A Detailed Examination of Algebraic Structures Derived from Arithmetic Rings by Sequential Globalization and Absorber Adjunction

Abstract

We conduct a systematic investigation of two algebraic structures constructed sequentially from the ring of integers $\mathbb{Z}$. The first structure, $\mathcal{S} = G(\mathbb{Z})$, is obtained via the globalization functor G, adjoining an element $\mathcal{T}$ defined as the additive identity and multiplicative absorber. We establish through detailed verification that $\mathcal{S}$ is a commutative unital standard semiring. A comprehensive analysis demonstrates that $\mathcal{S}$ is a Principal Ideal Semiring (PIS), an integral semidomain, and zerosumfree, with all ideals being subtractive. Its Krull dimension is 2. We provide a complete classification of its congruences and compute its ideal zeta function as $\zeta_{\mathcal{S}}(s) = \zeta(s)$. We analyze its factorization properties, proving that $\mathcal{S}$ is not a Unique Factorization Domain (UFD); specifically, the element $0_{\mathbb{Z}}$ is shown to be prime but reducible, as it lacks a factorization into irreducibles. The localization behavior is analyzed, revealing that the semifield of fractions $\mathrm{Frac}(\mathcal{S})$ collapses to the Boolean semiring $\mathbb{B}$. The action of the unit group $U(\mathcal{S}) \cong \mathbb{Z}/2\mathbb{Z}$ identifies the set of fixed points (singlets) $\mathcal{A} = \{0_{\mathbb{Z}}, \mathcal{T}\}$, which is isomorphic to $\mathbb{B}$.

We develop the scheme theory associated with $\mathcal{S}$, defining the structure sheaf on $\mathrm{Spec}(\mathcal{S})$ and proving it is a locally (semi)ringed space. The stalks are computed, revealing $\mathbb{B}$ at the generic point.

The second structure, $\mathcal{S}' = A(G(\mathbb{Z}))$, is constructed by applying the absorber adjunction functor A to $\mathcal{S}$, adjoining a universal absorbing element Ω. We prove that $\mathcal{S}'$ is a commutative unital hemiring, but not a standard semiring. Its Krull dimension is 3. The singlets of $\mathcal{S}'$ form an idempotent subhemiring $\mathcal{A}'$, isomorphic to the extended Boolean semiring $\mathbb{B}_{\text{ext}}$.

Generalizations to rings of integers $\mathcal{O}_K$ in algebraic number fields are examined, proving that the class semigroup of $G(\mathcal{O}_K)$ is isomorphic to the class group $\mathrm{Cl}(K)$. We conclude with an analysis of the categorical implications, demonstrating how the functors G and A systematically increase the Krull dimension by introducing new generic points. We discuss the connections of these constructions to $\mathbb{F}_1$-geometry, highlighting the emergence of characteristic 1 structures from characteristic 0 objects via symmetry analysis and localization, and situating these structures within the context of the Arithmetic Site.

Contents

1 Axiomatic Foundations and the Construction of Standard Algebraic Systems

We stipulate that the foundational system for the subsequent mathematical developments is Zermelo-Fraenkel set theory augmented by the Axiom of Choice (ZFC). The construction of the algebraic structures central to this work relies upon the existence and properties of the standard number systems. We commence by detailing the construction of these systems, ensuring definitional precision for the algebraic structures subsequently investigated.

1.1 Set Theoretic Preliminaries

We presuppose familiarity with the standard axioms of ZFC. We briefly recall the definitions essential for algebraic constructions.

Definition 1.1. A relation R on a set A is an *equivalence relation* if it is reflexive (aRa), symmetric ($aRb \implies bRa$), and transitive (aRb and $bRc \implies aRc$). The equivalence class of a is $[a]_R = \{x \in A \mid xRa\}$. The quotient set is A/R.

Definition 1.2. A relation $\leq$ on a set A is a *partial order* if it is reflexive, antisymmetric ($a \leq b$ and $b \leq a \implies a = b$), and transitive. It is a *total order* if, additionally, for all $a, b \in A$, either $a \leq b$ or $b \leq a$.

1.2 The Natural Numbers ω

We employ the von Neumann construction of the natural numbers, which identifies each natural number with the set of its predecessors. This construction relies upon the Axiom of Infinity.

Axiom 1 (Axiom of Infinity). There exists a set S such that the empty set $\emptyset \in S$, and for every $x \in S$, the successor $S(x) := x \cup \{x\}$ is also an element of S. Such a set is called an *inductive set.*

Definition 1.3. The set of natural numbers, denoted by ω (or $\mathbb{N}_0$), is defined as the intersection of all inductive sets:

$$\omega := \bigcap \{S \mid S \text{ is an inductive set}\}. \tag{1.1}$$

Lemma 1.4. *The set ω is an inductive set, and it is the smallest such set with respect to the inclusion relation; that is, if S' is an inductive set, then $\omega \subseteq S'$.*

Proof. Let $\mathcal{I}$ denote the collection of all inductive sets. By the Axiom of Infinity, $\mathcal{I}$ is non-empty. We verify that $\omega = \bigcap \mathcal{I}$ satisfies the defining properties of an inductive set.

(i) Verification of $\emptyset \in \omega$. By the definition of an inductive set, $\emptyset \in S$ for every $S \in \mathcal{I}$. By the definition of set intersection, $\emptyset$ is an element of the intersection of all elements of $\mathcal{I}$. Consequently, $\emptyset \in \omega$.

(ii) Verification of closure under the successor operation. Let $x \in \omega$. By the definition of intersection, $x \in S$ for every $S \in \mathcal{I}$. Since each $S \in \mathcal{I}$ possesses the property of being inductive, it follows that the successor $S(x) = x \cup \{x\}$ is an element of S for every $S \in \mathcal{I}$. Consequently, $S(x)$ is an element of the intersection $\bigcap \mathcal{I}$. Therefore, $S(x) \in \omega$.

The two conditions being satisfied, ω is an inductive set.

To demonstrate the minimality property, let S' be an arbitrary inductive set. By definition, $S' \in \mathcal{I}$. By the properties of set intersection, $\bigcap \mathcal{I} \subseteq S'$. Therefore, $\omega \subseteq S'$. $\square$

Notation 1.5. We adopt the standard numeral notation: $0 := \emptyset$, $1 := S(0) = \{0\}$, $2 := S(1) = \{0, 1\}$, and so forth.

The structure of ω allows for the Principle of Mathematical Induction, which is a direct consequence of its definition.

Theorem 1.6 (Principle of Mathematical Induction). *Let A be a subset of ω. If the following conditions hold:*

1. $0 \in A$.

2. *For all $n \in \omega$, the condition $n \in A$ implies $S(n) \in A$.*

Then $A = \omega$.

Proof. The hypotheses imposed upon the set A precisely state that A is an inductive set. By the minimality established in Lemma 1.4, we have the inclusion $\omega \subseteq A$. Since $A \subseteq \omega$ by hypothesis, we conclude by the Axiom of Extensionality that $A = \omega$. $\square$

The binary operations of addition and multiplication on ω are established via the Recursion Theorem, which guarantees the existence and uniqueness of functions defined recursively on ω.

Definition 1.7 (Operations on ω). For $m \in \omega$:

1. Addition (+): Defined recursively by the equations $m + 0 := m$ and $m + S(n) := S(m + n)$.
2. Multiplication ($\times$): Defined recursively by the equations $m \times 0 := 0$ and $m \times S(n) := (m \times n) + m$.

We verify the algebraic properties of these operations using the Principle of Mathematical Induction.

Lemma 1.8 (Properties of Addition). *The structure $(\omega, +, 0)$ is a commutative monoid. That is, for all $m, n, l \in \omega$:*

1. *Identity: $m + 0 = m$.*
2. *Associativity: $(m + n) + l = m + (n + l)$.*
3. *Commutativity: $m + n = n + m$.*

Proof. 1. Identity. This is immediate from the base case of Definition 1.7.1.

2. Associativity. We proceed by induction on l. Let $P(l)$ be the statement $(m+n)+l = m+(n+l)$. Base Case $P(0)$: LHS = $(m + n) + 0 = m + n$. RHS = $m + (n + 0) = m + n$. $P(0)$ holds.

Inductive Step: Assume $P(l)$ holds. We examine $P(S(l))$. LHS = $(m+n)+S(l)$. By Definition 1.7.1, this is $S((m + n) + l)$. By the inductive hypothesis $P(l)$, this is $S(m + (n + l))$. RHS = $m + (n + S(l))$. By Definition 1.7.1, this is $m + S(n + l)$. Applying the definition again, this is $S(m + (n + l))$. Since LHS=RHS, $P(S(l))$ holds. By Theorem 1.6, associativity holds for all $l \in \omega$.

3. Commutativity. This requires intermediate steps, established by induction. Step 3a: $0 + m = m$. Induction on m. Base case $0 + 0 = 0$. Inductive step: Assume $0 + m = m$. $0 + S(m) = S(0 + m)$ (by Definition 1.7.1). By hypothesis, $S(m)$. Step 3b: $S(n) + m = S(n + m)$. Induction on m. Base case $m = 0$: $S(n) + 0 = S(n)$. $S(n + 0) = S(n)$. Inductive step: Assume $S(n) + m = S(n + m)$. $S(n) + S(m) = S(S(n) + m)$ (by Definition 1.7.1). By hypothesis, $S(S(n + m))$. Also $S(n + S(m)) = S(S(n + m))$. Step 3c: $m + n = n + m$. Induction on n. Base case $n = 0$: $m + 0 = m$. By Step 3a, $0 + m = m$. Inductive step: Assume $m + n = n + m$. $m + S(n) = S(m + n)$. By hypothesis, $S(n + m)$. By Step 3b, $S(n + m) = S(n) + m$. $\square$

Lemma 1.9 (Properties of Multiplication and Distributivity). *The structure $(\omega, \times, 1)$ is a commutative monoid, and multiplication distributes over addition. Furthermore, 0 is the multiplicative absorber.*

Proof. The verification of these properties (Identity, Distributivity, Associativity, Commutativity) proceeds by systematic application of induction, utilizing the definitions and the properties of addition established in Lemma 1.8. The property $m \times 0 = 0$ is by Definition 1.7.2. $\square$

Proposition 1.10. *The structure $(\omega, +, \times, 0, 1)$ is a commutative unital standard semiring (Definition 2.5). It is zerosumfree and satisfies the cancellation laws:*

1. *Zerosumfree: $m + n = 0 \implies m = 0$ and $n = 0$.*
2. *Additive cancellation: $m + l = n + l \implies m = n$.*
3. *Multiplicative cancellation (for $l \neq 0$): $m \times l = n \times l \implies m = n$.*

Proof. The semiring axioms are established in Lemmas 1.8 and 1.9.

1. Zerosumfree. If $n \neq 0$, then $n = S(k)$ for some $k \in \omega$. $m + n = m + S(k) = S(m + k)$. The successor $S(x)$ is never 0 (this follows from the construction, as $0 = \emptyset$ and $S(x) = x \cup \{x\} \neq \emptyset$). Thus $m + n \neq 0$. Therefore, $m + n = 0$ necessitates $n = 0$, which implies $m + 0 = m = 0$.

2. Additive cancellation law. We proceed by induction on l. Base Case ($l = 0$): $m + 0 = n + 0 \implies m = n$.

Inductive Step: Assume cancellation holds for l. Suppose $m + S(l) = n + S(l)$. By definition, $S(m+l) = S(n+l)$. The successor function S is injective (a property derived from the ZFC construction of ordinals). Thus $m + l = n + l$. By the inductive hypothesis, $m = n$.

3. Multiplicative cancellation law. This relies first on establishing the absence of zero divisors in ω. Lemma (Absence of Zero Divisors): If $m \times l = 0$, then $m = 0$ or $l = 0$. Proof of Lemma: Assume $l \neq 0$. Then $l = S(k)$. $mS(k) = mk + m = 0$. By the zerosumfree property (Part 1), $m = 0$ and $mk = 0$.

Suppose $m \times l = n \times l$ and $l \neq 0$. We utilize the standard total order relation on ω: $n \leq m$ if and only if there exists $k \in \omega$ such that $m = n + k$. Assume without loss of generality $m \geq n$, so $m = n + k$. The equality becomes $(n + k)l = nl$. By distributivity, $nl + kl = nl$. We write $nl + kl = nl + 0$. By the additive cancellation law (Part 2), we conclude $kl = 0$. Since $l \neq 0$, by the absence of zero divisors, we must have $k = 0$. Thus $m = n + 0 = n$. □

1.3 The Ring of Integers $\mathbb{Z}$

We construct the ring of integers $\mathbb{Z}$ from the semiring ω utilizing the method of Grothendieck group completion applied to the commutative, cancellative monoid $(\omega, +)$.

Definition 1.11. Define the relation $\sim$ on the Cartesian product $\omega \times \omega$ by the condition $(a, b) \sim (c, d)$ if and only if $a + d = b + c$ in ω.

Lemma 1.12. *The relation $\sim$ is an equivalence relation on $\omega \times \omega$.*

Proof. Reflexivity: $(a, b) \sim (a, b)$ since $a + b = b + a$ by commutativity in ω (Lemma 1.8.3). Symmetry: $(a, b) \sim (c, d) \implies a + d = b + c$. By commutativity, $c + b = d + a$, so $(c, d) \sim (a, b)$. Transitivity: Assume $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. Then $a + d = b + c$ (Eq. 1) and $c + f = d + e$ (Eq. 2). We wish to show $a + f = b + e$. We sum Eq. 1 and Eq. 2: $(a + d) + (c + f) = (b + c) + (d + e)$. We rearrange terms using associativity and commutativity: $(a + f) + (d + c) = (b + e) + (c + d)$. We apply the additive cancellation law (Proposition 1.10.2) to cancel the term $(c + d)$ from both sides. This yields $a + f = b + e$. Thus $(a, b) \sim (e, f)$. □

Definition 1.13. The set of integers, $\mathbb{Z}$, is defined as the set of equivalence classes $\mathbb{Z} := (\omega \times \omega)/\sim$. We denote the equivalence class containing the pair (a, b) by $[a, b]$. This represents the formal difference $a - b$.

Definition 1.14 (Operations on $\mathbb{Z}$). Addition ($+_{\mathbb{Z}}$) and multiplication ($\times_{\mathbb{Z}}$) on $\mathbb{Z}$ are defined as follows:

$$[a, b] +_{\mathbb{Z}} [c, d] := [a + c, b + d], \tag{1.2}$$

$$[a, b] \times_{\mathbb{Z}} [c, d] := [ac + bd, ad + bc]. \tag{1.3}$$

Lemma 1.15. *The operations $+_{\mathbb{Z}}$ and $\times_{\mathbb{Z}}$ are well-defined on the set of equivalence classes $\mathbb{Z}$.*

Proof. We must demonstrate independence from the choice of representatives.

1. Addition. Assume $[a, b] = [a', b']$ and $[c, d] = [c', d']$. This means $a + b' = b + a'$ and $c + d' = d + c'$. We must verify $[a+c, b+d] = [a'+c', b'+d']$. This requires the equality $(a+c)+(b'+d') = (b+d)+(a'+c')$. We rearrange the LHS: $(a+b')+(c+d')$. By the assumptions, this equals $(b+a')+(d+c')$. We rearrange the RHS: $(b+d)+(a'+c')$. By commutativity and associativity in ω, $(b+a')+(d+c') = (b+d)+(a'+c')$. Thus the equality holds.

2. Multiplication. Assume $[a, b] = [a', b']$, so $a+b' = b+a'$. We verify that $[a, b]\times[c, d] = [a', b']\times[c, d]$. This requires $[ac + bd, ad + bc] = [a'c + b'd, a'd + b'c]$. The condition is $(ac + bd) + (a'd + b'c) = (ad+bc)+(a'c+b'd)$. We rearrange the LHS: $c(a+b')+d(b+a')$. We rearrange the RHS: $c(b+a')+d(a+b')$.

By the assumption $a + b' = b + a'$, LHS=RHS. By symmetry of the definition of multiplication (which relies on the commutativity of multiplication in ω, Lemma 1.9), independence from the second argument also holds. $\square$

Theorem 1.16. *The structure $(\mathbb{Z}, +_{\mathbb{Z}}, \times_{\mathbb{Z}})$ is a commutative ring with unity. Furthermore, it is an integral domain of characteristic 0, and a Principal Ideal Domain (PID).*

Proof. The verification of the axioms of a commutative ring relies systematically on the established properties of the semiring ω.

(i) $(\mathbb{Z}, +_{\mathbb{Z}})$ is an abelian group. Associativity and commutativity follow directly from the corresponding properties in ω. The additive identity is $0_{\mathbb{Z}} = [0, 0]$. Verification: $[a, b] + [0, 0] = [a + 0, b + 0] = [a, b]$. The additive inverse of $[a, b]$ is $-[a, b] = [b, a]$. Verification: $[a, b] + [b, a] = [a + b, b + a]$. Since $(a + b) + 0 = (b + a) + 0$, $[a + b, b + a] = [0, 0] = 0_{\mathbb{Z}}$.

(ii) $(\mathbb{Z}, \times_{\mathbb{Z}})$ is a commutative monoid. Associativity and commutativity are verified by explicit computation using the definitions and properties of ω. The multiplicative identity is $1_{\mathbb{Z}} = [1, 0]$. Verification: $[a, b] \times [1, 0] = [a \cdot 1 + b \cdot 0, a \cdot 0 + b \cdot 1] = [a, b]$.

(iii) Distributivity. Verified by explicit computation.

The integral domain property (absence of zero divisors) relies on the cancellation laws in ω. Characteristic 0: The n-fold sum $n \cdot 1_{\mathbb{Z}} = [n, 0]$. $[n, 0] = 0_{\mathbb{Z}}$ implies $n + 0 = 0 + 0$, so $n = 0$. PID: $\mathbb{Z}$ is a Euclidean Domain with the Euclidean function being the absolute value, and every Euclidean Domain is a PID. $\square$

We identify $n \in \omega$ with the equivalence class $[n, 0] \in \mathbb{Z}$. This defines an injective homomorphism of semirings $\iota : \omega \to \mathbb{Z}$. We henceforth utilize standard notation for the elements and operations of $\mathbb{Z}$.

1.4 The Field of Rational Numbers $\mathbb{Q}$

Definition 1.17. The field of rational numbers, $\mathbb{Q}$, is the field of fractions of the integral domain $\mathbb{Z}$. It is constructed via localization at the multiplicative subset $D = \mathbb{Z} \setminus \{0\}$.

2 Algebraic Preliminaries: Semirings and Associated Structures

We establish the precise definitions and foundational results for the algebraic structures central to this investigation. The study of semirings requires careful attention to concepts that differ from their counterparts in ring theory due to the potential absence of additive inverses.

2.1 Definitions of Semirings and Hemirings

Definition 2.1. A *semiring* is an algebraic structure $(R, +, \times)$ consisting of a non-empty set R equipped with two binary operations, addition $(+)$ and multiplication $(\times)$, satisfying the following axioms:

(S1) $(R, +)$ is a commutative semigroup (addition is associative and commutative).

(S2) $(R, \times)$ is a semigroup (multiplication is associative).

(S3) Multiplication distributes over addition: $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$.

Definition 2.2. Let $(M, *)$ be a semigroup.

1. An element $e \in M$ is an *identity element* if $a * e = e * a = a$ for all $a \in M$.
2. An element $z \in M$ is an *absorbing element* (or *annihilator*) if $a * z = z * a = z$ for all $a \in M$.

Lemma 2.3. *In a semigroup, an identity element, if it exists, is unique. An absorbing element, if it exists, is unique.*

Proof. Uniqueness of identity: Let e_1, e_2 be identities. $e_1 = e_1 * e_2$ (as e_2 is an identity). $e_1 * e_2 = e_2$ (as e_1 is an identity). Thus $e_1 = e_2$. Uniqueness of absorber: Let z_1, z_2 be absorbers. $z_1 = z_1 * z_2$ (as z_1 absorbs). $z_1 * z_2 = z_2$ (as z_2 absorbs). Thus $z_1 = z_2$. $\square$

Definition 2.4. A semiring $(R, +, \times)$ is further characterized as follows:

1. *Commutative* if $(R, \times)$ is commutative.
2. *Unital* if $(R, \times)$ possesses an identity element 1_R.
3. A *hemiring* if $(R, +)$ possesses an identity element 0_R.

A crucial distinction concerns the relationship between the additive identity and the multiplicative absorber.

Definition 2.5. A semiring R is called a *standard semiring* (or a *semiring with zero*) if it is a hemiring and its unique additive identity 0_R is also the unique multiplicative absorbing element.

We shall analyze structures that are hemirings but may or may not satisfy the conditions of a standard semiring.

Definition 2.6. Let R be a commutative unital hemiring with additive identity 0_R.

1. R is *zerosumfree* if $a + b = 0_R$ implies $a = 0_R$ and $b = 0_R$.
2. R is *additively cancellative* if $a + c = b + c$ implies $a = b$.
3. R is an *integral semidomain relative to* 0_R if $1_R \neq 0_R$ and it has no zero divisors relative to 0_R (i.e., if $a \times b = 0_R$, then $a = 0_R$ or $b = 0_R$).

If R is a standard semiring, we simply refer to it as an integral semidomain.

Definition 2.7. Let R be a commutative unital semiring possessing a multiplicative absorber z_R. R is a *z-integral semidomain* if $1_R \neq z_R$ and it has no z-divisors (i.e., if $a \times b = z_R$, then $a = z_R$ or $b = z_R$).

2.2 Characteristic One and Idempotent Structures

Definition 2.8. A semiring R is *idempotent* if $a + a = a$ for all $a \in R$. A unital hemiring is said to be of *characteristic one* if $1_R + 1_R = 1_R$.

Lemma 2.9. *A unital hemiring R is of characteristic 1 if and only if it is idempotent.*

Proof. ($\implies$) Assume $1_R + 1_R = 1_R$. Let $a \in R$. We compute $a + a$. $a + a = a(1_R) + a(1_R)$. By the distributive axiom (S3), $a(1_R + 1_R)$. By hypothesis, $a(1_R) = a$. ($\impliedby$) Assume R is idempotent. By definition, $1_R + 1_R = 1_R$. □

Example 2.10 (The Boolean Semiring $\mathbb{B}$). The Boolean semiring $\mathbb{B} = (\{0, 1\}, \vee, \wedge)$ has addition as logical OR ($1 \vee 1 = 1$) and multiplication as logical AND. It is a standard commutative unital idempotent semiring. It is often considered the simplest non-trivial characteristic one structure, sometimes denoted $\mathbb{F}_{1^2}$.

Definition 2.11. In an idempotent semiring R, the *algebraic order* $\leq$ is defined by the relation $a \leq b$ if and only if $a + b = b$.

Lemma 2.12. *The algebraic order on an idempotent semiring R is a partial order. Furthermore, $(R, +)$ forms a join-semilattice where $a + b$ is the least upper bound of $\{a, b\}$.*

Proof. Reflexivity: $a + a = a$ (idempotence). $a \leq a$. Antisymmetry: $a \leq b$ and $b \leq a$. $a + b = b$ and $b + a = a$. By commutativity (S1), $a = b$. Transitivity: $a \leq b$ and $b \leq c$. $a + b = b$ and $b + c = c$. We compute $a + c = a + (b + c)$. By associativity (S1), $(a + b) + c = b + c = c$. Thus $a \leq c$.

To show $a + b$ is the join: $a \leq a + b$ since $a + (a + b) = (a + a) + b = a + b$. Similarly $b \leq a + b$. If $a \leq c$ and $b \leq c$, then $a + c = c, b + c = c$. $(a + b) + c = a + (b + c) = a + c = c$. Thus $a + b \leq c$. □

2.3 Ideals, Congruences, and Quotients

Definition 2.13. Let R be a commutative semiring. An *ideal* I of R is a non-empty subset $I \subseteq R$ such that $I + I \subseteq I$ and $R \times I \subseteq I$.

Lemma 2.14. *If a commutative semiring R possesses a multiplicative absorbing element z_R, then $z_R \in I$ for any ideal I.*

Proof. Let I be an ideal. Since I is non-empty, let $x \in I$. By the absorption property of ideals, $z_R \times x \in I$. By the definition of an absorbing element (Definition 2.2), $z_R \times x = z_R$. Thus $z_R \in I$. □

Definition 2.15. A commutative unital semiring R is a *Principal Ideal Semiring* (PIS) if every ideal is principal, i.e., generated by a single element $(a)_R = Ra$.

Definition 2.16. An ideal I of a semiring R is called *subtractive* (or a *k-ideal*) if for all $a, b \in R$, whenever $a \in I$ and $a + b \in I$, it follows that $b \in I$.

The presence of an additive absorbing element imposes severe restrictions on the existence of subtractive ideals.

Definition 2.17. An element $y \in R$ is an *additive absorber* if $y + b = y$ for all $b \in R$.

Lemma 2.18. *Let R be a semiring. If an ideal I contains an additive absorber y, then I is subtractive if and only if $I = R$.*

Proof. ($\Longleftarrow$) If $I = R$, it is trivially subtractive.

($\Longrightarrow$) Assume I is subtractive and $y \in I$. We show $I = R$. Let $b \in R$ be an arbitrary element. Let $a = y$. We have $a \in I$. Consider the sum $a + b = y + b$. By the property of the additive absorber y, $y + b = y$. Thus $a + b \in I$. By the definition of a subtractive ideal (Definition 2.16), since $a \in I$ and $a + b \in I$, we must conclude $b \in I$. Since b was arbitrary, $I = R$. □

Definition 2.19. A *congruence* ρ on a semiring R is an equivalence relation on R compatible with the operations: if $a\rho b$ and $c\rho d$, then $(a + c)\rho(b + d)$ and $(a \times c)\rho(b \times d)$.

Definition 2.20. Let I be an ideal of a commutative hemiring R. The *Bourne relation* ρ_I associated with I is defined by $a\rho_I b$ if and only if there exist $x, y \in I$ such that $a + x = b + y$.

Proposition 2.21. *The Bourne relation ρ_I is a congruence on R. The quotient $R/I := R/\rho_I$ is a hemiring.*

Proof. We verify ρ_I is an equivalence relation. Reflexivity: $a + 0_R = a + 0_R$. $0_R \in I$. Symmetry: $a+x = b+y \implies b+y = a+x$. Transitivity: Assume $a\rho_I b$ and $b\rho_I c$. $a+x_1 = b+y_1$ and $b+x_2 = c+y_2$ $(x_i, y_i \in I)$. We compute: $a + (x_1 + x_2) = (a + x_1) + x_2 = (b + y_1) + x_2$. By commutativity and associativity: $(b + x_2) + y_1 = (c + y_2) + y_1 = c + (y_1 + y_2)$. Since $x_1 + x_2 \in I$ and $y_1 + y_2 \in I$, $a\rho_I c$.

Compatibility with operations is verified similarly. The quotient inherits the hemiring structure. □

Theorem 2.22. *Let R be a standard commutative semiring. An ideal I is subtractive if and only if I is the kernel of the Bourne congruence ρ_I, i.e., $I = [0_R]_{\rho_I}$.*

Proof. The kernel is $K = \{a \mid a\rho_I 0_R\}$. $a\rho_I 0_R$ means $a + x = 0_R + y = y$ for some $x, y \in I$.

($\Longrightarrow$) Assume I is subtractive. $I \subseteq K$: If $a \in I$. $a + 0_R = a$. Since $0_R \in I$ and $a \in I$, we have $a\rho_I 0_R$. $K \subseteq I$: If $a \in K$, $a + x = y$ $(x, y \in I)$. We have $x \in I$ and $x + a = y \in I$. By the subtractive property, $a \in I$.

($\Longleftarrow$) Assume $K = I$. Let $a \in I$ and $a + b \in I$. Then $a\rho_I 0_R$ and $(a + b)\rho_I 0_R$. By compatibility of the congruence, $(a + b)\rho_I(0_R + b) = b$. By transitivity (since $b\rho_I(a + b)$ by symmetry and $(a + b)\rho_I 0_R$), $b\rho_I 0_R$. Thus $b \in K = I$. □

2.4 The Spectrum and Dimension

Definition 2.23. Let R be a commutative unital semiring.

1. An ideal P is ***prime*** if P is proper ($P \neq R$) and if $a \times b \in P$ implies $a \in P$ or $b \in P$. The set of prime ideals is $\mathrm{Spec}(R)$.

2. An ideal M is ***maximal*** if M is proper and there is no ideal I such that $M \subsetneq I \subsetneq R$. The set of maximal ideals is $\mathrm{MaxSpec}(R)$.

Definition 2.24. The ***Krull dimension*** of R, $\mathrm{Kdim}(R)$, is the supremum of the lengths n of chains of distinct prime ideals $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$.

2.5 Factorization Theory in Integral Semidomains

We define concepts related to factorization. The relationship between associates depends on the presence of cancellative elements.

Definition 2.25. Let R be an integral semidomain (relative to 0_R). $U(R)$ denotes the group of units.

1. Divisibility: $a|b$ if there exists $c \in R$ such that $b = ac$.

2. Associates: a and b are associates ($a \sim b$) if $a|b$ and $b|a$.

3. Irreducible element: A non-zero ($x \neq 0_R$), non-unit element x is ***irreducible*** if $x = ab$ implies $a \in U(R)$ or $b \in U(R)$.

4. Prime element: A non-zero, non-unit element p is ***prime*** if $p|ab$ implies $p|a$ or $p|b$.

5. Unique Factorization Domain (UFD): R is a UFD if every non-zero, non-unit element can be written uniquely (up to order and associates) as a product of irreducible elements.

Definition 2.26. An element $c \in R$ is ***multiplicatively cancellative*** if $ca = cb$ implies $a = b$.

Lemma 2.27. *Let R be an integral semidomain. If a, b are associates and a is multiplicatively cancellative, then $b = ua$ for some unit $u \in U(R)$.*

Proof. $a|b \implies b = ca$. $b|a \implies a = db$. Substituting, $a = d(ca) = (dc)a$. Since a is cancellative, $1_R = dc$. Thus c and d are units. $b = ca$. □

In rings that are integral domains, prime elements are always irreducible. This implication may fail in semirings if the element is not cancellative.

Lemma 2.28. *Let R be an integral semidomain. If p is a prime element and p is multiplicatively cancellative, then p is irreducible.*

Proof. Let p be prime and cancellative. Suppose $p = ab$. Since $p|ab$, by primality, $p|a$ or $p|b$. Assume without loss of generality $p|a$. Then $a = pc$ for some $c \in R$. Substituting into $p = ab$, we get $p = (pc)b = p(cb)$. We write this as $p \cdot 1_R = p(cb)$. Since p is cancellative, we may cancel p from both sides, yielding $1_R = cb$. This implies b is a unit. Therefore, p is irreducible. □

2.6 Group Actions and Symmetries

Definition 2.29. Let M be an algebraic structure. A ***G-action*** on M by automorphisms is a group homomorphism $\rho : G \to \mathrm{Aut}(M)$. An element $x \in M$ is a ***fixed point*** (or ***singlet***) if $\rho(g)(x) = x$ for all $g \in G$. The set of fixed points is denoted M^G.

Proposition 2.30. *If M is a semiring, the set of fixed points M^G forms a sub-semiring of M.*

Proof. We verify closure under the operations. Let $x, y \in M^G$. Addition: $g \cdot (x + y) = g \cdot x + g \cdot y$ (since $\rho(g)$ is a homomorphism). Since x, y are fixed, $g \cdot x = x$ and $g \cdot y = y$. Thus $g \cdot (x + y) = x + y$. So $x + y \in M^G$. Multiplication: $g \cdot (xy) = (g \cdot x)(g \cdot y) = xy$. So $xy \in M^G$. □

3 The Globalization Functor G and the Construction of $\mathcal{S}$

We introduce the globalization construction applied to commutative rings, which systematically adjoins a new element functioning simultaneously as the additive identity and the multiplicative absorber. We then apply this construction to the ring of integers $\mathbb{Z}$ to define the structure $\mathcal{S}$.

3.1 The Globalization Functor G

Let **CRing** denote the category of commutative unital rings and $\mathbf{CSRing}_0$ the category of standard commutative unital semirings.

Definition 3.1 (Globalization Functor G). The globalization functor $G : \mathbf{CRing} \to \mathbf{CSRing}_0$ is defined as follows. For $R \in \mathbf{CRing}$, $G(R)$ is constructed by adjoining a formal element $\mathcal{T}_R \notin R$. The underlying set is $G(R) = R \cup \{\mathcal{T}_R\}$. The operations are defined such that $\mathcal{T}_R$ acts as the additive identity and the multiplicative absorber, extending the operations of R.

Addition ($+_{G(R)}$): $a +_{G(R)} b$ is $a +_R b$ if $a, b \in R$; otherwise it is the element that is not $\mathcal{T}_R$ (if one exists), or $\mathcal{T}_R$ if both are $\mathcal{T}_R$.

Multiplication ($\times_{G(R)}$): $a \times_{G(R)} b$ is $a \times_R b$ if $a, b \in R$; otherwise it is $\mathcal{T}_R$.

For a ring homomorphism $f : R_1 \to R_2$, $G(f) : G(R_1) \to G(R_2)$ is defined by $G(f)(x) = f(x)$ if $x \in R_1$, and $G(f)(\mathcal{T}_{R_1}) = \mathcal{T}_{R_2}$.

Proposition 3.2. *The construction G defines a covariant functor from* **CRing** *to* $\mathbf{CSRing}_0$.

Proof. The verification that $G(R)$ satisfies the axioms of a standard commutative unital semiring is performed by systematic case analysis, relying on the ring axioms of R and the defining properties of $\mathcal{T}_R$. This is detailed for $R = \mathbb{Z}$ in Theorem 3.6 and generalizes immediately. The verification that $G(f)$ is a homomorphism and that composition and identities are preserved is straightforward. □

3.2 Definition of $\mathcal{S} = G(\mathbb{Z})$ and Operations

We apply the globalization functor to the ring of integers $\mathbb{Z}$.

Construction 3.3. Let $\mathcal{S} := G(\mathbb{Z})$. We denote the adjoined element $\mathcal{T}_\mathbb{Z}$ simply by $\mathcal{T}$. Thus $\mathcal{S} = \mathbb{Z} \cup \{\mathcal{T}\}$.

The operations on $\mathcal{S}$ are explicitly given by Definition 3.1.

Definition 3.4 (Addition on $\mathcal{S}$). For $a, b \in \mathcal{S}$:

$$a + b := \begin{cases} a +_\mathbb{Z} b & \text{if } a \in \mathbb{Z}, b \in \mathbb{Z} \quad \text{(Case A1)} \\ a & \text{if } a \in \mathbb{Z}, b = \mathcal{T} \quad \text{(Case A2)} \\ b & \text{if } a = \mathcal{T}, b \in \mathbb{Z} \quad \text{(Case A3)} \\ \mathcal{T} & \text{if } a = \mathcal{T}, b = \mathcal{T} \quad \text{(Case A4)} \end{cases} \tag{3.1}$$

Definition 3.5 (Multiplication on $\mathcal{S}$). For $a, b \in \mathcal{S}$:

$$a \times b := \begin{cases} a \times_\mathbb{Z} b & \text{if } a \in \mathbb{Z}, b \in \mathbb{Z} \quad \text{(Case M1)} \\ \mathcal{T} & \text{if } a \in \mathbb{Z}, b = \mathcal{T} \quad \text{(Case M2)} \\ \mathcal{T} & \text{if } a = \mathcal{T}, b \in \mathbb{Z} \quad \text{(Case M3)} \\ \mathcal{T} & \text{if } a = \mathcal{T}, b = \mathcal{T} \quad \text{(Case M4)} \end{cases} \tag{3.2}$$

3.3 Verification of the Semiring Structure

Theorem 3.6. *The structure $(\mathcal{S}, +, \times)$ is a standard commutative unital semiring. The additive identity is $0_\mathcal{S} = \mathcal{T}$. The multiplicative identity is $1_\mathcal{S} = 1_\mathbb{Z}$.*

Proof. We systematically verify the axioms through an examination of all possible cases.

Part I: $(\mathcal{S}, +)$ is a commutative monoid.

I.1. Commutativity (Axiom S1). $a + b = b + a$. Case I.1.1: $a, b \in \mathbb{Z}$. $a + b = a +_{\mathbb{Z}} b$. $b + a = b +_{\mathbb{Z}} a$. Equality holds by commutativity of $+_{\mathbb{Z}}$. Case I.1.2: $a \in \mathbb{Z}, b = \mathcal{T}$. $a + b = a$ (Case A2). $b + a = a$ (Case A3).

I.2. Associativity (Axiom S1). $(a + b) + c = a + (b + c)$.

Case I.2.1: $a, b, c \in \mathbb{Z}$. Equality holds by associativity of $+_{\mathbb{Z}}$.

Case I.2.2: $a, b \in \mathbb{Z}, c = \mathcal{T}$. LHS: $(a +_{\mathbb{Z}} b) + \mathcal{T}$. Since $a +_{\mathbb{Z}} b \in \mathbb{Z}$, by Case A2, this equals $a +_{\mathbb{Z}} b$. RHS: $a + (b + \mathcal{T})$. $b + \mathcal{T} = b$ (Case A2). $a + b = a +_{\mathbb{Z}} b$.

Case I.2.3: $a \in \mathbb{Z}, b = \mathcal{T}, c \in \mathbb{Z}$. LHS: $(a + \mathcal{T}) + c = a + c$. RHS: $a + (\mathcal{T} + c) = a + c$.

Case I.2.4: $a = \mathcal{T}, b, c \in \mathbb{Z}$. LHS: $(\mathcal{T} + b) + c = b + c$. RHS: $\mathcal{T} + (b + c) = b + c$.

Case I.2.5: Two elements are $\mathcal{T}$. E.g., $a \in \mathbb{Z}, b = \mathcal{T}, c = \mathcal{T}$. LHS: $(a + \mathcal{T}) + \mathcal{T} = a + \mathcal{T} = a$. RHS: $a + (\mathcal{T} + \mathcal{T}) = a + \mathcal{T} = a$.

Case I.2.8: $a = b = c = \mathcal{T}$. LHS = RHS = $\mathcal{T}$.

I.3. Additive Identity. By inspection of Definition 3.4 (Cases A2, A3, A4), $\mathcal{T}$ satisfies $a + \mathcal{T} = a$ for all $a \in \mathcal{S}$. Thus $0_{\mathcal{S}} = \mathcal{T}$.

Part II: $(\mathcal{S}, \times)$ is a commutative monoid.

II.1. Commutativity. $a \times b = b \times a$. Case II.1.1: $a, b \in \mathbb{Z}$. Equality holds by commutativity of $\times_{\mathbb{Z}}$. Case II.1.2: $a \in \mathbb{Z}, b = \mathcal{T}$. $a \times b = \mathcal{T}$ (Case M2). $b \times a = \mathcal{T}$ (Case M3).

II.2. Associativity (Axiom S2). $(a \times b) \times c = a \times (b \times c)$.

Case II.2.1: $a, b, c \in \mathbb{Z}$. Equality holds by associativity of $\times_{\mathbb{Z}}$.

Case II.2.2: At least one element is $\mathcal{T}$. By inspection of Definition 3.5 (Cases M2, M3, M4), $\mathcal{T}$ is a multiplicative absorbing element. If any variable is $\mathcal{T}$, the product involving that variable is $\mathcal{T}$. Suppose $a = \mathcal{T}$. LHS: $(\mathcal{T} \times b) \times c = \mathcal{T} \times c = \mathcal{T}$. RHS: $\mathcal{T} \times (b \times c) = \mathcal{T}$.

II.3. Multiplicative Identity. We verify $1_{\mathbb{Z}}$ is the identity. If $a \in \mathbb{Z}$. $1_{\mathbb{Z}} \times a = 1_{\mathbb{Z}} \times_{\mathbb{Z}} a = a$ (Case M1). If $a = \mathcal{T}$. $1_{\mathbb{Z}} \times \mathcal{T} = \mathcal{T}$ (Case M2). Thus $1_{\mathcal{S}} = 1_{\mathbb{Z}}$.

Part III: Distributivity (Axiom S3). We verify $a \times (b + c) = (a \times b) + (a \times c)$.

Case III.1: $a, b, c \in \mathbb{Z}$. Equality holds by distributivity in $\mathbb{Z}$.

Case III.2: $a \in \mathbb{Z}$. Subcase III.2.a: $b \in \mathbb{Z}, c = \mathcal{T}$. LHS: $a \times (b + \mathcal{T}) = a \times b$. RHS: $(a \times b) + (a \times \mathcal{T})$. $a \times \mathcal{T} = \mathcal{T}$. $(a \times b) + \mathcal{T}$. Since $a \times b \in \mathbb{Z}$, this equals $a \times b$. Subcase III.2.c: $b = \mathcal{T}, c = \mathcal{T}$. LHS: $a \times (\mathcal{T} + \mathcal{T}) = a \times \mathcal{T} = \mathcal{T}$. RHS: $(a \times \mathcal{T}) + (a \times \mathcal{T}) = \mathcal{T} + \mathcal{T} = \mathcal{T}$.

Case III.3: $a = \mathcal{T}$. LHS: $\mathcal{T} \times (b + c) = \mathcal{T}$. RHS: $(\mathcal{T} \times b) + (\mathcal{T} \times c) = \mathcal{T} + \mathcal{T} = \mathcal{T}$.

Part IV: Standard Semiring. The additive identity $0_{\mathcal{S}} = \mathcal{T}$ (Part I.3) is the multiplicative absorber (Part II.2.2). □

4 Algebraic Properties of the Semiring $\mathcal{S}$

4.1 Basic Element Properties

Proposition 4.1. *The additive idempotents of $\mathcal{S}$ ($x + x = x$) are $\{0_{\mathbb{Z}}, \mathcal{T}\}$. The multiplicative idempotents of $\mathcal{S}$ ($x \times x = x$) are $\{0_{\mathbb{Z}}, 1_{\mathbb{Z}}, \mathcal{T}\}$.*

Proof. 1. Additive idempotents. If $x \in \mathbb{Z}$. $x + x = x +_{\mathbb{Z}} x = 2x$. The equation $2x = x$ in $\mathbb{Z}$ implies $x = 0_{\mathbb{Z}}$. If $x = \mathcal{T}$. $\mathcal{T} + \mathcal{T} = \mathcal{T}$.

2. Multiplicative idempotents. If $x \in \mathbb{Z}$. $x \times x = x \times_{\mathbb{Z}} x = x^2$. The equation $x^2 = x$ in $\mathbb{Z}$ implies $x(x - 1) = 0_{\mathbb{Z}}$. Thus $x = 0_{\mathbb{Z}}$ or $x = 1_{\mathbb{Z}}$. If $x = \mathcal{T}$. $\mathcal{T} \times \mathcal{T} = \mathcal{T}$. □

Proposition 4.2. *The group of units of $\mathcal{S}$ is $U(\mathcal{S}) = \{1_{\mathbb{Z}}, -1_{\mathbb{Z}}\}$.*

Proof. We seek $x, y \in \mathcal{S}$ such that $x \times y = 1_{\mathcal{S}} = 1_{\mathbb{Z}}$. If $x = \mathcal{T}$ or $y = \mathcal{T}$, then $x \times y = \mathcal{T}$. Since $1_{\mathbb{Z}} \neq \mathcal{T}$, neither x nor y can be $\mathcal{T}$. Thus $x, y \in \mathbb{Z}$. The condition becomes $x \times_{\mathbb{Z}} y = 1_{\mathbb{Z}}$. This implies $x \in U(\mathbb{Z})$. $U(\mathbb{Z}) = \{1_{\mathbb{Z}}, -1_{\mathbb{Z}}\}$. □

Proposition 4.3. *The semiring $\mathcal{S}$ is an integral semidomain.*

Proof. We verify the absence of zero divisors. The zero element is $0_{\mathcal{S}} = \mathcal{T}$. Analyze $a \times b = \mathcal{T}$. If $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$. $a \times b = a \times_{\mathbb{Z}} b \in \mathbb{Z}$. Thus $a \times b \neq \mathcal{T}$. Therefore, $a \times b = \mathcal{T}$ implies that at least one of a or b must be $\mathcal{T}$. □

Proposition 4.4. *The semiring $\mathcal{S}$ is zerosumfree. It is not additively cancellative.*

Proof. We analyze the condition $a + b = 0_{\mathcal{S}} = \mathcal{T}$. Case 1: $a, b \in \mathbb{Z}$. $a + b = a +_{\mathbb{Z}} b \in \mathbb{Z}$. Thus $a + b \neq \mathcal{T}$. Case 2: $a \in \mathbb{Z}, b = \mathcal{T}$. $a + b = a$. We require $a = \mathcal{T}$. Contradiction. The only solution is $a = \mathcal{T}, b = \mathcal{T}$. Thus $\mathcal{S}$ is zerosumfree.

We demonstrate the failure of additive cancellation. Consider $a = 0_{\mathbb{Z}}, b = \mathcal{T}, c = 1_{\mathbb{Z}}$. $a + c = 0_{\mathbb{Z}} + 1_{\mathbb{Z}} = 1_{\mathbb{Z}}$ (Case A1). $b + c = \mathcal{T} + 1_{\mathbb{Z}} = 1_{\mathbb{Z}}$ (Case A3). We have $a + c = b + c$, but $a = 0_{\mathbb{Z}} \neq \mathcal{T} = b$. □

4.2 The Ideal Structure of $\mathcal{S}$

We characterize the ideals of $\mathcal{S}$, establishing a correspondence with the ideals of $\mathbb{Z}$.

Theorem 4.5. *The ideals of $\mathcal{S}$ are precisely the following sets:*

1. *The zero ideal $I_0 = \{\mathcal{T}\}$.*
2. *The sets of the form $I_J = J \cup \{\mathcal{T}\}$, where J is a (non-empty) ideal of the ring $\mathbb{Z}$.*

Proof. Part I: Characterization of an arbitrary ideal. Let I be an ideal of $\mathcal{S}$. Since $\mathcal{T}$ is the multiplicative absorber, by Lemma 2.14, $\mathcal{T} \in I$. Define $J = I \cap \mathbb{Z}$. Then $I = J \cup \{\mathcal{T}\}$.

Case 1: $J = \emptyset$. Then $I = \{\mathcal{T}\} = I_0$.

Case 2: $J \neq \emptyset$. We demonstrate that J is an ideal of $\mathbb{Z}$. (i) Closure under $+_{\mathbb{Z}}$. Let $a, b \in J$. Then $a, b \in I$. $a + b \in I$. Since $a, b \in \mathbb{Z}$, $a + b = a +_{\mathbb{Z}} b \in \mathbb{Z}$. Thus $a +_{\mathbb{Z}} b \in J$. (ii) Absorption under $\times_{\mathbb{Z}}$. Let $r \in \mathbb{Z}, a \in J$. $r \in \mathcal{S}, a \in I$. $r \times a \in I$. Since $r, a \in \mathbb{Z}$, $r \times a = r \times_{\mathbb{Z}} a \in \mathbb{Z}$. Thus $r \times_{\mathbb{Z}} a \in J$.

Part II: Verification that the forms define ideals. 1. I_0 is readily verified as an ideal.

2. Let J be a non-empty ideal of $\mathbb{Z}$. $I_J = J \cup \{\mathcal{T}\}$. (i) Additive closure. If $a, b \in J$, $a + b \in J$. If $a \in J, b = \mathcal{T}$, $a + \mathcal{T} = a \in J$. (ii) Absorption by $\mathcal{S}$. $r \in \mathcal{S}, a \in I_J$. If $r \in \mathbb{Z}$. If $a \in J$, $r \times a \in J$. If $a = \mathcal{T}$, $r \times a = \mathcal{T}$. If $r = \mathcal{T}$. $r \times a = \mathcal{T}$. I_J is an ideal of $\mathcal{S}$. □

Corollary 4.6. *The map $J \mapsto I_J$ defines an isomorphism between the lattice of ideals of $\mathbb{Z}$ and the lattice of non-zero ideals of $\mathcal{S}$.*

Theorem 4.7. *The semiring $\mathcal{S}$ is a Principal Ideal Semiring (PIS).*

Proof. We utilize the characterization in Theorem 4.5 and the fact that $\mathbb{Z}$ is a PID (Theorem 1.16).

1. $I_0 = (\mathcal{T})_{\mathcal{S}}$.

2. Ideals I_J. Since J is an ideal of $\mathbb{Z}$, $J = (n)_{\mathbb{Z}}$ for some $n \geq 0$. We compute the principal ideal generated by n in $\mathcal{S}$. $(n)_{\mathcal{S}} = \{r \times n \mid r \in \mathcal{S}\}$. If $r \in \mathbb{Z}$: $r \times n = r \times_{\mathbb{Z}} n$. The collection of these elements is $n\mathbb{Z} = J$. If $r = \mathcal{T}$: $\mathcal{T} \times n = \mathcal{T}$. Thus, $(n)_{\mathcal{S}} = J \cup \{\mathcal{T}\} = I_J$. □

Corollary 4.8. *The semiring $\mathcal{S}$ is Noetherian.*

Proof. Let $I_1 \subseteq I_2 \subseteq \dots$ be an ascending chain of ideals in $\mathcal{S}$. Let $J_k = I_k \cap \mathbb{Z}$. We obtain an ascending chain $J_1 \subseteq J_2 \subseteq \dots$. If the chain is not eventually I_0, the J_k (for k sufficiently large) form an ascending chain of ideals in $\mathbb{Z}$. Since $\mathbb{Z}$ is Noetherian, the chain J_k stabilizes at some J_M. Consequently, $I_k = J_k \cup \{\mathcal{T}\}$ stabilizes at I_M. □

4.3 Subtractive Ideals in $\mathcal{S}$

Theorem 4.9. *Every ideal of the semiring $\mathcal{S}$ is subtractive.*

Proof. Let I be an ideal of $\mathcal{S}$. We must show that if $a \in I$ and $a + b \in I$, then $b \in I$.

Case 1: $I = I_0 = \{\mathcal{T}\}$. $a = \mathcal{T}$. $a + b = \mathcal{T} + b = b$. $b \in I_0$.

Case 2: $I = I_J = J \cup \{\mathcal{T}\}$. Subcase 2a: $a = \mathcal{T}$. $a + b = b$. $b \in I_J$.

Subcase 2b: $a \in J$. (i) $b = \mathcal{T}$. $b \in I_J$. (ii) $b \in \mathbb{Z}$. $a + b = a +_{\mathbb{Z}} b$. Since $a + b \in I_J$ and $a + b \in \mathbb{Z}$, we have $a +_{\mathbb{Z}} b \in J$. Since J is an ideal of the ring $\mathbb{Z}$, it is an additive subgroup. $a \in J$ and $a +_{\mathbb{Z}} b \in J$ implies that the difference $b = (a +_{\mathbb{Z}} b) -_{\mathbb{Z}} a$ must belong to J. Thus $b \in I_J$. □

5 The Spectrum and Geometric Structure of $\mathcal{S}$

We determine the structure of the prime spectrum $\mathrm{Spec}(\mathcal{S})$, analyze the associated Zariski topology, compute the Krull dimension, and construct the associated semiring scheme.

5.1 The Spectrum of Prime Ideals

Theorem 5.1. *The spectrum* $\mathrm{Spec}(\mathcal{S})$ *consists precisely of the following ideals:*

1. *The zero ideal* $P_{\mathcal{T}} = \{\mathcal{T}\}$.

2. *The ideals* $P_J = J \cup \{\mathcal{T}\}$, *where* J *is a prime ideal of* $\mathbb{Z}$. *Explicitly:* $P_{(0)} = \{0_{\mathbb{Z}}, \mathcal{T}\}$ *and* $P_{(p)} = p\mathbb{Z} \cup \{\mathcal{T}\}$ *(where* p *is a prime number).*

Proof. We utilize the characterization from Theorem 4.5.

1. $I_0 = \{\mathcal{T}\}$. It is proper since $1_{\mathbb{Z}} \notin I_0$. The condition $a \times b = \mathcal{T}$ implies $a = \mathcal{T}$ or $b = \mathcal{T}$ by Proposition 4.3. Thus I_0 is prime.

2. Ideals I_J. I_J is proper if and only if J is proper in $\mathbb{Z}$. We establish the equivalence: I_J is prime in $\mathcal{S}$ if and only if J is prime in $\mathbb{Z}$.

($\Longrightarrow$) Assume I_J is prime. Let $a, b \in \mathbb{Z}$ such that $a \times_{\mathbb{Z}} b \in J$. Then $a \times b \in I_J$. Since I_J is prime, $a \in I_J$ or $b \in I_J$. Since $a, b \in \mathbb{Z}$, $a \in J$ or $b \in J$.

($\Longleftarrow$) Assume J is prime in $\mathbb{Z}$. Let $a, b \in \mathcal{S}$ such that $a \times b \in I_J$. If $a = \mathcal{T}$ or $b = \mathcal{T}$, we are done. If $a, b \in \mathbb{Z}$. $a \times b \in I_J \cap \mathbb{Z} = J$. Since J is prime, $a \in J$ or $b \in J$.

The prime ideals of $\mathbb{Z}$ are (0) and (p). □

Theorem 5.2. *The maximal spectrum* $\mathrm{MaxSpec}(\mathcal{S})$ *consists of the ideals* $P_{(p)}$. *The minimal spectrum* $\mathrm{MinSpec}(\mathcal{S})$ *consists solely of* $P_{\mathcal{T}}$.

Proof. We analyze the inclusions in the poset $(\mathrm{Spec}(\mathcal{S}), \subseteq)$. (i) $P_{\mathcal{T}} \subsetneq P_{(0)}$. $0_{\mathbb{Z}} \in P_{(0)}$ and $0_{\mathbb{Z}} \notin P_{\mathcal{T}}$. (ii) $P_{(0)} \subsetneq P_{(p)}$. Since $(0) \subsetneq p\mathbb{Z}$, the inclusion holds. It is strict because $p \in P_{(p)}$ and $p \notin P_{(0)}$. (iii) $P_{(p)} \subseteq P_{(q)}$ iff $p\mathbb{Z} \subseteq q\mathbb{Z}$. This occurs iff $q|p$, so $p = q$.

The ideals $P_{(p)}$ are the maximal elements. $P_{\mathcal{T}}$ is the unique minimum element. □

Theorem 5.3. *The Krull dimension of the semiring* $\mathcal{S}$ *is 2.*

Proof. Based on the inclusion analysis in the proof of Theorem 5.2, the maximal chains of distinct prime ideals are of the form:

$$P_{\mathcal{T}} \subsetneq P_{(0)} \subsetneq P_{(p)}. \tag{5.1}$$

The length of this chain is $n = 2$. Therefore, $\mathrm{Kdim}(\mathcal{S}) = 2$. □

5.2 The Zariski Topology on $\mathrm{Spec}(\mathcal{S})$

Let $X = \mathrm{Spec}(\mathcal{S})$. The closed sets are $V(I) = \{P \in X \mid I \subseteq P\}$. We denote $I_{(n)} = (n)_{\mathcal{S}}$ for $n \geq 0$.

Lemma 5.4. *The closed sets in the Zariski topology on* $\mathrm{Spec}(\mathcal{S})$ *are characterized as follows:*

1. $V(P_{\mathcal{T}}) = X$.

2. $V(P_{(0)}) = \{P_{(0)}\} \cup \{P_{(p)} \mid p \text{ prime}\}$.

3. *If* $n > 1$. $V(I_{(n)}) = \{P_{(p)} \mid p \text{ is a prime divisor of } n\}$.

4. $V(I_{(1)}) = V(\mathcal{S}) = \emptyset$.

Proof. 1. $I = P_{\mathcal{T}}$. Contained in every ideal.

2. $I = P_{(0)}$. We require $0_{\mathbb{Z}} \in P$. $P_{\mathcal{T}}$ does not contain $0_{\mathbb{Z}}$. P_J contains $0_{\mathbb{Z}}$ since J is an ideal of $\mathbb{Z}$.

3. $I = I_{(n)}, n > 1$. Requires $n \in P$. This excludes $P_{\mathcal{T}}$ and $P_{(0)}$. We must have $P = P_{(p)}$ such that $n \in p\mathbb{Z}$, i.e., $p|n$.

4. $I = \mathcal{S}$. Prime ideals are proper. □

Theorem 5.5. *The topological space* $\mathrm{Spec}(\mathcal{S})$ *exhibits the following properties:*

1. *The closed points are the maximal ideals* $P_{(p)}$.
2. *The point* $P_{\mathcal{T}}$ *is the unique generic point. Consequently,* $\mathrm{Spec}(\mathcal{S})$ *is irreducible.*
3. $\mathrm{Spec}(\mathcal{S})$ *is a Noetherian topological space.*
4. $\mathrm{Spec}(\mathcal{S})$ *is* T_0 *but not* T_1.

Proof. 1. A point P is closed if $\{P\} = V(I)$. This is equivalent to P being maximal.

2. The closure of $\{P_{\mathcal{T}}\}$ is $V(P_{\mathcal{T}}) = X$ (Lemma 5.4.1). The existence of a unique generic point implies irreducibility.

3. Follows since $\mathcal{S}$ is Noetherian (Corollary 4.8).

4. Since $P_{\mathcal{T}}$ and $P_{(0)}$ are not closed, the space is not T_1. It is T_0 because the specialization order (inclusion) distinguishes the points. □

5.3 The Structure Sheaf and the Semiring Scheme

We define the structure sheaf on $X = \mathrm{Spec}(\mathcal{S})$, utilizing the theory of localization for semirings.

Definition 5.6. Let R be a commutative unital semiring and M a multiplicative subset ($1_R \in M, M \times M \subseteq M$). The localization R_M is the set of equivalence classes $[r, m]$ ($r \in R, m \in M$) under the relation $(r_1, m_1) \sim (r_2, m_2)$ if there exists $k \in M$ such that $k(r_1 m_2) = k(r_2 m_1)$. The operations are defined by $[r_1, m_1] + [r_2, m_2] = [r_1 m_2 + r_2 m_1, m_1 m_2]$ and $[r_1, m_1] \times [r_2, m_2] = [r_1 r_2, m_1 m_2]$.

Lemma 5.7. *The localization* R_M *is a commutative unital semiring.*

Proof. The verification that the operations are well-defined and satisfy the axioms is standard, analogous to the proof for rings. □

We define the structure sheaf on the basis of distinguished open sets $D(f) = \{P \in X \mid f \notin P\}$.

Definition 5.8. The structure sheaf $\mathcal{O}_{\mathrm{Spec}(\mathcal{S})}$ on $X = \mathrm{Spec}(\mathcal{S})$ is the unique sheaf of semirings such that for any $f \in \mathcal{S}$, $\mathcal{O}_{\mathrm{Spec}(\mathcal{S})}(D(f)) = \mathcal{S}_f$, the localization of $\mathcal{S}$ at the multiplicative set $\{f^n \mid n \geq 0\}$.

The existence and uniqueness of such a sheaf on the base $D(f)$ extending to a sheaf on X is a standard result in sheaf theory applicable to sheaves of algebraic structures.

We analyze the structure of the localizations $\mathcal{S}_f$.

Proposition 5.9. *Let* $f \in \mathcal{S}$.

1. *If* $f = \mathcal{T}$. $D(\mathcal{T}) = \emptyset$. $\mathcal{S}_{\mathcal{T}}$ *is the zero semiring* $\{0\}$.
2. *If* $f \in \mathbb{Z}$. $\mathcal{S}_f \cong G(\mathbb{Z}_f)$.

Proof. 1. $M = \{\mathcal{T}^n\}$. Since $\mathcal{T}$ is absorbing, $M = \{\mathcal{T}, 1\}$ (if we include $f^0 = 1$). The equivalence relation identifies all elements. $[r_1, m_1] \sim [r_2, m_2]$. Take $k = \mathcal{T}$. $k(r_1 m_2) = \mathcal{T}$. $k(r_2 m_1) = \mathcal{T}$. Thus there is only one element.

2. $f \in \mathbb{Z}$. $M = \{f^n\}$. Since $\mathcal{T} \notin M$, this is a localization at a subset of $\mathbb{Z}$. We apply Theorem 8.4 (proven below in Section 8.2). The localization $\mathcal{S}_M$ is isomorphic to $G(\mathbb{Z}_M) = G(\mathbb{Z}_f)$. □

We examine the stalks of the structure sheaf. The stalk at a point $P \in X$ is the direct limit of the sections over open neighborhoods of P. It is isomorphic to the localization at the prime ideal P: $\mathcal{O}_{\mathrm{Spec}(\mathcal{S}), P} = \mathcal{S}_P$, where the localization is taken at the multiplicative set $M = \mathcal{S} \setminus P$.

Proposition 5.10. *The stalks of the structure sheaf* $\mathcal{O}_{\mathrm{Spec}(\mathcal{S})}$ *are as follows:*

1. *At the generic point* $P_{\mathcal{T}}$. *The stalk is isomorphic to the Boolean semiring* $\mathbb{B}$.
2. *At the codimension 1 point* $P_{(0)}$. *The stalk is isomorphic to* $G(\mathbb{Q})$.

3. *At a closed point $P_{(p)}$. The stalk is isomorphic to $G(\mathbb{Z}_{(p)})$.*

Proof. 1. $P = P_{\mathcal{T}} = \{\mathcal{T}\}$. $M = \mathcal{S} \setminus P_{\mathcal{T}} = \mathbb{Z}$. The localization $\mathcal{S}_{\mathbb{Z}}$ is the semifield of fractions $\mathrm{Frac}(\mathcal{S})$. This is computed in Theorem 8.8 (Section 8.2) to be $\mathbb{B}$.

2. $P = P_{(0)} = \{0_{\mathbb{Z}}, \mathcal{T}\}$. $M = \mathcal{S} \setminus P_{(0)} = \mathbb{Z} \setminus \{0_{\mathbb{Z}}\}$. This is the restricted semifield of fractions $Q^*(\mathcal{S})$. By Corollary 8.6 (Section 8.2), this is $G(\mathbb{Q})$.

3. $P = P_{(p)} = p\mathbb{Z} \cup \{\mathcal{T}\}$. $M = \mathcal{S} \setminus P_{(p)} = \mathbb{Z} \setminus p\mathbb{Z}$. By Theorem 8.4 (Section 8.2), this is $G(\mathbb{Z}_M)$. $\mathbb{Z}_M$ is the localization of $\mathbb{Z}$ at the prime ideal (p), denoted $\mathbb{Z}_{(p)}$. The stalk is $G(\mathbb{Z}_{(p)})$. $\square$

Definition 5.11. A *local semiring* is a commutative unital semiring possessing a unique maximal ideal. A *locally (semi)ringed space* is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of semirings such that the stalks $\mathcal{O}_X, x$ are local semirings for all $x \in X$.

Theorem 5.12. *The pair $(\mathrm{Spec}(\mathcal{S}), \mathcal{O}_{\mathrm{Spec}(\mathcal{S})})$ is a locally (semi)ringed space.*

Proof. We must verify that the stalks identified in Proposition 5.10 are local semirings.

1. $\mathbb{B} = \{0, 1\}$. The ideals are $\{0\}$ and $\mathbb{B}$. The unique maximal ideal is $\{0\}$.

2. $G(\mathbb{Q}) = \mathbb{Q} \cup \{\mathcal{T}\}$. The ideals correspond to the ideals of $\mathbb{Q}$. Since $\mathbb{Q}$ is a field, its only ideals are (0) and $\mathbb{Q}$. The ideals of $G(\mathbb{Q})$ are $I_0 = \{\mathcal{T}\}$, $I_{(0)} = \{0_{\mathbb{Q}}, \mathcal{T}\}$, and $I_{\mathbb{Q}} = G(\mathbb{Q})$. The unique maximal ideal is $I_{(0)}$. $G(\mathbb{Q})$ is a local semiring.

3. $G(\mathbb{Z}_{(p)})$. The ideals correspond to the ideals of the local ring $\mathbb{Z}_{(p)}$. The ring $\mathbb{Z}_{(p)}$ has a unique maximal ideal $M = (p)\mathbb{Z}_{(p)}$. By the lattice isomorphism (Corollary 4.6), $G(\mathbb{Z}_{(p)})$ has a unique maximal ideal $I_M = M \cup \{\mathcal{T}\}$. $\square$

Remark 5.13. The construction $(\mathrm{Spec}(\mathcal{S}), \mathcal{O}_{\mathrm{Spec}(\mathcal{S})})$ defines an affine scheme in the category of semiring schemes. The globalization construction provides a method to construct specific examples of such schemes from standard arithmetic rings. The appearance of $\mathbb{B}$ at the generic point highlights the connection to characteristic 1 geometry at the deepest level of the spectrum.

6 Divisibility and Factorization in $\mathcal{S}$

We examine concepts of divisibility and factorization within the integral semidomain $\mathcal{S}$. This analysis reveals significant deviations from the factorization theory in $\mathbb{Z}$.

6.1 Divisibility and Cancellative Elements

We analyze the divisibility relation (Definition 2.25.1). Recall $0_{\mathcal{S}} = \mathcal{T}$.

Lemma 6.1. *Let $a, b \in \mathcal{S}$.*

1. *$\mathcal{T}|a$ if and only if $a = \mathcal{T}$.*
2. *$a|\mathcal{T}$ for all $a \in \mathcal{S}$.*
3. *If $a, b \in \mathbb{Z}$ and $a \neq 0_{\mathbb{Z}}$. Then $a|_{\mathcal{S}}b$ if and only if $a|_{\mathbb{Z}}b$.*
4. *$0_{\mathbb{Z}}|a$ if and only if $a \in \{0_{\mathbb{Z}}, \mathcal{T}\}$.*
5. *If $a \in \mathbb{Z} \setminus \{0_{\mathbb{Z}}\}$, then $a|0_{\mathbb{Z}}$.*

Proof. 1. If $\mathcal{T}|a$, $a = \mathcal{T} \times c = \mathcal{T}$. Conversely, $\mathcal{T}|\mathcal{T}$.

2. $\mathcal{T} = a \times \mathcal{T}$ by the absorbing property.

3. Let $a, b \in \mathbb{Z}, a \neq 0_{\mathbb{Z}}$. ($\Longrightarrow$) If $a|_{\mathcal{S}}b$, $b = ac$. If $c = \mathcal{T}$, $b = a\mathcal{T} = \mathcal{T}$. Contradiction as $b \in \mathbb{Z}$. So $c \in \mathbb{Z}$. $b = a \times_{\mathbb{Z}} c$. ($\Longleftarrow$) If $a|_{\mathbb{Z}}b$, $b = ac$ in $\mathbb{Z}$. This holds in $\mathcal{S}$.

4. $0_{\mathbb{Z}}|a$ means $a = 0_{\mathbb{Z}} \times c$. If $c \in \mathbb{Z}$. $a = 0_{\mathbb{Z}} \times_{\mathbb{Z}} c = 0_{\mathbb{Z}}$. If $c = \mathcal{T}$. $a = 0_{\mathbb{Z}} \times \mathcal{T} = \mathcal{T}$.

5. $0_{\mathbb{Z}} = a \times_{\mathbb{Z}} 0_{\mathbb{Z}}$. This holds in $\mathcal{S}$. $\square$

We examine the cancellative property.

Proposition 6.2. *The multiplicatively cancellative elements of $\mathcal{S}$ are precisely the elements in $\mathbb{Z} \setminus \{0_\mathbb{Z}\}$. The elements $0_\mathbb{Z}$ and $\mathcal{T}$ are not cancellative.*

Proof. We check the condition $ca = cb \implies a = b$.

If $c = \mathcal{T}$. $ca = \mathcal{T}, cb = \mathcal{T}$. $\mathcal{T} = \mathcal{T}$ does not imply $a = b$.

If $c = 0_\mathbb{Z}$. Let $a = 1_\mathbb{Z}, b = 2_\mathbb{Z}$. $0_\mathbb{Z} \times 1_\mathbb{Z} = 0_\mathbb{Z}$. $0_\mathbb{Z} \times 2_\mathbb{Z} = 0_\mathbb{Z}$. But $1_\mathbb{Z} \neq 2_\mathbb{Z}$. $0_\mathbb{Z}$ is not cancellative.

If $c \in \mathbb{Z} \setminus \{0_\mathbb{Z}\}$. Suppose $ca = cb$. Case 1: $a, b \in \mathbb{Z}$. $c \times_\mathbb{Z} a = c \times_\mathbb{Z} b$. Since $\mathbb{Z}$ is an integral domain and $c \neq 0_\mathbb{Z}$, $a = b$. Case 2: $a \in \mathbb{Z}, b = \mathcal{T}$. $ca \in \mathbb{Z}$. $cb = c\mathcal{T} = \mathcal{T}$. Contradiction as $\mathbb{Z} \cap \{\mathcal{T}\} = \emptyset$. Case 3: $a = \mathcal{T}, b = \mathcal{T}$. $a = b$. □

Proposition 6.3. *Elements $a, b \in \mathcal{S}$ are associates if and only if $a = ub$ for some unit $u \in U(\mathcal{S}) = \{\pm 1_\mathbb{Z}\}$.*

Proof. ($\implies$) Assume $a|b$ and $b|a$.

Case 1: $a = \mathcal{T}$. $a|b \implies b = \mathcal{T}$. $a = 1_\mathbb{Z} b$.

Case 2: $a \in \mathbb{Z} \setminus \{0_\mathbb{Z}\}$. a is cancellative. We show $b \in \mathbb{Z} \setminus \{0_\mathbb{Z}\}$. If $b = \mathcal{T}$. $\mathcal{T}|a \implies a = \mathcal{T}$, contradiction. If $b = 0_\mathbb{Z}$. $a|0_\mathbb{Z}$ holds. $0_\mathbb{Z}|a \implies a \in \{0_\mathbb{Z}, \mathcal{T}\}$, contradiction. So $b \in \mathbb{Z} \setminus \{0_\mathbb{Z}\}$. Since a (and b) are cancellative, by Lemma 2.27, $a = ub$ for a unit u.

Case 3: $a = 0_\mathbb{Z}$. $0_\mathbb{Z}|b \implies b \in \{0_\mathbb{Z}, \mathcal{T}\}$. If $b = \mathcal{T}$. $\mathcal{T}|0_\mathbb{Z} \implies 0_\mathbb{Z} = \mathcal{T}$, contradiction. If $b = 0_\mathbb{Z}$. $0_\mathbb{Z} = 1_\mathbb{Z} \times 0_\mathbb{Z}$. □

6.2 Irreducibles and Primes

The set of non-zero ($\neq \mathcal{T}$), non-unit elements is $\mathbb{Z} \setminus \{1_\mathbb{Z}, -1_\mathbb{Z}\}$.

Proposition 6.4. *The irreducible elements of $\mathcal{S}$ are precisely the elements $\pm p$, where p is a prime number. The element $0_\mathbb{Z}$ is reducible.*

Proof. Case 1: $x = 0_\mathbb{Z}$. We seek a factorization $0_\mathbb{Z} = ab$ into non-units. Consider $a = 2$ and $b = 0_\mathbb{Z}$. Both are non-units. $2 \times 0_\mathbb{Z} = 0_\mathbb{Z}$. $0_\mathbb{Z}$ is reducible.

Case 2: $x \in \mathbb{Z} \setminus \{0_\mathbb{Z}, \pm 1_\mathbb{Z}\}$. Suppose $x = ab$ in $\mathcal{S}$. Since $x \neq \mathcal{T}$, $a, b \in \mathbb{Z}$. The factorization is equivalent to $x = a \times_\mathbb{Z} b$ in $\mathbb{Z}$. Since $U(\mathcal{S}) = U(\mathbb{Z})$, irreducibility in $\mathcal{S}$ is equivalent to irreducibility in $\mathbb{Z}$. □

Proposition 6.5. *The prime elements of $\mathcal{S}$ are precisely the irreducible elements ($\pm p$) and the element $0_\mathbb{Z}$.*

Proof. 1. Irreducibles $x = \pm p$. Suppose $x|ab$. If $a, b \in \mathbb{Z}$. If $ab \neq 0_\mathbb{Z}$, $x|_\mathbb{Z} ab \implies x|_\mathbb{Z} a$ or $x|_\mathbb{Z} b$. If $ab = 0_\mathbb{Z}$, say $a = 0_\mathbb{Z}$. $x|0_\mathbb{Z}$ holds (Lemma 6.1.5). If $a = \mathcal{T}$ or $b = \mathcal{T}$. $ab = \mathcal{T}$. $x|\mathcal{T}$ holds (Lemma 6.1.2).

2. Reducible element $x = 0_\mathbb{Z}$. Suppose $0_\mathbb{Z}|ab$. By Lemma 6.1.4, $ab \in \{0_\mathbb{Z}, \mathcal{T}\}$.

Case 2a: $ab = 0_\mathbb{Z}$. $a, b \in \mathbb{Z}$ and $a \times_\mathbb{Z} b = 0_\mathbb{Z}$. $a = 0_\mathbb{Z}$ or $b = 0_\mathbb{Z}$.

Case 2b: $ab = \mathcal{T}$. $a = \mathcal{T}$ or $b = \mathcal{T}$. $0_\mathbb{Z}|\mathcal{T}$ holds. Thus $0_\mathbb{Z}$ is a prime element in $\mathcal{S}$. □

Theorem 6.6. *In the integral semidomain $\mathcal{S}$, there exist prime elements that are not irreducible.*

Proof. The element $0_\mathbb{Z}$ is prime (Proposition 6.5) but reducible (Proposition 6.4). This occurs because $0_\mathbb{Z}$ is not multiplicatively cancellative (Proposition 6.2), hence Lemma 2.28 does not apply. □

6.3 Unique Factorization

Theorem 6.7. *The semiring $\mathcal{S}$ is not a Unique Factorization Domain (UFD).*

Proof. We demonstrate that the existence condition for factorization fails. Consider the element $0_\mathbb{Z}$ (non-zero, non-unit). Suppose $0_\mathbb{Z} = x_1 x_2 \dots x_k$, where x_i are irreducible. By Proposition 6.4, $x_i = \pm p_i$. Thus $x_i \in \mathbb{Z}$ and $x_i \neq 0_\mathbb{Z}$. The product $P = x_1 \times_\mathbb{Z} \cdots \times_\mathbb{Z} x_k$. Since $\mathbb{Z}$ is an integral domain, $P \neq 0_\mathbb{Z}$. This contradicts the assumption that the product equals $0_\mathbb{Z}$. Therefore, $0_\mathbb{Z}$ cannot be factored into a product of irreducible elements. □

7 Congruences and Quotient Structures of $\mathcal{S}$

We provide a complete classification of the congruences on $\mathcal{S}$ and analyze the resulting quotient semirings.

7.1 Classification of Congruences

Let $\rho \in \mathrm{Cong}(\mathcal{S})$. The kernel is $\mathrm{Ker}(\rho) = [\mathcal{T}]_\rho$.

Lemma 7.1. *If $\rho \in \mathrm{Cong}(\mathcal{S})$, the kernel $I = [\mathcal{T}]_\rho$ is an ideal of $\mathcal{S}$.*

Proof. Standard verification using compatibility of ρ and the properties of $\mathcal{T}$. Additive closure: $a, b \in I \implies a\rho\mathcal{T}, b\rho\mathcal{T} \implies (a+b)\rho(\mathcal{T}+\mathcal{T}) = \mathcal{T}$. Absorption: $r \in \mathcal{S}, a \in I \implies a\rho\mathcal{T} \implies (ra)\rho(r\mathcal{T}) = \mathcal{T}$. □

Let $\rho_\mathbb{Z} = \rho \cap (\mathbb{Z}\times\mathbb{Z})$. $\rho_\mathbb{Z}$ is a congruence on the ring $\mathbb{Z}$, corresponding uniquely to an ideal $(n)_\mathbb{Z}, n \geq 0$.

Lemma 7.2. *Let $\rho \in \mathrm{Cong}(\mathcal{S})$. Let $I = \mathrm{Ker}(\rho)$ and $J = I \cap \mathbb{Z}$. Let $\rho_\mathbb{Z}$ correspond to $(n)_\mathbb{Z}$.*

1. *If $I = \{\mathcal{T}\}$ (i.e., $J = \emptyset$), any $n \geq 0$ defines a valid congruence ρ_n.*
2. *If $I \neq \{\mathcal{T}\}$ (i.e., $J = (m)_\mathbb{Z} \neq \emptyset$). Then we must have $(n) = (m)$.*

Proof. 1. If $I = \{\mathcal{T}\}$. The relation ρ_n is defined by: $x\rho_n y$ iff ($x, y \in \mathbb{Z}$ and $x \equiv y(n)$) or ($x = y = \mathcal{T}$). Compatibility is verified by case analysis. If $a\rho_n b, c\rho_n d$. If $a, b, c, d \in \mathbb{Z}$, compatibility follows from $\mathbb{Z}$. If $a = \mathcal{T}$, then $b = \mathcal{T}$. $(\mathcal{T}+c)\rho_n(\mathcal{T}+d) \iff c\rho_n d$. $(\mathcal{T}\times c)\rho_n(\mathcal{T}\times d) \iff \mathcal{T}\rho_n\mathcal{T}$.

2. If $I \neq \{\mathcal{T}\}$. $J \neq \emptyset$. Since J is an ideal of $\mathbb{Z}$, $0_\mathbb{Z} \in J$. Thus $0_\mathbb{Z} \in I$, so $0_\mathbb{Z}\rho\mathcal{T}$.

We show $(n) = (m)$. (i) $(n) \subseteq (m)$. Let $x \in (n)$. $x\rho_\mathbb{Z}0_\mathbb{Z}$. So $x\rho 0_\mathbb{Z}$. Since $0_\mathbb{Z}\rho\mathcal{T}$, by transitivity, $x\rho\mathcal{T}$. So $x \in I$. Since $x \in \mathbb{Z}$, $x \in J = (m)$.

(ii) $(m) \subseteq (n)$. Let $x \in (m)$. $x \in J$, so $x\rho\mathcal{T}$. Since $0_\mathbb{Z}\rho\mathcal{T}$, by symmetry and transitivity, $x\rho 0_\mathbb{Z}$. Since $x, 0_\mathbb{Z} \in \mathbb{Z}$, $x\rho_\mathbb{Z}0_\mathbb{Z}$, so $x \in (n)$. □

Theorem 7.3. *The congruences on $\mathcal{S}$ are completely classified into the following types, parameterized by $n \geq 0$:*

1. *Type A (Trivial Kernel): Congruences ρ_n ($n \geq 0$). $\mathrm{Ker}(\rho_n) = \{\mathcal{T}\}$.*
2. *Type B (Non-Trivial Kernel): Congruences ρ'_n ($n \geq 0$). $\mathrm{Ker}(\rho'_n) = I_{(n)} = (n) \cup \{\mathcal{T}\}$. The congruence is defined by the partition consisting of the kernel $I_{(n)}$ and the distinct classes $C_k = \{x \in \mathbb{Z} \mid x \equiv k(n), x \notin (n)\}$ for $k = 1, \ldots, n-1$.*

(Note: For $n = 0$, Type B is the Rees congruence associated with $I_{(0)}$.)

Proof. The necessity follows from Lemma 7.2. Sufficiency requires verification that Type B relations are indeed congruences.

Type B Verification (ρ'_n). We verify compatibility across the partition classes. Let $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Addition: If $A = C_j, B = C_k$. $a_i + b_i \in \mathbb{Z}$. $a_i + b_i \equiv j + k(n)$. They belong to the same class (C_{j+k} or $I_{(n)}$). If $A = I_{(n)}, B = C_k$. If $a_i \in (n)$, $a_i + b_i \equiv k(n) \in C_k$. If $a_i = \mathcal{T}$, $a_i + b_i = b_i \in C_k$. The sum is always in C_k. Multiplication: Products belong to the same class (either C_{jk} or $I_{(n)}$ if one factor is in $I_{(n)}$ due to the ideal property). □

7.2 Quotient Structures

Theorem 7.4. *The quotient semirings of $\mathcal{S}$ are characterized up to isomorphism as follows:*

1. *$\mathcal{S}/\rho_n \cong G(\mathbb{Z}/n\mathbb{Z})$ (The globalization of the ring $\mathbb{Z}/n\mathbb{Z}$). ($n \geq 0$).*
2. *$\mathcal{S}/\rho'_n \cong \mathbb{Z}/n\mathbb{Z}$. ($n \geq 0$).*

Proof. 1. $\mathcal{S}/\rho_n$. The classes are $[\mathcal{T}]$ (zero element) and the classes $[k]_n = \{x \in \mathbb{Z} \mid x \equiv k(n)\}$. The operations match the definition of $G(\mathbb{Z}/n\mathbb{Z})$.

2. $\mathcal{S}/\rho'_n$. Case $n = 0$. The classes are $I_{(0)}$ (zero element) and $\{k\}$ for $k \in \mathbb{Z} \setminus \{0\}$. Let $Q_0 = \mathcal{S}/\rho'_0$. Define $\phi : \mathbb{Z} \to Q_0$ by $\phi(0) = I_{(0)}$ and $\phi(k) = \{k\}$ for $k \neq 0$. ϕ is a bijection. Homomorphism verification: $\phi(a+b) = \phi(a) + \phi(b)$. If $a, b, a+b \neq 0$. LHS $= \{a+b\}$. RHS $= \{a\} + \{b\}$. The class containing $a+b$ is $\{a+b\}$. If $a+b = 0$. LHS $= I_{(0)}$. RHS $= \{a\} + \{-a\}$. The class containing $a + (-a) = 0_{\mathbb{Z}}$ is $I_{(0)}$. Thus $Q_0 \cong \mathbb{Z}$.

Case $n \geq 1$. The classes are $I_{(n)}$ (zero element) and C_k ($k = 1, \ldots, n-1$). Define $\phi : \mathbb{Z}/n\mathbb{Z} \to \mathcal{S}/\rho'_n$ by $\phi(\bar{0}) = I_{(n)}$ and $\phi(\bar{k}) = C_k$ for $k \neq 0$. This is a bijective homomorphism, as the operations on the classes C_k mirror the operations in $\mathbb{Z}/n\mathbb{Z}$. □

8 Number Theoretic Aspects and Localization of $\mathcal{S}$

8.1 Ideal Zeta Function of $\mathcal{S}$

We define and compute the ideal zeta function associated with $\mathcal{S}$. Since all ideals are subtractive (Theorem 4.9), the norm is defined via the Bourne congruence (Definition 2.20).

Definition 8.1. Let R be a commutative unital hemiring. The ideal zeta function of R is $\zeta_R(s) = \sum_I (N(I))^{-s}$, where the sum is over all subtractive ideals I of finite norm $N(I) = |R/I|$.

We first establish the identification of the Bourne congruences.

Lemma 8.2. *Let I be an ideal of $\mathcal{S}$.*

1. *If $I = I_0 = \{\mathcal{T}\}$, the Bourne congruence ρ_{I_0} is the identity congruence ρ_0.*
2. *If $I = I_{(n)}, n \geq 0$, the Bourne congruence $\rho_{I_{(n)}}$ is the Type B congruence ρ'_n.*

Proof. 1. $I = I_0$. $a\rho_{I_0}b \iff a + x = b + y$ with $x = y = \mathcal{T}$. $a = b$. This is ρ_0.

2. $I = I_{(n)}$. Since $I_{(n)}$ is subtractive, by Theorem 2.22, $I_{(n)}$ is the kernel of $\rho_{I_{(n)}}$. We compare $\rho_{I_{(n)}}$ with ρ'_n, which also has kernel $I_{(n)}$. Let $a, b \in \mathbb{Z}$. $a\rho_{I_{(n)}}b \iff a + x = b + y$ for $x, y \in I_{(n)}$. If $x, y \in (n)$. $a - b = y - x \in (n)$. $a \equiv b(n)$. If $x = \mathcal{T}, y \in (n)$. $a = b + y$. $a \equiv b(n)$. If $x = \mathcal{T}, y = \mathcal{T}$. $a = b$. In all cases, $a \equiv b(n)$. The partition induced by $\rho_{I_{(n)}}$ restricted to $\mathbb{Z}$ is the congruence modulo n. This matches the definition of ρ'_n. □

Theorem 8.3. *The ideal zeta function of the semiring $\mathcal{S}$ is identical to the Riemann zeta function $\zeta(s)$.*

Proof. We determine the norms of the ideals based on the quotients identified in Theorem 7.4 and Lemma 8.2.

1. I_0. The quotient is $\mathcal{S}/\rho_0 \cong \mathcal{S}$. $N(I_0) = |\mathcal{S}|$, infinite.
2. $I_{(0)}$. The quotient is $\mathcal{S}/\rho'_0 \cong \mathbb{Z}$. $N(I_{(0)}) = |\mathbb{Z}|$, infinite.
3. $I_{(n)}, n \geq 1$. The quotient is $\mathcal{S}/\rho'_n \cong \mathbb{Z}/n\mathbb{Z}$. The norm is $N(I_{(n)}) = n$. This is finite.

We sum over the ideals of finite norm:

$$\zeta_{\mathcal{S}}(s) = \sum_{n=1}^{\infty} (N(I_{(n)}))^{-s} = \sum_{n=1}^{\infty} n^{-s}. \tag{8.1}$$

This series is the definition of the Riemann zeta function $\zeta(s)$. □

8.2 Localization of $\mathcal{S}$

We investigate the localization of $\mathcal{S}$ (Definition 5.6).

Theorem 8.4. *Let $M \subseteq \mathbb{Z}$ be a multiplicative subset of $\mathbb{Z}$. The localization $\mathcal{S}_M$ is isomorphic to the globalization of the localized ring $\mathbb{Z}_M$. $\mathcal{S}_M \cong G(\mathbb{Z}_M)$.*

Proof. $\mathcal{S}_M$ consists of classes $[r, m], r \in \mathcal{S}, m \in M$.

We analyze the structure of the classes. Case 1: $r \in \mathbb{Z}$. The subset of classes $C_{\mathbb{Z}} = \{[r, m] \mid r \in \mathbb{Z}, m \in M\}$. The equivalence relation restricted to this subset coincides with the definition of $\mathbb{Z}_M$. $[r_1, m_1] \sim [r_2, m_2]$ means $\exists k \in M$ such that $k(r_1 m_2) = k(r_2 m_1)$. Since all elements are in $\mathbb{Z}$, the operations are those of $\mathbb{Z}$. Thus $C_{\mathbb{Z}} \cong \mathbb{Z}_M$.

Case 2: $r = \mathcal{T}$. The class $[\mathcal{T}, m]$. We show this class is unique. Let $m_1, m_2 \in M$. $[\mathcal{T}, m_1] \sim [\mathcal{T}, m_2]$? We need $k(\mathcal{T} m_2) = k(\mathcal{T} m_1)$. Since $k, m_i \in M \subseteq \mathbb{Z}$, by Case M2, $\mathcal{T} m_i = \mathcal{T}$ and $k\mathcal{T} = \mathcal{T}$. Thus $\mathcal{T} = \mathcal{T}$. Let $\mathcal{T}_M$ denote this unique class.

The structure of $\mathcal{S}_M$ is $\mathbb{Z}_M \cup \{\mathcal{T}_M\}$. We verify the operations match the globalization $G(\mathbb{Z}_M)$. $\mathcal{T}_M$ is the additive identity: $[\mathcal{T}, 1] + [r, m] = [\mathcal{T} m + r1, m] = [\mathcal{T} + r, m]$. If $r \in \mathbb{Z}$, $\mathcal{T} + r = r$. The sum is $[r, m]$. $\mathcal{T}_M$ is the multiplicative absorber: $[\mathcal{T}, 1] \times [r, m] = [\mathcal{T} r, m] = [\mathcal{T}, m] = \mathcal{T}_M$. The structure is $G(\mathbb{Z}_M)$. ☐

Definition 8.5. The restricted semifield of fractions of $\mathcal{S}$ is $Q^*(\mathcal{S}) = \mathcal{S}_{M^*}$, where $M^* = \mathbb{Z} \setminus \{0_{\mathbb{Z}}\}$ (the set of cancellative elements, Proposition 6.2).

Corollary 8.6. $Q^*(\mathcal{S}) \cong G(\mathbb{Q})$.

Proof. $\mathbb{Z}_{M^*} = \mathbb{Q}$. The result follows from Theorem 8.4. ☐

We now consider the standard definition of the semifield of fractions for an integral semidomain.

Definition 8.7. Let R be an integral semidomain. The semifield of fractions $\mathrm{Frac}(R)$ is the localization R_M where $M = R \setminus \{0_R\}$.

In $\mathcal{S}$, $0_{\mathcal{S}} = \mathcal{T}$. The set of non-zero elements is $M = \mathcal{S} \setminus \{\mathcal{T}\} = \mathbb{Z}$. Crucially, $0_{\mathbb{Z}} \in M$.

Theorem 8.8. *The semifield of fractions of $\mathcal{S}$ is* $\mathrm{Frac}(\mathcal{S}) \cong \mathbb{B}$.

Proof. We analyze the localization $\mathcal{S}_{\mathbb{Z}}$. Elements are $[r, m], r \in \mathcal{S}, m \in \mathbb{Z}$. Since $0_{\mathbb{Z}} \in M$, we can utilize $k = 0_{\mathbb{Z}}$ in the definition of the equivalence relation.

We identify the equivalence classes. Class 1: The zero element $C_T = [\mathcal{T}, 1_{\mathbb{Z}}]$. It consists of all $[\mathcal{T}, m]$ for $m \in \mathbb{Z}$.

Class 2: Elements $[r, m]$ where $r \in \mathbb{Z}$. Consider two such elements $[r_1, m_1]$ and $[r_2, m_2]$. We check for equivalence using $k = 0_{\mathbb{Z}}$. LHS: $k(r_1 m_2) = 0_{\mathbb{Z}} \times (r_1 m_2)$. Since $r_1 m_2 \in \mathbb{Z}$, the product is $0_{\mathbb{Z}} \times_{\mathbb{Z}} (r_1 m_2) = 0_{\mathbb{Z}}$ (Case M1). RHS: $k(r_2 m_1) = 0_{\mathbb{Z}} \times (r_2 m_1) = 0_{\mathbb{Z}}$. Since LHS=RHS, all such elements are equivalent. We denote this class by $C_Z = [1_{\mathbb{Z}}, 1_{\mathbb{Z}}]$.

We verify that the two classes are distinct. Is $C_T \sim C_Z$? $[\mathcal{T}, 1] \sim [1, 1]$ requires $\exists k \in \mathbb{Z}$ such that $k(\mathcal{T}1) = k(1 \cdot 1)$. $k\mathcal{T} = k$. If $k \in \mathbb{Z}$, $k\mathcal{T} = \mathcal{T}$ (Case M2). So we require $\mathcal{T} = k$. This contradicts $k \in \mathbb{Z}$. The localization $\mathrm{Frac}(\mathcal{S})$ has exactly two elements, $\{C_T, C_Z\}$.

We analyze the operations. C_T is the zero element. C_Z is the unity element. We check addition $C_Z + C_Z$. $C_Z + C_Z = [1, 1] + [1, 1] = [1 \cdot 1 + 1 \cdot 1, 1 \cdot 1] = [1 +_{\mathbb{Z}} 1, 1] = [2_{\mathbb{Z}}, 1_{\mathbb{Z}}]$. Since $2_{\mathbb{Z}} \in \mathbb{Z}$, $[2_{\mathbb{Z}}, 1_{\mathbb{Z}}] \in C_Z$. Thus $C_Z + C_Z = C_Z$.

The structure $(\{C_T, C_Z\}, +, \times)$ is an idempotent semiring with two elements. It is isomorphic to the Boolean semiring $\mathbb{B}$. ☐

9 Symmetry Analysis and Singlets in $\mathcal{S}$

We analyze the symmetries of $\mathcal{S}$ arising from its group of units.

9.1 The Canonical $\mathbb{Z}/2\mathbb{Z}$ Action

By Proposition 4.2, $U(\mathcal{S}) = \{1_{\mathbb{Z}}, -1_{\mathbb{Z}}\} \cong \mathbb{Z}/2\mathbb{Z}$.

Definition 9.1. The canonical $\mathbb{Z}/2\mathbb{Z}$ action on $\mathcal{S}$ is the action $\Psi : U(\mathcal{S}) \times \mathcal{S} \to \mathcal{S}$ defined by multiplication: $\Psi(g, s) := g \times s$.

Theorem 9.2. *The action Ψ is an action by automorphisms of the additive monoid $(\mathcal{S}, +)$.*

Proof. We verify the conditions of Definition 2.29. Let $\Sigma_g(s) = g \times s$.

1. Σ_g is an additive homomorphism. $\Sigma_g(a+b) = g(a+b)$. By distributivity in $\mathcal{S}$ (Theorem 3.6), this equals $ga + gb = \Sigma_g(a) + \Sigma_g(b)$.
2. Σ_g is bijective. The inverse is $\Sigma_{g^{-1}}$. $\Sigma_{g^{-1}}(\Sigma_g(s)) = g^{-1}(gs) = (g^{-1}g)s = 1_{\mathbb{Z}}s = s$.
3. $g \mapsto \Sigma_g$ is a group homomorphism. $\Sigma_{g_1g_2}(s) = (g_1g_2)s = g_1(g_2s) = \Sigma_{g_1}(\Sigma_{g_2}(s))$. □

9.2 Characterization of Singlets

We determine the fixed points (singlets) of the action Ψ. An element s is a singlet if and only if it is fixed by the generator $-1_{\mathbb{Z}}$, i.e., $(-1_{\mathbb{Z}}) \times s = s$.

Theorem 9.3. *The set of singlets in $\mathcal{S}$ under the canonical $\mathbb{Z}/2\mathbb{Z}$ action Ψ is $\mathcal{A} = \{0_{\mathbb{Z}}, \mathcal{T}\}$.*

Proof. We solve the equation $(-1_{\mathbb{Z}}) \times s = s$.

Case 1: $s = n \in \mathbb{Z}$. $(-1_{\mathbb{Z}}) \times n = (-1_{\mathbb{Z}}) \times_{\mathbb{Z}} n = -n$. We require $-n = n$. This implies $2n = 0_{\mathbb{Z}}$. Since $\mathrm{char}(\mathbb{Z}) = 0$, $n = 0_{\mathbb{Z}}$.

Case 2: $s = \mathcal{T}$. $(-1_{\mathbb{Z}}) \times \mathcal{T}$. By Case M2, this equals $\mathcal{T}$. □

9.3 The Algebraic Structure of the Set of Singlets $\mathcal{A}$

Theorem 9.4. *The subset $\mathcal{A} = \{0_{\mathbb{Z}}, \mathcal{T}\}$ forms a sub-semiring of $\mathcal{S}$ (by Proposition 2.30). This sub-semiring is isomorphic to the Boolean semiring $\mathbb{B}$.*

Proof. We examine the operation tables restricted to $\mathcal{A}$.

Addition (+): $0_{\mathbb{Z}} + 0_{\mathbb{Z}} = 0_{\mathbb{Z}}$. $0_{\mathbb{Z}} + \mathcal{T} = 0_{\mathbb{Z}}$. $\mathcal{T} + \mathcal{T} = \mathcal{T}$. Multiplication ($\times$): $0_{\mathbb{Z}} \times 0_{\mathbb{Z}} = 0_{\mathbb{Z}}$. $0_{\mathbb{Z}} \times \mathcal{T} = \mathcal{T}$. $\mathcal{T} \times \mathcal{T} = \mathcal{T}$.

Identities in $\mathcal{A}$: Additive Identity $0_{\mathcal{A}} = \mathcal{T}$. Multiplicative Identity $1_{\mathcal{A}} = 0_{\mathbb{Z}}$.

The structure is a standard commutative unital semiring. It is idempotent since $0_{\mathbb{Z}} + 0_{\mathbb{Z}} = 0_{\mathbb{Z}}$ and $\mathcal{T} + \mathcal{T} = \mathcal{T}$. The isomorphism $\psi : \mathcal{A} \to \mathbb{B}$ is defined by mapping the identities: $\psi(\mathcal{T}) = 0$ and $\psi(0_{\mathbb{Z}}) = 1$. □

Proposition 9.5. *The algebraic order (Definition 2.11) on the idempotent semiring $\mathcal{A}$ is $\mathcal{T} \leq 0_{\mathbb{Z}}$.*

Proof. We check the condition $a + b = b$. $\mathcal{T} + 0_{\mathbb{Z}} = 0_{\mathbb{Z}}$. □

10 The Absorber Adjunction: The Hemiring $\mathcal{S}'$

We investigate the structure obtained by applying the absorber adjunction construction $A(\mathcal{S})$ to $\mathcal{S}$. This involves adjoining an element Ω defined to be absorbing for both addition and multiplication.

10.1 The Absorber Adjunction Functor A

Let **CHemiring** denote the category of commutative unital hemirings.

Definition 10.1 (Absorber Adjunction Functor A). The functor A : **CHemiring** $\to$ **CHemiring** is defined as follows. For $S \in$ **CHemiring**, $A(S) = S \cup \{\Omega_S\}$, where $\Omega_S \notin S$. The operations are extended such that Ω_S is a universal absorbing element. For $a, b \in A(S)$ and $* \in \{+, \times\}$:

$$a *_{A(S)} b := \begin{cases} \Omega_S & \text{if } a = \Omega_S \text{ or } b = \Omega_S \\ a *_S b & \text{if } a, b \in S \end{cases} \tag{10.1}$$

Proposition 10.2. *A is a well-defined covariant functor on* **CHemiring**. *$A(S)$ is a commutative unital hemiring.*

Proof. The verification of axioms (associativity, commutativity, distributivity) follows immediately from the definition, as Ω_S absorbs all operations, and otherwise the properties are inherited from S. The additive identity 0_S and multiplicative identity 1_S remain the identities in $A(S)$. Functoriality is straightforward. □

10.2 Construction and Verification of $\mathcal{S}'$

Construction 10.3. Let $\mathcal{S}' := A(\mathcal{S}) = A(G(\mathbb{Z}))$. We denote the adjoined element by Ω. $\mathcal{S}' = \mathbb{Z} \cup \{\mathcal{T}, \Omega\}$.

Theorem 10.4. *The structure $(\mathcal{S}', +, \times)$ is a commutative unital hemiring. It is not a standard semiring.*

Proof. By Proposition 10.2, $\mathcal{S}'$ is a commutative unital hemiring. Additive Identity $0_{\mathcal{S}'} = 0_{\mathcal{S}} = \mathcal{T}$. Multiplicative Identity $1_{\mathcal{S}'} = 1_{\mathcal{S}} = 1_{\mathbb{Z}}$. Multiplicative Absorber. By construction, this is Ω.

A standard semiring requires the additive identity to be the multiplicative absorber (Definition 2.5). Since $\mathcal{T} \neq \Omega$, $\mathcal{S}'$ is not a standard semiring. Explicitly, the axiom $a \times 0_R = 0_R$ fails for $a = \Omega$: $\Omega \times \mathcal{T}$. By Equation (10.1), the result is Ω. Thus $\Omega \times \mathcal{T} = \Omega \neq \mathcal{T}$. □

10.3 Algebraic Properties of $\mathcal{S}'$

Proposition 10.5. *The hemiring $\mathcal{S}'$ is zerosumfree (relative to $\mathcal{T}$). It is an integral semidomain relative to $\mathcal{T}$. It is also a z-integral semidomain relative to Ω.*

Proof. 1. Zerosumfree. $a + b = \mathcal{T}$. If $a = \Omega$ or $b = \Omega$, $a + b = \Omega \neq \mathcal{T}$. Thus $a, b \in \mathcal{S}$. $a + b = \mathcal{T}$ in $\mathcal{S}$ implies $a = b = \mathcal{T}$ (Proposition 4.4).

2. Integral relative to $\mathcal{T}$. $a \times b = \mathcal{T}$. If $a = \Omega$ or $b = \Omega$, $a \times b = \Omega \neq \mathcal{T}$. Thus $a, b \in \mathcal{S}$. $a \times b = \mathcal{T}$ in $\mathcal{S}$ implies $a = \mathcal{T}$ or $b = \mathcal{T}$ (Proposition 4.3).

3. z-integral relative to Ω. $a \times b = \Omega$. By definition (10.1), this occurs if and only if $a = \Omega$ or $b = \Omega$. □

Proposition 10.6. $U(\mathcal{S}') = \{1_{\mathbb{Z}}, -1_{\mathbb{Z}}\}$.

Proof. $xy = 1_{\mathbb{Z}}$. If $x = \Omega$ or $y = \Omega$, $xy = \Omega \neq 1_{\mathbb{Z}}$. Thus $x, y \in \mathcal{S}$. $U(\mathcal{S}') = U(\mathcal{S})$. □

10.4 The Ideal Structure of $\mathcal{S}'$

Theorem 10.7. *The ideals of $\mathcal{S}'$ are precisely the following sets:*

1. *The minimal ideal $I_\Omega = \{\Omega\}$.*
2. *The sets of the form $I' = I \cup \{\Omega\}$, where I is an ideal of the semiring $\mathcal{S}$.*

Proof. *Part I: Characterization.* Let I' be an ideal. $\Omega \in I'$ (Lemma 2.14). Define $I = I' \cap \mathcal{S}$. $I' = I \cup \{\Omega\}$. If $I = \emptyset$, $I' = I_\Omega$. If $I \neq \emptyset$. We show I is an ideal of $\mathcal{S}$. Closure and absorption by $\mathcal{S}$ follow since operations restricted to $\mathcal{S}$ remain in $\mathcal{S}$.

Part II: Verification. 1. I_Ω is an ideal as Ω absorbs everything. 2. $I' = I \cup \{\Omega\}$. Closure and absorption follow from the properties of I in $\mathcal{S}$ and the absorbing nature of Ω in $\mathcal{S}'$. □

Theorem 10.8. *The semiring $\mathcal{S}'$ is a PIS.*

Proof. 1. $I_\Omega = (\Omega)_{\mathcal{S}'}$. 2. $I' = I \cup \{\Omega\}$. Since $\mathcal{S}$ is a PIS (Theorem 4.7), $I = (a)_{\mathcal{S}}$. We compute $(a)_{\mathcal{S}'} = \mathcal{S}'a = (\mathcal{S}a) \cup (\{\Omega\}a)$. $\mathcal{S}a = I$. $\Omega a = \Omega$. Thus, $(a)_{\mathcal{S}'} = I \cup \{\Omega\} = I'$. □

10.5 Subtractive Ideals in $\mathcal{S}'$

Theorem 10.9. *The only subtractive ideal of $\mathcal{S}'$ is the improper ideal $\mathcal{S}'$.*

Proof. The element Ω is an additive absorbing element in $\mathcal{S}'$ (by Equation (10.1)). Furthermore, Ω is the multiplicative absorber, so $\Omega \in I$ for any ideal I. By Lemma 2.18, if an ideal I contains an additive absorber, it is subtractive if and only if $I = R$. □

10.6 The Spectrum of $\mathcal{S}'$

Theorem 10.10. *The spectrum* $\operatorname{Spec}(\mathcal{S}')$ *consists of the ideal* I_Ω *together with the ideals* $P' = P \cup \{\Omega\}$*, where* $P \in \operatorname{Spec}(\mathcal{S})$.

Proof. 1. $I_\Omega = \{\Omega\}$. Prime by the z-integral property (Proposition 10.5.3).

2. Ideals $I' = I \cup \{\Omega\}$. I' is proper iff I is proper. We establish the equivalence: I' is prime in $\mathcal{S}'$ if and only if I is prime in $\mathcal{S}$.

($\Longrightarrow$) Assume I' is prime. $a, b \in \mathcal{S}, ab \in I$. $ab \in I'$. $a \in I'$ or $b \in I'$. Since $a, b \in \mathcal{S}$, $a \in I$ or $b \in I$.

($\Longleftarrow$) Assume I is prime. $ab \in I'$. If $a = \Omega$ or $b = \Omega$, done. If $a, b \in \mathcal{S}$. $ab \in I' \cap \mathcal{S} = I$. $a \in I$ or $b \in I$. $\square$

Notation 10.11. We denote the prime ideals of $\mathcal{S}'$ corresponding to the primes of $\mathcal{S}$ ($P_\mathcal{T}, P_{(0)}, P_{(p)}$) as $Q_\mathcal{T}, Q_{(0)}, Q_{(p)}$, and $Q_\Omega = I_\Omega$. $Q_\mathcal{T} = \{\mathcal{T}, \Omega\}$. $Q_{(0)} = \{0_\mathbb{Z}, \mathcal{T}, \Omega\}$. $Q_{(p)} = p\mathbb{Z} \cup \{\mathcal{T}, \Omega\}$.

Theorem 10.12. *The Krull dimension of the semiring* $\mathcal{S}'$ *is 3.*

Proof. The map $\Psi : \operatorname{Spec}(\mathcal{S}) \to \operatorname{Spec}(\mathcal{S}') \setminus \{Q_\Omega\}$, $\Psi(P) = P \cup \{\Omega\}$, is an order-preserving bijection. Q_Ω is the unique minimum element. A maximal chain in $\operatorname{Spec}(\mathcal{S}')$ extends a maximal chain in $\operatorname{Spec}(\mathcal{S})$ (length 2, Theorem 5.3) by prepending Q_Ω.

$$Q_\Omega \subsetneq Q_\mathcal{T} \subsetneq Q_{(0)} \subsetneq Q_{(p)}. \tag{10.2}$$

We verify the inclusions are strict. $Q_\Omega \subsetneq Q_\mathcal{T}$ ($\mathcal{T} \neq \Omega$). $Q_\mathcal{T} \subsetneq Q_{(0)}$ ($0_\mathbb{Z} \notin \{\mathcal{T}, \Omega\}$). $Q_{(0)} \subsetneq Q_{(p)}$ ($p \notin \{0_\mathbb{Z}, \mathcal{T}, \Omega\}$).

The length of this chain is 3. $\square$

11 Symmetry Analysis and Singlets of $\mathcal{S}'$

11.1 The Canonical $\mathbb{Z}/2\mathbb{Z}$ Action and Singlets

$U(\mathcal{S}') = \{\pm 1_\mathbb{Z}\}$ (Proposition 10.6). The canonical action $\Psi'(g, s) = g \times s$ is by automorphisms of $(\mathcal{S}', +)$.

Theorem 11.1. *The set of singlets in* $\mathcal{S}'$ *under the canonical* $\mathbb{Z}/2\mathbb{Z}$ *action* Ψ' *is* $\mathcal{A}' = \{0_\mathbb{Z}, \mathcal{T}, \Omega\}$.

Proof. We solve $(-1_\mathbb{Z}) \times s = s$.

Case 1: $s \in \mathcal{S}$. Solutions are $\{0_\mathbb{Z}, \mathcal{T}\}$ (Theorem 9.3).

Case 2: $s = \Omega$. $(-1_\mathbb{Z}) \times \Omega$. By Definition 10.1, this product is Ω. Ω is a singlet. $\square$

11.2 The Structure of the Singlets $\mathcal{A}'$

Definition 11.2. The extended Boolean semiring $\mathbb{B}_{\text{ext}} = \{0, 1, \infty\}$ is obtained by applying the absorber adjunction functor to $\mathbb{B}$. $A(\mathbb{B})$.

Theorem 11.3. *The subset* $\mathcal{A}'$ *forms a commutative unital sub-hemiring of* $\mathcal{S}'$*. It is isomorphic to the extended Boolean semiring* $\mathbb{B}_{\text{ext}}$*. It is an idempotent semiring (characteristic one).*

Proof. We examine the operation tables restricted to $\mathcal{A}' = \{0_\mathbb{Z}, \mathcal{T}, \Omega\}$.

Addition (+):

$+$	$0_\mathbb{Z}$	$\mathcal{T}$	Ω
$0_\mathbb{Z}$	$0_\mathbb{Z}$	$0_\mathbb{Z}$	Ω
$\mathcal{T}$	$0_\mathbb{Z}$	$\mathcal{T}$	Ω
Ω	Ω	Ω	Ω

Operations within $\{0_\mathbb{Z}, \mathcal{T}\}$ follow $\mathcal{A}$ (Theorem 9.4). Any sum involving Ω results in Ω.

Multiplication ($\times$):

$\times$	$0_{\mathbb{Z}}$	$\mathcal{T}$	Ω
$0_{\mathbb{Z}}$	$0_{\mathbb{Z}}$	$\mathcal{T}$	Ω
$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	Ω
Ω	Ω	Ω	Ω

Any product involving Ω results in Ω.

Identities: $0_{\mathcal{A}'} = \mathcal{T}$. $1_{\mathcal{A}'} = 0_{\mathbb{Z}}$. Isomorphism $\psi' : \mathcal{A}' \to \mathbb{B}_{\text{ext}}$. $\psi'(\mathcal{T}) = 0, \psi'(0_{\mathbb{Z}}) = 1, \psi'(\Omega) = \infty$. Idempotency: $x + x = x$ holds by inspection of the addition table. $\square$

Proposition 11.4. *The algebraic order (Definition 2.11) on the idempotent semiring $\mathcal{A}'$ is a total order: $\mathcal{T} \leq 0_{\mathbb{Z}} \leq \Omega$.*

Proof. $\mathcal{T} + 0_{\mathbb{Z}} = 0_{\mathbb{Z}} \implies \mathcal{T} \leq 0_{\mathbb{Z}}$. $0_{\mathbb{Z}} + \Omega = \Omega \implies 0_{\mathbb{Z}} \leq \Omega$. $\square$

12 Generalization to Algebraic Number Fields

We generalize the constructions to the ring of integers $\mathcal{O}_K$ of an algebraic number field K. We recall that $\mathcal{O}_K$ is a Dedekind domain.

12.1 The Globalization $G(\mathcal{O}_K)$

Definition 12.1. Let $\mathcal{O}_K$ be the ring of integers of K. Define $G(\mathcal{O}_K) = \mathcal{O}_K \cup \{\mathcal{T}_K\}$, where the operations are defined by the globalization functor (Definition 3.1).

Theorem 12.2. *$G(\mathcal{O}_K)$ is a standard commutative unital semiring, integral semidomain, and zerosumfree. The ideals are $\{\mathcal{T}_K\}$ and $I_J = J \cup \{\mathcal{T}_K\}$, where J is a non-zero ideal of $\mathcal{O}_K$. All ideals are subtractive.*

Proof. The verification of the axioms and basic properties generalizes the analysis for $\mathcal{S} = G(\mathbb{Z})$ (Theorems 3.6, 4.3, 4.4, 4.5, 4.9). The proof that ideals are subtractive relies crucially on the fact that J is an additive subgroup of $\mathcal{O}_K$, allowing for subtraction within the ring structure to establish membership in J. $\square$

12.2 Ideal Theory and the Class Group

Theorem 12.3. *$G(\mathcal{O}_K)$ is a Principal Ideal Semiring (PIS) if and only if $\mathcal{O}_K$ is a Principal Ideal Domain (PID) (i.e., the class number $h_K = 1$).*

Proof. The proof generalizes the argument of Theorem 4.7. The principal ideal generated by $a \in \mathcal{O}_K$ in $G(\mathcal{O}_K)$ is $(a)_{G(\mathcal{O}_K)} = (a)_{\mathcal{O}_K} \cup \{\mathcal{T}_K\}$. Therefore, I_J is principal in $G(\mathcal{O}_K)$ if and only if J is principal in $\mathcal{O}_K$. $\square$

We analyze the class semigroup when $h_K \geq 1$.

Definition 12.4. Let R be an integral semidomain. The class semigroup $\mathrm{Cl}(R)$ is the set of non-zero ideals $\mathcal{I}^*(R)$ modulo the equivalence $I_1 \sim I_2$ if there exist $a, b \in R \setminus \{0_R\}$ such that $aI_1 = bI_2$.

In $G(\mathcal{O}_K)$, the set of non-zero elements is $R \setminus \{0_R\} = \mathcal{O}_K$.

Theorem 12.5. *The class semigroup $\mathrm{Cl}(G(\mathcal{O}_K))$ is isomorphic to the ideal class group $\mathrm{Cl}(K)$ of the number field K.*

Proof. We establish an isomorphism $\Phi : \mathrm{Cl}(K) \to \mathrm{Cl}(G(\mathcal{O}_K))$. We identify $\mathrm{Cl}(K)$ with the equivalence classes of non-zero integral ideals of $\mathcal{O}_K$.

Let $[J] \in \mathrm{Cl}(K)$. Define $\Phi([J]) = [I_J]_\sim$.

1. Well-defined. If $[J_1] = [J_2]$, then $\alpha J_1 = \beta J_2$ for $\alpha, \beta \in \mathcal{O}_K \setminus \{0\}$. We verify $\alpha I_{J_1} = \beta I_{J_2}$ in $G(\mathcal{O}_K)$. $\alpha I_{J_1} = \alpha(J_1 \cup \{\mathcal{T}_K\}) = (\alpha J_1) \cup (\alpha \times \mathcal{T}_K)$. Since $\alpha \in \mathcal{O}_K$, $\alpha \times \mathcal{T}_K = \mathcal{T}_K$. $\alpha I_{J_1} = (\alpha J_1) \cup \{\mathcal{T}_K\}$. Since $\alpha J_1 = \beta J_2$, the equality holds.

2. Homomorphism. We verify the multiplication of ideals: $I_{J_1} I_{J_2} = I_{J_1 J_2}$. $I_{J_1} I_{J_2} = (J_1 \cup \{\mathcal{T}_K\})(J_2 \cup \{\mathcal{T}_K\}) = (J_1 J_2) \cup (J_1 \mathcal{T}_K) \cup (\mathcal{T}_K J_2) \cup (\mathcal{T}_K \mathcal{T}_K)$. Since $\mathcal{T}_K$ is absorbing, this simplifies to $(J_1 J_2) \cup \{\mathcal{T}_K\} = I_{J_1 J_2}$.

3. Injectivity. Suppose $\Phi([J_1]) = \Phi([J_2])$. $aI_{J_1} = bI_{J_2}$ for $a, b \in \mathcal{O}_K \setminus \{0\}$. $(aJ_1) \cup \{\mathcal{T}_K\} = (bJ_2) \cup \{\mathcal{T}_K\}$. Intersecting both sides with $\mathcal{O}_K$ yields $aJ_1 = bJ_2$. Thus $[J_1] = [J_2]$.

4. Surjectivity. Any non-zero ideal of $G(\mathcal{O}_K)$ is of the form I_J. □

12.3 Spectrum and Dimension

Theorem 12.6. *The Krull dimension of $G(\mathcal{O}_K)$ is 2. The spectrum $\operatorname{Spec}(G(\mathcal{O}_K))$ consists of $P_{\mathcal{T}_K} = \{\mathcal{T}_K\}$, $P_{(0)} = \{0_{\mathcal{O}_K}, \mathcal{T}_K\}$, and $P_{\mathfrak{p}} = \mathfrak{p} \cup \{\mathcal{T}_K\}$ where $\mathfrak{p}$ is a non-zero prime ideal of $\mathcal{O}_K$.*

Proof. The characterization of prime ideals follows the methodology of Theorem 5.1: I_J is prime in $G(\mathcal{O}_K)$ iff J is prime in $\mathcal{O}_K$. The prime ideals of the Dedekind domain $\mathcal{O}_K$ are (0) and the non-zero prime (maximal) ideals $\mathfrak{p}$. A maximal chain in $G(\mathcal{O}_K)$ is $P_{\mathcal{T}_K} \subsetneq P_{(0)} \subsetneq P_{\mathfrak{p}}$. The length is 2. □

12.4 Symmetry Actions in $G(\mathcal{O}_K)$

The group of units $U(\mathcal{O}_K)$ acts on $G(\mathcal{O}_K)$ by multiplication: $\Psi_K(g, s) = g \times s$.

Theorem 12.7. *The set of singlets in $G(\mathcal{O}_K)$ under the action of $U(\mathcal{O}_K)$ is $A_K = \{0_{\mathcal{O}_K}, \mathcal{T}_K\}$. A_K forms a sub-semiring isomorphic to $\mathbb{B}$.*

Proof. We seek s such that $g \times s = s$ for all $g \in U(\mathcal{O}_K)$. If $s = x \in \mathcal{O}_K$. $gx = x$, so $(g - 1_{\mathcal{O}_K})x = 0_{\mathcal{O}_K}$. Since the characteristic of K is 0, $-1_{\mathcal{O}_K} \in U(\mathcal{O}_K)$ and $-1_{\mathcal{O}_K} \neq 1_{\mathcal{O}_K}$. Taking $g = -1_{\mathcal{O}_K}$ yields $(-2_{\mathcal{O}_K})x = 0_{\mathcal{O}_K}$. Since $\mathcal{O}_K$ is an integral domain and $-2_{\mathcal{O}_K} \neq 0_{\mathcal{O}_K}$, we must have $x = 0_{\mathcal{O}_K}$. If $s = \mathcal{T}_K$. $g \times \mathcal{T}_K = \mathcal{T}_K$. The isomorphism to $\mathbb{B}$ follows as in Theorem 9.4, with $0_{A_K} = \mathcal{T}_K$ and $1_{A_K} = 0_{\mathcal{O}_K}$. □

12.5 The Extended Construction $A(G(\mathcal{O}_K))$

Definition 12.8. Define $A(G(\mathcal{O}_K)) = G(\mathcal{O}_K) \cup \{\Omega_K\}$, where Ω_K is the universal absorbing element.

Theorem 12.9. *$A(G(\mathcal{O}_K))$ possesses the following properties:*

1. *It is a commutative unital hemiring (not standard).*
2. $\operatorname{Kdim}(A(G(\mathcal{O}_K))) = 3$.
3. *It is a PIS if and only if $\mathcal{O}_K$ is a PID.*
4. *No proper ideal is subtractive.*
5. *The singlets under $U(\mathcal{O}_K)$ are $\{0_{\mathcal{O}_K}, \mathcal{T}_K, \Omega_K\}$, isomorphic to $\mathbb{B}_{\text{ext}}$.*

Proof. These properties generalize the results established for $\mathcal{S}'$. The ideal structure follows Theorem 10.7. The PIS property follows by combining Theorem 10.8 and Theorem 12.3. The non-subtractive property follows from Lemma 2.18. The Krull dimension increases by 1 from $\operatorname{Kdim}(G(\mathcal{O}_K)) = 2$. The singlet analysis follows Theorem 11.1. □

13 Categorical Aspects and the Geometry of the Spectra

13.1 Functorial Properties of Globalization and Absorber Adjunction

We analyze the properties of the functors $G : \mathbf{CRing} \to \mathbf{CSRing}_0$ and $A : \mathbf{CHemiring} \to \mathbf{CHemiring}$.

Theorem 13.1. *The Krull dimension behaves additively under the functors G and A when applied to suitable Noetherian structures.*

1. *If R is a Noetherian commutative unital ring,* $\operatorname{Kdim}(G(R)) = \operatorname{Kdim}(R) + 1$.

2. *If S is a Noetherian commutative unital hemiring,* $\mathrm{Kdim}(A(S)) = \mathrm{Kdim}(S) + 1$.

Proof. 1. The analysis of $\mathrm{Spec}(G(R))$ generalizes the arguments of Section 5.1. The spectrum consists of $P_{\mathcal{T}_R}$ and ideals I_J where $J \in \mathrm{Spec}(R)$. The map $J \mapsto I_J$ is an order-preserving bijection. $P_{\mathcal{T}_R}$ is the unique minimum element, adjoined below the image of $\mathrm{Spec}(R)$. This increases the length of maximal chains by 1.

2. The analysis of $\mathrm{Spec}(A(S))$ generalizes the arguments of Section 9.6. The spectrum consists of I_{Ω_S} and ideals I'_P where $P \in \mathrm{Spec}(S)$. I_{Ω_S} is the unique minimum element. This increases the length of maximal chains by 1. □

Corollary 13.2. *For a Dedekind domain $\mathcal{O}_K$ (where $\mathrm{Kdim}(\mathcal{O}_K) = 1$), we have $\mathrm{Kdim}(A(G(\mathcal{O}_K))) = 1 + 1 + 1 = 3$.*

13.2 The Spectral Sequence and Geometric Interpretation

The sequence of constructions $R \to G(R) \to A(G(R))$ induces a sequence of spectra where each step corresponds geometrically to the addition of a generic point.

Theorem 13.3. *Let R be a Noetherian commutative unital ring.*

1. $\mathrm{Spec}(G(R))$ *is homeomorphic to the space obtained by augmenting* $\mathrm{Spec}(R)$ *with a new generic point* $P_{\mathcal{T}_R}$.

2. $\mathrm{Spec}(A(G(R)))$ *is homeomorphic to the space obtained by augmenting* $\mathrm{Spec}(G(R))$ *with a new generic point* I_{Ω_R}.

Proof. 1. Let $X = \mathrm{Spec}(R)$ and $X_G = \mathrm{Spec}(G(R))$. The map $\pi_G : X_G \setminus \{P_{\mathcal{T}_R}\} \to X$ corresponding to the order-preserving bijection $I_J \mapsto J$ is a homeomorphism, as the Zariski topology on Noetherian spaces is determined by the specialization order. Since $P_{\mathcal{T}_R}$ is the unique generic point of X_G, X_G is X augmented by a generic point.

2. The argument is analogous for the functor A. □

The structure of $\mathrm{Spec}(A(G(\mathbb{Z})))$ (using Notation 10.11) provides a visualization of this hierarchy of generic points, suggesting a stratification related to absolute arithmetic:

$$\begin{array}{ccc}
\text{Codimension 0 (Closed Points)} & Q_{(p)} & \text{Points of } \mathrm{Spec}(\mathbb{Z}) \\
 & \rotatebox{90}{$\subsetneq$} & \\
\text{Codimension 1} & Q_{(0)} & \text{Generic point of } \mathrm{Spec}(\mathbb{Z}) \\
 & \rotatebox{90}{$\subsetneq$} & \\
\text{Codimension 2} & Q_{\mathcal{T}} & \text{Generic point of } \mathrm{Spec}(G(\mathbb{Z})) \\
 & \rotatebox{90}{$\subsetneq$} & \\
\text{Codimension 3} & Q_{\Omega} & \text{Generic point of } \mathrm{Spec}(A(G(\mathbb{Z})))
\end{array}$$

14 Connections to $\mathbb{F}_1$-Geometry and Adelic Structures

The constructions analyzed exhibit connections to the study of geometry over the "field with one element" $\mathbb{F}_1$.

14.1 Emergence of Characteristic 1 Structures

We summarize how characteristic 1 (idempotent) structures emerge from the analysis of the characteristic 0 structure $G(\mathbb{Z})$.

Theorem 14.1. *The globalization of $\mathbb{Z}$, $G(\mathbb{Z})$, gives rise to the characteristic 1 structure $\mathbb{B}$ in two distinct procedures:*

1. *Via symmetry analysis: The sub-semiring of singlets* $G(\mathbb{Z})^{U(G(\mathbb{Z}))} \cong \mathbb{B}$.

2. *Via localization: The semifield of fractions* $\mathrm{Frac}(G(\mathbb{Z})) \cong \mathbb{B}$.

Proof. 1. Proven in Theorem 9.4. 2. Proven in Theorem 8.8. □

This dual emergence underscores a relationship between the globalization process, the reduction induced by symmetry (often viewed as a descent to a base related to $\mathbb{F}_1$), and the transition to characteristic 1 behavior. The localization result demonstrates that the presence of the non-cancellative prime element $0_{\mathbb{Z}}$ within the multiplicative set $\mathbb{Z}$ forces the collapse of the arithmetic structure into the simplest idempotent structure when inverted.

14.2 Interpretation in $\mathbb{F}_1$ Geometry

In the context of $\mathbb{F}_1$ geometry, the Boolean semiring $\mathbb{B}$ is often identified with $\mathbb{F}_{1^2}$. The structure $G(\mathbb{Z})$ can be viewed as an object relating $\mathbb{Z}$ (characteristic 0) and $\mathbb{B}$ (characteristic 1).

14.2.1 Monoidal Spectra (Deitmar Schemes)

We examine the multiplicative monoid structure of $\mathcal{S}$. $M = (\mathcal{S}, \times)$. This is the monoid $(\mathbb{Z}, \times)$ with an adjoined zero element $\mathcal{T}$. The spectrum of this monoid $\mathrm{Spec}(M)$ (the set of prime ideals of the monoid) consists of the zero ideal $\{\mathcal{T}\}$, the ideal $\{0_{\mathbb{Z}}, \mathcal{T}\}$, and the ideals $p\mathbb{Z} \cup \{\mathcal{T}\}$. This structure is isomorphic to $\mathrm{Spec}(\mathcal{S})$. This indicates that for $G(\mathbb{Z})$, the semiring spectrum captures the same information as the monoidal spectrum.

14.2.2 The Arithmetic Site

The emergence of $\mathbb{B}$ via symmetry analysis aligns with the perspective of Connes and Consani, where the Arithmetic Site is related to the action of $\mathbb{R}_+^\times$ on the adeles. The analysis of singlets under the unit group $U(\mathbb{Z})$ provides a discrete analogue of this process, filtering the characteristic 0 structure to reveal the characteristic 1 base.

The sequence of singlets $\mathbb{B} \to \mathbb{B}_{\text{ext}}$ obtained from $G(\mathbb{Z}) \to A(G(\mathbb{Z}))$ suggests that the functors G and A correspond to operations constructing higher-dimensional or extended structures within the framework of characteristic 1 geometry.

14.3 Valuations and the Adelic Semiring

We explore the adelic realization associated with $G(\mathbb{Z})$. We utilize the restricted semifield of fractions $K = G(\mathbb{Q})$ (Corollary 8.6).

Lemma 14.2. *The valuation semirings of $K = G(\mathbb{Q})$ (defined analogously to valuation rings) are precisely of the form $G(W)$, where W is a valuation ring of $\mathbb{Q}$.*

Proof. Let V be a valuation semiring of $G(\mathbb{Q})$. Let $W = V \cap \mathbb{Q}$. W is a subring of $\mathbb{Q}$. The invertible elements of $G(\mathbb{Q})$ are $\mathbb{Q}^\times$. For $x \in \mathbb{Q}^\times$, either $x \in V$ or $x^{-1} \in V$. Thus W is a valuation ring of $\mathbb{Q}$. Since $0_{G(\mathbb{Q})} = \mathcal{T} \in V$, $V = W \cup \{\mathcal{T}\} = G(W)$. The converse is straightforward. □

The valuation rings of $\mathbb{Q}$ correspond to the places v of $\mathbb{Q}$. Let $\mathbb{Q}_v$ be the completion, and $\mathcal{O}_v$ the valuation ring of the completion.

Definition 14.3. The adele semiring $\mathbb{A}_{G(\mathbb{Q})}$ is the restricted product of the semirings $G(\mathbb{Q}_v)$ with respect to the sub-semirings $G(\mathcal{O}_v)$:

$$\mathbb{A}_{G(\mathbb{Q})} = \prod_v{}' (G(\mathbb{Q}_v), G(\mathcal{O}_v)). \tag{14.1}$$

This construction provides a geometric object encoding the arithmetic information of $G(\mathbb{Z})$, consistent with the observation that $\zeta_{G(\mathbb{Z})}(s) = \zeta(s)$. The study of this space relates to the structure of the Arithmetic Site, examining how the characteristic 1 components (the $\mathcal{T}$ elements in each factor) interact with the characteristic 0 components within this globalized adelic structure.

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