The absorber adjunction construction defines a functor $A : \mathbf{SRing} \to \mathbf{SRing}$. $A(S) = S \cup \{\Omega_S\}$.

Theorem 13.3. *The Krull dimension behaves additively under the functors G and A when applied to suitable Noetherian structures.*

1. *If R is a Noetherian ring,* $\mathrm{Kdim}(G(R)) = \mathrm{Kdim}(R) + 1$.

2. If S is a Noetherian hemiring, $\mathrm{Kdim}(A(S)) = \mathrm{Kdim}(S) + 1$.

Proof. The proofs rely on the generalized topological analysis in Theorem 13.1. The functors augment the spectrum by adjoining a new unique generic point below the existing spectrum, increasing the length of maximal chains by 1. □

13.3 Connections to Idempotent Structures

The analysis of the singlets reveals a connection to idempotent semirings.

Definition 13.4. A semiring R is *idempotent* if $a + a = a$ for all $a \in R$.

Theorem 13.5. *The sub-semirings of singlets $\mathcal{A} \cong \mathbb{B}$ and $\mathcal{A}' \cong \mathbb{B}_{\text{ext}}$ are idempotent semirings.*

Proof. Verified in Theorems 9.4 and 11.4. □

In an idempotent semiring, addition defines the algebraic partial order: $a \leq b$ iff $a + b = b$.

Proposition 13.6. *The algebraic order on $\mathcal{A}'$ is a total order: $\mathcal{T} \leq 0_{\mathbb{Z}} \leq \Omega$.*

Proof. We verify the relations based on the addition table in Theorem 11.4. $\mathcal{T} + 0_{\mathbb{Z}} = 0_{\mathbb{Z}}$ (since $\mathcal{T}$ is the identity in $\mathcal{A}$). So $\mathcal{T} \leq 0_{\mathbb{Z}}$. $0_{\mathbb{Z}} + \Omega = \Omega$ (since Ω is the additive absorber in $\mathcal{S}'$). So $0_{\mathbb{Z}} \leq \Omega$. By transitivity, the order is established. □

References

[1] A. Asok, P. A. Østvær. $\mathbb{A}^1$-homotopy theory and contractible varieties: a survey. arXiv:1903.07851, 2019.

[2] T. Browning. Cubic Forms and the Circle Method. Progress in Mathematics, Vol. 343. Birkhäuser, 2021.

[3] P. Cartier, B. Julia, P. Moussa, P. Vanhove (Eds.). Frontiers in Number Theory, Physics, and Geometry II. Springer-Verlag, 2007.

[4] E. Blanchard, D. Ellwood, M. Khalkhali, M. Marcolli, H. Moscovici, S. Popa (Eds.). Quanta of Maths. Clay Mathematics Proceedings, Vol. 11. American Mathematical Society, 2010.

[5] J. S. Golan. Semirings and their Applications. Springer Netherlands, 1999.

[6] G. H. Hardy, E. M. Wright. An Introduction to the Theory of Numbers. 6th Edition, revised by D. R. Heath-Brown and J. H. Silverman. Oxford University Press, 2008.

[7] T. Jech. Set Theory: The Third Millennium Edition, revised and expanded. Springer Monographs in Mathematics. Springer-Verlag, 2003.

[8] P. Kumam, K. Chakraborty, H. M. Srivastava, P. Debnath (Eds.). Advances in Number Theory and Applied Analysis. World Scientific, 2023.

[9] S. Lang. Algebra. Revised Third Edition, Graduate Texts in Mathematics, Vol. 211. Springer-Verlag, 2002.

[10] J.S. Milne. Algebraic Number Theory. Version 3.08, 2020. Available at www.jmilne.org/math/.

[11] M. Ram Murty, P. Rath. Transcendental Numbers. Springer New York, 2014.

[12] R. Abdellatif, V. Karemaker, L. Smajlovic (Eds.). Women in Numbers Europe IV: Research Directions in Number Theory. Association for Women in Mathematics Series, Vol. 32. Springer, 2024.