



THE HIGHER POWER MOMENTS OF THE COEFFICIENTS OF THE DEDEKIND ZETA FUNCTION OVER A POLYNOMIAL IN SIX VARIABLES

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Abstract

Let \mathbb{K} be a non-normal field of degree 3 over \mathbb{Q} . Let $\ell \geq 2$ be an integer. In this paper, we investigate the ℓ th power moment of the coefficients attached to the Dedekind zeta function $\zeta_{\mathbb{K}}(s)$ over a sequence. In particular, we consider the following:

$$S_{\ell}(x) := \sum_{\substack{n=x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2 \leq x \\ (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{Z}^6}} a_{\mathbb{K}}^{\ell}(n)$$

and establish an asymptotic result, where $\zeta_{\mathbb{K}}(s) = \sum_{n=1}^{\infty} \frac{a_{\mathbb{K}}(n)}{n^s}$ is the Dedekind zeta function.

1. Introduction

Let \mathbb{K} be a number field of degree $[\mathbb{K} : \mathbb{Q}] = d$ and $\mathcal{O}_{\mathbb{K}}$ be its ring of integers. Then the *Dedekind zeta function* attached to \mathbb{K} is defined as

$$\zeta_{\mathbb{K}}(s) := \prod_{\substack{\mathfrak{p} \subseteq \mathcal{O}_{\mathbb{K}} \\ \mathfrak{p} \neq 0}} \left(1 - \frac{1}{(N\mathfrak{p})^s}\right)^{-1} = \sum_{\substack{\mathfrak{u} \subseteq \mathcal{O}_{\mathbb{K}} \\ \mathfrak{u} \neq 0}} \frac{1}{(N\mathfrak{u})^s} \quad (1)$$

for $\text{Re}(s) > 1$, where the product is over non-zero prime ideals in $\mathcal{O}_{\mathbb{K}}$ and $N\mathfrak{u}$ denotes the absolute norm of \mathfrak{u} . For $\mathbb{K} = \mathbb{Q}$, the Dedekind zeta function $\zeta_{\mathbb{K}}(s)$ is

the Riemann zeta function. The function $\zeta_{\mathbb{K}}(s)$ extends analytically to the entire complex plane except for a simple pole at $s = 1$, and the residue at $s = 1$ is given by the analytic class number formula

$$\lim_{s \rightarrow 1^+} (s-1)\zeta_{\mathbb{K}}(s) = \frac{2^{r_1}(2\pi)^{r_2}hR}{\omega\sqrt{|D_{\mathbb{K}}|}},$$

where r_1 is the number of real embeddings of \mathbb{K} , $2r_2$ is the number of complex embeddings of \mathbb{K} , h denotes the class number, R is the regulator, ω is the number of roots of unity in \mathbb{K} , and $D_{\mathbb{K}}$ is the discriminant of \mathbb{K} . The Dedekind zeta function satisfies a functional equation similar to the Riemann zeta function,

$$\xi_{\mathbb{K}}(s) = \xi_{\mathbb{K}}(s-1),$$

where

$$\xi_{\mathbb{K}}(s) := \left(\frac{\sqrt{|D_{\mathbb{K}}|}}{2^{r_2}\pi^{n/2}} \right)^s \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} \zeta_{\mathbb{K}}(s),$$

which is analytic in the whole complex plane except for the simple poles at $s = 0$ and $s = 1$.

We rewrite Equation (1) as a Dirichlet series

$$\zeta_{\mathbb{K}}(s) = \sum_{n=1}^{\infty} \frac{a_{\mathbb{K}}(n)}{n^s}, \quad \operatorname{Re}(s) > 1, \quad (2)$$

where $a_{\mathbb{K}}(n)$ denotes the number of integral ideals in \mathbb{K} with norm n . The coefficients $a_{\mathbb{K}}(n)$ of the Dedekind zeta function $\zeta_{\mathbb{K}}(s)$ in Equation (2) satisfy the following properties: for all $n \geq 1$, we have $a_{\mathbb{K}}(n) \geq 0$; for all coprime integers m and n , $a_{\mathbb{K}}(mn) = a_{\mathbb{K}}(m)a_{\mathbb{K}}(n)$; and for any $\epsilon > 0$,

$$a_{\mathbb{K}}(n) \leq d(n)^{[\mathbb{K}:\mathbb{Q}]} \ll n^{\epsilon},$$

where $d(n)$ denotes the divisor function. Since the coefficients $a_{\mathbb{K}}(n)$ are multiplicative, the Dedekind zeta function has the Euler product

$$\zeta_{\mathbb{K}}(s) = \sum_{n=1}^{\infty} \frac{a_{\mathbb{K}}(n)}{n^s} = \prod_p \left(1 + \frac{a_{\mathbb{K}}(p)}{p^s} + \frac{a_{\mathbb{K}}(p^2)}{p^{2s}} + \cdots + \frac{a_{\mathbb{K}}(p^k)}{p^{ks}} + \cdots \right) \quad (3)$$

for $\operatorname{Re}(s) > 1$.

In 1949, Landau [13] investigated the first moment of $a_{\mathbb{K}}(n)$ for a number field \mathbb{K} with degree $d \geq 2$ and proved that

$$\sum_{n \leq x} a_{\mathbb{K}}(n) = c_{\mathbb{K}}x + \mathcal{O}(x^{1-\frac{2}{d+1}+\epsilon})$$

for some constant $c_{\mathbb{K}}$ depending on \mathbb{K} . Later, Chandrasekharan and Narasimhan [4] studied the second moment of $a_{\mathbb{K}}(n)$ and proved that

$$\sum_{n \leq x} a_{\mathbb{K}}^2(n) \ll x(\log x)^{d-1}.$$

For a Galois extension \mathbb{K} over \mathbb{Q} of degree $d > 1$, Chandrasekharan and Good [3] proved that

$$\sum_{n \leq x} a_{\mathbb{K}}^{\ell}(n) = xP_{\ell}(\log x) + \mathcal{O}(x^{1-2d^{-\ell}+\epsilon}),$$

for every $\epsilon > 0$ and integer $\ell \geq 2$, where $P_{\ell}(\log x)$ denotes a suitable polynomial of degree $d^{\ell-1} - 1$. For a non-normal field extension \mathbb{K} over \mathbb{Q} of degree 3 given by an irreducible polynomial $x^3 + Ax^2 + Bx + C$ of discriminant $D < 0$, Fomenko [6] investigated the first and second moments and proved

$$\sum_{n \leq x} a_{\mathbb{K}}^2(n) = c_1 x \log x + c_2 x + \mathcal{O}(x^{9/11+\epsilon})$$

and

$$\sum_{n \leq x} a_{\mathbb{K}}^3(n) = xP_3(\log x) + \mathcal{O}(x^{73/79+\epsilon}),$$

where $P_3(\log x)$ is a polynomial in $\log x$ of degree 4. Lü [17] improved the error terms, and Liu [16] further improved the above-stated result of Fomenko. For a more recent development, see [7].

Let \mathbb{K} be a non-normal field over \mathbb{Q} of degree 3 given by an irreducible polynomial $x^3 + Ax^2 + Bx + C$ of discriminant $D < 0$. For a given natural number $\ell \geq 2$, we consider the ℓ th power moment of the Dedekind zeta function associated with \mathbb{K} given by

$$S_{\ell}(x) := \sum_{\substack{n=x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2 \leq x \\ (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{Z}^6}} a_{\mathbb{K}}^{\ell}(n). \quad (4)$$

In this paper, we prove the following results.

Theorem 1. *Let \mathbb{K} be a non-normal field over \mathbb{Q} of degree 3 given by an irreducible polynomial $x^3 + Ax^2 + Bx + C$ of discriminant $D < 0$. Then for any $\epsilon > 0$, we have*

$$S_{\ell}(x) = \begin{cases} c_1 x^3 + c_2 x^3 \log x + \mathcal{O}(x^{\frac{25}{9}+\epsilon}) & \text{if } \ell = 2, \\ x^3 P_3(\log x) + \mathcal{O}(x^{\frac{79}{27}+\epsilon}) & \text{if } \ell = 3, \end{cases}$$

where c_1, c_2 are some suitable constants and $P_3(\log x)$ is a polynomial in $\log x$ of degree 4.

Theorem 2. *Let \mathbb{K} be a non-normal field over \mathbb{Q} of degree 3 given by an irreducible polynomial $x^3 + Ax^2 + Bx + C$ of discriminant $D < 0$, and let $\ell \geq 4$ be an integer. Then for any $\epsilon > 0$, we have*

$$S_\ell(x) = \begin{cases} x^3 P_\ell(\log x) + \mathcal{O}(x^{3-\frac{2}{3\ell}+\epsilon}) & \text{for even } \ell \geq 4, \\ x^3 Q_\ell(\log x) + \mathcal{O}(x^{3-\frac{2}{3\ell}+\epsilon}) & \text{for odd } \ell \geq 5, \end{cases}$$

where $P_\ell(\log x), Q_\ell(\log x)$ are polynomials in $\log x$ of degree $a_{0,\ell} + a_{3,\ell} - 1$. Here,

$$a_{0,\ell} = 1 + \sum_{i=1}^{\lfloor \frac{\ell}{2} \rfloor} \binom{\ell}{2i} \frac{(2i)!}{i!(i+1)!}$$

and

$$a_{3,\ell} = \sum_{i=1}^{\lceil \frac{\ell}{2}-1 \rceil} \binom{\ell}{2i+1} \frac{4(2i+1)!}{(i-1)!(i+3)!}.$$

2. Preliminaries

Throughout the paper, ϵ denotes an arbitrarily small positive constant, but not necessarily the same one as others, and any implied constant may depend on ϵ .

Let $H_k(SL(2, \mathbb{Z}))$ be the set of normalized Hecke eigenforms of even integral weight k for the full modular group $SL(2, \mathbb{Z})$. Let $\{\lambda_f(n)\}_{n \in \mathbb{N}}$ be the normalized Fourier coefficients of the cusp form $f \in H_k(SL(2, \mathbb{Z}))$ at infinity, i.e.,

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e^{2\pi i n z}$$

for all $z \in \mathbb{H}$, where \mathbb{H} is the Poincaré upper half-plane. The Fourier coefficients $\lambda_f(n)$ are the Hecke eigenvalues of f , and these are real numbers. Also, $\lambda_f(n)$ satisfies the following relation:

$$\lambda_f(m)\lambda_f(n) = \sum_{d|\gcd(m,n)} \lambda_f\left(\frac{mn}{d^2}\right). \quad (5)$$

As a consequence of Deligne's [5] proof of Weil's conjectures, we have

$$|\lambda_f(n)| \leq d(n) \ll_{\epsilon} n^{\epsilon}$$

for any $\epsilon > 0$, where $d(n)$ is the divisor function.

The L -function associated with the normalized Hecke eigenform $f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e^{2\pi i n z}$ is defined as

$$L(f, s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} \quad (6)$$

for $\operatorname{Re}(s) > 1$. Since $\lambda_f(mn) = \lambda_f(m)\lambda_f(n)$ for all positive integers m and n such that $\gcd(m, n) = 1$, the L -function has the Euler product

$$L(f, s) = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{1}{p^{2s}} \right)^{-1} = \prod_p \left(1 - \frac{\alpha_p}{p^s} \right)^{-1} \left(1 - \frac{\beta_p}{p^s} \right)^{-1},$$

where $\alpha_p + \beta_p = \lambda_f(p)$ and $\alpha_p\beta_p = |\alpha_p| = |\beta_p| = 1$.

For a given Dirichlet character χ of modulus N , the *twisted L -function* is defined as

$$L(f \otimes \chi, s) := \sum_{n=1}^{\infty} \frac{\lambda_f(n)\chi(n)}{n^s}$$

for $\operatorname{Re}(s) > 1$. Note that both the L -functions $L(f, s)$ and $L(f \otimes \chi, s)$ have analytic continuations to the whole complex plane (see, [11, Section 7.2]).

For $j \geq 2$, the j th *symmetric power L -function* of degree $j + 1$ is defined as

$$L(\operatorname{sym}^j f, s) := \prod_p \prod_{i=0}^j (1 - \alpha_p^{j-i} \beta_p^i p^{-s})^{-1} = \sum_{n=1}^{\infty} \frac{\lambda_{\operatorname{sym}^j f}(n)}{n^s} \quad \text{for } \operatorname{Re}(s) > 1, \quad (7)$$

where $\lambda_{\operatorname{sym}^j f}(n)$ is a real-valued multiplicative function. From Deligne's bound [5], we have

$$|\lambda_{\operatorname{sym}^j f}(n)| \leq d_{j+1}(n) \ll_{\epsilon} n^{\epsilon}$$

for any $\epsilon > 0$, where $d_j(n)$ denotes the number of j factors of a positive integer n .

For $j \geq 2$, we define the *twisted j th symmetric power L -function* as

$$L(\operatorname{sym}^j f \otimes \chi, s) := \sum_{n=1}^{\infty} \frac{\lambda_{\operatorname{sym}^j f}(n)\chi(n)}{n^s} \quad \text{for } \operatorname{Re}(s) > 1. \quad (8)$$

Also, both the L -functions $L(\operatorname{sym}^j f, s)$ and $L(\operatorname{sym}^j f \otimes \chi, s)$ have analytic continuations to the whole complex plane and satisfy nice functional equations (for more details, see [18, 19]).

For a given Dirichlet character χ of modulus N , the *Dirichlet L -function* is defined as

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \quad \text{for } \operatorname{Re}(s) > 1.$$

Denote

$$L(\operatorname{sym}^j f, s) := \begin{cases} \zeta(s) & \text{if } j = 0, \\ L(f, s) & \text{if } j = 1 \end{cases}$$

and

$$L(\operatorname{sym}^j f \otimes \chi, s) := \begin{cases} L(s, \chi) & \text{if } j = 0, \\ L(f \otimes \chi, s) & \text{if } j = 1. \end{cases}$$

Let $f = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} q^n \in H_k(SL(2, \mathbb{Z}))$. The j th symmetric power L -function can be written as

$$L(\text{sym}^j f, s) = \prod_p \left(1 + \frac{\lambda_{\text{sym}^j f}(p)}{p^s} + \frac{\lambda_{\text{sym}^j f}(p^2)}{p^{2s}} + \cdots + \frac{\lambda_{\text{sym}^j f}(p^k)}{p^{ks}} + \cdots \right) \quad (9)$$

for $\text{Re}(s) > 1$. The coefficients $\lambda_{\text{sym}^j f}(n)$ of the Dirichlet series in Equation (7) and the Fourier coefficients $\lambda_f(n)$ satisfy

$$\lambda_f(p^j) = \lambda_{\text{sym}^j f}(p) = \frac{\alpha_p^{j+1} - \beta_p^{j+1}}{\alpha_p - \beta_p}. \quad (10)$$

Let $r_6(n)$ denote the number of representations of a positive integer n by a polynomial $x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 \in \mathbb{Q}[x_1, x_2, x_3, x_4, x_5, x_6]$, i.e.,

$$r_6(n) := \#\{\underline{x} := (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{Z}^6 \mid n = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2\}.$$

Note that

$$\begin{aligned} r_6(n) &= 16 \sum_{m|n} \chi\left(\frac{n}{m}\right) m^2 - 4 \sum_{m|n} \chi(m) m^2 \\ &:= 16r(n) - 4t(n), \end{aligned} \quad (11)$$

where χ is a non-principal Dirichlet character of modulus 4. Note that both $r(n)$ and $t(n)$ are multiplicative functions. Using Equation (11), the sum $S_\ell(x)$ defined in Equation (4) can be expressed as

$$\begin{aligned} S_\ell(x) &= \sum_{n=x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2 \leq x} a_{\mathbb{K}}^\ell(n) = \sum_{n \leq x} a_{\mathbb{K}}^\ell(n) r_6(n) \\ &= 16 \sum_{n \leq x} a_{\mathbb{K}}^\ell(n) r(n) - 4 \sum_{n \leq x} a_{\mathbb{K}}^\ell(n) t(n). \end{aligned} \quad (12)$$

Also, notice that

$$r(p) = p^2 + \chi(p), \quad t(p) = 1 + p^2 \chi(p). \quad (13)$$

From [6], we learn that

$$\zeta_{\mathbb{K}}(s) = \zeta(s) L(f, s),$$

where f is a holomorphic cusp form of weight 1 for the congruence subgroup $\Gamma_0(|D|)$. This implies

$$a_{\mathbb{K}}(n) = \sum_{m|n} \lambda_f(m)$$

and consequently, we get

$$a_{\mathbb{K}}(p) = 1 + \lambda_f(p). \quad (14)$$

Recent years have witnessed an increasing focus on the average behaviour of arithmetical functions; refer to [2, 21] and the associated references for further details.

3. Auxiliary Results

For a real number m , we write $\lfloor m \rfloor$ and $\lceil m \rceil$ for the floor and ceiling of m , respectively.

Since $a_{\mathbb{K}}^{\ell}(p) = (1 + \lambda_f(p))^{\ell}$, we have

$$a_{\mathbb{K}}^{\ell}(p) = \sum_{i=0}^{\ell} \binom{\ell}{i} \lambda_f^i(p). \quad (15)$$

Following Equation (10) and [14], we write

$$a_{\mathbb{K}}^{\ell}(p) = a_{0,\ell} + a_{1,\ell} \lambda_f(p) + \sum_{i=2}^{\ell} a_{i,\ell} \lambda_{sym^i f}(p). \quad (16)$$

Let $\ell = 2m$ for some $m \geq 1$. From [14, Lemma 7.1, Equation (38)], we have

$$\lambda_f^{\ell}(p) = \lambda_f^{2m}(p) = \frac{(2m)!}{m!(m+1)!} + \sum_{r=1}^{m-1} \left(\frac{(2m)!(2r+1)!}{(m-r)!(m+r+1)!} \right) \lambda_{sym^{2r}f}(p) + \lambda_{sym^{2m}f}(p). \quad (17)$$

Let $\ell = 2m + 1$ be an odd integer for some $m \geq 1$. Then we have

$$\lambda_f^{2m+1}(p) = \frac{2(2m+1)!}{m!(m+2)!} + \sum_{r=1}^{m-1} \left(\frac{(2m+1)!(2r+2)}{(m-r)!(m+r+2)!} \right) \lambda_{sym^{2r+1}f}(p) + \lambda_{sym^{2m+1}f}(p). \quad (18)$$

From Equations (15), (16), (17), and (18), we get

$$a_{0,\ell} = \begin{cases} 1 + \sum_{i=1}^{\frac{\ell}{2}} \binom{\ell}{2i} \frac{(2i)!}{i!(i+1)!} & \text{for even } \ell, \\ 1 + \sum_{i=1}^{\frac{\ell-1}{2}} \binom{\ell}{2i} \frac{(2i)!}{i!(i+1)!} & \text{for odd } \ell \end{cases} \quad (19)$$

and

$$a_{3,\ell} = \begin{cases} \sum_{i=1}^{\frac{\ell}{2}-1} \binom{\ell}{2i+1} \frac{4(2i+1)!}{(i-1)!(i+3)!} & \text{for even } \ell, \\ \sum_{i=1}^{\frac{\ell-1}{2}} \binom{\ell}{2i+1} \frac{4(2i+1)!}{(i-1)!(i+3)!} & \text{for odd } \ell. \end{cases} \quad (20)$$

Next, we consider the Dirichlet series associated with $a_{\mathbb{K}}^{\ell}(n)r(n)$ and $a_{\mathbb{K}}^{\ell}(n)t(n)$ given by

$$R_{\ell}(s) = \sum_{n=1}^{\infty} \frac{a_{\mathbb{K}}^{\ell}(n)r(n)}{n^s} \quad (21)$$

and

$$T_{\ell}(s) = \sum_{n=1}^{\infty} \frac{a_{\mathbb{K}}^{\ell}(n)t(n)}{n^s} \quad (22)$$

for $\operatorname{Re}(s) > 3$.

In this section, we find the L -decompositions of $R_{\ell}(s)$ and $T_{\ell}(s)$ involving known automorphic L -functions. Throughout, $s \in \mathbb{C}$ denotes the complex number $s = \gamma + it$, where $\operatorname{Re}(s) = \gamma$ and $\operatorname{Im}(s) = t$.

Lemma 1. *We have*

$$R_{\ell}(s) = \begin{cases} \zeta(s-2)^{a_{0,\ell}} L(\operatorname{sym}^3 f, s-2)^{a_{3,\ell}} W_{\ell}(s) H_{\ell}(s) & \text{for even } \ell \geq 4, \\ \zeta(s-2)^{a_{0,\ell}} L(\operatorname{sym}^3 f, s-2)^{a_{3,\ell}} W'_{\ell}(s) H'_{\ell}(s) & \text{for odd } \ell \geq 5, \end{cases}$$

where

$$W_{\ell}(s) = \prod_{\substack{1 \leq t_1 \leq \ell \\ t_1 \neq 3}} L(\operatorname{sym}^{t_1} f, s-2)^{a_{t_1,\ell}} \prod_{0 \leq t_2 \leq \ell} L(\operatorname{sym}^{t_2} f \otimes \chi, s)^{a_{t_2,\ell}}$$

and

$$W'_{\ell}(s) = \prod_{\substack{1 \leq t'_1 \leq \ell \\ t'_1 \neq 3}} L(\operatorname{sym}^{t'_1} f, s-2)^{a_{t'_1,\ell}} \prod_{0 \leq t'_2 \leq \ell} L(\operatorname{sym}^{t'_2} f \otimes \chi, s)^{a_{t'_2,\ell}}$$

for some suitable constants $a_{t_1,\ell}, a_{t_2,\ell}, a_{t'_1,\ell}, a_{t'_2,\ell}$. Here, $H_{\ell}(s), H'_{\ell}(s)$ converge absolutely and uniformly in the right half-plane $\operatorname{Re}(s) > \frac{5}{2}$ with $H_{\ell}(s), H'_{\ell}(s)$ non-zero for $\operatorname{Re}(s) = 3$, where $a_{0,\ell}$ and $a_{3,\ell}$ are given in Equations (19) and (20).

Proof. Since $a_{\mathbb{K}}^{\ell}(n)r(n)$ is a multiplicative function, $R_{\ell}(s)$ has the Euler product

$$R_{\ell}(s) = \prod_p \left(1 + \frac{a_{\mathbb{K}}^{\ell}(p)r(p)}{p^s} + \sum_{k=2}^{\infty} \frac{a_{\mathbb{K}}^{\ell}(p^k)r(p^k)}{p^{ks}} \right) \quad (23)$$

for $\operatorname{Re}(s) > 3$.

Let $\ell \geq 4$ be an even integer. Note that

$$a_{\mathbb{K}}^{\ell}(p)r(p) = (a_{0,\ell} + a_{1,\ell}\lambda_f(p) + \sum_{i=2}^{\ell} a_{i,\ell}\lambda_{\operatorname{sym}^i f}(p))(p^2 + \chi(p)) := c(p),$$

where $a_{0,\ell}$ and $a_{3,\ell}$ are given in Equations (19) and (20). From the structure of $c(p)$, we define the Dirichlet series associated with the coefficients $c(n)$ as

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{c(n)}{n^s} \\ &= \zeta(s-2)^{a_{0,\ell}} L(\operatorname{sym}^3 f, s-2)^{a_{3,\ell}} \prod_{\substack{1 \leq t_1 \leq \ell \\ t_1 \neq 3}} L(\operatorname{sym}^{t_1} f, s-2)^{a_{t_1,\ell}} \prod_{0 \leq t_2 \leq \ell} L(\operatorname{sym}^{t_2} f \otimes \chi, s)^{a_{t_2,\ell}}, \end{aligned}$$

which is absolutely convergent in $\operatorname{Re}(s) > 3$, where $a_{t_1,\ell}, a_{t_2,\ell}, a_{t'_1,\ell}, a_{t'_2,\ell}$ are some suitable constants. We also note that

$$\begin{aligned} & \prod_p \left(1 + \frac{c(p)}{p^s} + \cdots + \frac{c(p^m)}{p^{ms}} + \cdots \right) \\ &= \zeta(s-2)^{a_{0,\ell}} L(\operatorname{sym}^3 f, s-2)^{a_{3,\ell}} \prod_{\substack{1 \leq t_1 \leq \ell \\ t_1 \neq 3}} L(\operatorname{sym}^{t_1} f, s-2)^{a_{t_1,\ell}} \\ & \quad \times \prod_{0 \leq t_2 \leq \ell} L(\operatorname{sym}^{t_2} f \otimes \chi, s)^{a_{t_2,\ell}} \\ &=: G_{\ell}(s) \end{aligned}$$

for $\operatorname{Re}(s) > 3$. Observe that $c(n) \ll_{\epsilon} n^{2+\epsilon}$ for any $\epsilon > 0$ and

$$\left| \frac{c(p)}{p^s} + \frac{c(p^2)}{p^{2s}} + \cdots + \frac{c(p^m)}{p^{ms}} + \cdots \right| \ll \sum_{m=1}^{\infty} \frac{p^{(2+\epsilon)m}}{p^{m\gamma}} < 1$$

for $\operatorname{Re}(s) \geq 3 + \epsilon$.

Write

$$P = \frac{a_{\mathbb{K}}^{\ell}(p)r(p)}{p^s} + \cdots + \frac{a_{\mathbb{K}}^{\ell}(p^m)r(p^m)}{p^{ms}} + \cdots$$

and

$$Q = \frac{c(p)}{p^s} + \cdots + \frac{c(p^m)}{p^{ms}} + \cdots.$$

From the above calculations, we observe that $|Q| < 1$ in $\operatorname{Re}(s) \geq 3 + \epsilon$. Notice that,

for $\operatorname{Re}(s) \geq 3 + \epsilon$, we have

$$\begin{aligned} \frac{1+P}{1+Q} &= (1+P)(1-Q+Q^2-Q^3+\cdots) \\ &= 1+P-Q-PQ+\text{higher terms} \\ &= 1 + \frac{a_{\mathbb{K}}^{\ell}(p^2)r(p^2)-b(p^2)}{p^{2s}} + \cdots + \frac{c_m(p^m)}{p^{ms}} + \cdots \end{aligned}$$

with $c_m(n) \ll_{\epsilon} n^{2+\epsilon}$. For $\operatorname{Re}(s) > \frac{5}{2}$, we have

$$\prod_p \left(\frac{1+P}{1+Q} \right) = \prod_p \left(1 + \frac{a_{\mathbb{K}}^{\ell}(p^2)r(p^2)-c(p^2)}{p^{2s}} + \cdots + \frac{c_m(p^m)}{p^{ms}} + \cdots \right) \ll_{\epsilon} 1.$$

Hence, for $\operatorname{Re}(s) > \frac{5}{2}$, we define

$$\begin{aligned} H_{\ell}(s) &:= \frac{R_{\ell}(s)}{G_{\ell}(s)} \\ &= \prod_p \left(\frac{1+P}{1+Q} \right) \\ &\ll_{\epsilon} 1 \end{aligned}$$

and $H_{\ell}(s) \neq 0$ for $\operatorname{Re}(s) = 3$. The proof is similar for the other case. \square

We omit the proof of the next lemma since it is very similar to the proof of Lemma 1.

Lemma 2. *For any $\epsilon > 0$, we have*

$$T_{\ell}(s) = \begin{cases} \zeta(s)^{a_{0,\ell}} L(\operatorname{sym}^3 f, s)^{a_{3,\ell}} \widetilde{W}_{\ell}(s) \widetilde{H}_{\ell}(s) & \text{for even } \ell \geq 4, \\ \zeta(s)^{a_{0,\ell}} L(\operatorname{sym}^3 f, s)^{a_{3,\ell}} \widetilde{W}'_{\ell}(s) \widetilde{H}'_{\ell}(s) & \text{for odd } \ell \geq 5, \end{cases}$$

where

$$\widetilde{W}_{\ell}(s) = \prod_{\substack{1 \leq t_1 \leq \ell \\ t_1 \neq 3}} L(\operatorname{sym}^{t_1} f, s)^{a_{t_1,\ell}} \prod_{0 \leq t_2 \leq \ell} L(\operatorname{sym}^{t_2} f \otimes \chi, s-2)^{a_{t_2,\ell}}$$

and

$$\widetilde{W}'_{\ell}(s) = \prod_{\substack{1 \leq t'_1 \leq \ell \\ t'_1 \neq 3}} L(\operatorname{sym}^{t'_1} f, s)^{a_{t'_1,\ell}} \prod_{0 \leq t'_2 \leq \ell} L(\operatorname{sym}^{t'_2} f \otimes \chi, s-2)^{a_{t'_2,\ell}}$$

for some suitable constants $a_{t_1,\ell}, a_{t_2,\ell}, a_{t'_1,\ell}, a_{t'_2,\ell}$. Here, $\widetilde{H}_{\ell}(s)$ and $\widetilde{H}'_{\ell}(s)$ converge absolutely and uniformly in the right half-plane $\operatorname{Re}(s) \geq \frac{5}{2} + \epsilon$ with $\widetilde{H}_{\ell}(s) \neq 0$, $\widetilde{H}'_{\ell}(s) \neq 0$ for $\operatorname{Re}(s) = 3$, where $a_{0,\ell}$ and $a_{3,\ell}$ are given in Equations (19) and (20).

ℓ	$a_{0,\ell}$	$a_{1,\ell}$	$a_{2,\ell}$	$a_{3,\ell}$	$a_{4,\ell}$	$a_{5,\ell}$	$a_{6,\ell}$	$a_{7,\ell}$	$a_{8,\ell}$
1	1	1	0	0	0	0	0	0	0
2	2	2	1	0	0	0	0	0	0
3	4	5	3	1	0	0	0	0	0
4	9	12	9	4	1	0	0	0	0
5	21	30	25	14	5	1	0	0	0
6	51	96	69	44	20	6	1	0	0
7	127	196	189	133	70	27	7	1	0
8	323	512	518	392	230	104	35	8	1

Table 1: Coefficients of $a_{\mathbb{K}}^{\ell}(p)$

In the next two results, we explicitly compute the coefficients of the L -functions appearing in the decompositions of $R_{\ell}(s)$ and $T_{\ell}(s)$ for $1 \leq \ell \leq 8$. Table 1 lists the coefficients of $a_{\mathbb{K}}^{\ell}(p)$ (compare with Equation (16)).

Using this table, we state the following.

Lemma 3. *For any $\epsilon > 0$, we have*

$$R_{\ell}(s) = \begin{cases} \zeta(s-2)W_1(s)H_1(s) & \text{if } \ell = 1, \\ \zeta(s-2)^2W_2(s)H_2(s) & \text{if } \ell = 2, \\ \zeta(s-2)^4L(\text{sym}^3f, s-2)W_3(s)H_3(s) & \text{if } \ell = 3, \\ \zeta(s-2)^9L(\text{sym}^3f, s-2)^4W_4(s)H_4(s) & \text{if } \ell = 4, \\ \zeta(s-2)^{21}L(\text{sym}^3f, s-2)^{14}W_5(s)H_5(s) & \text{if } \ell = 5, \\ \zeta(s-2)^{51}L(\text{sym}^3f, s-2)^{44}W_6(s)H_6(s) & \text{if } \ell = 6, \\ \zeta(s-2)^{127}L(\text{sym}^3f, s-2)^{133}W_7(s)H_7(s) & \text{if } \ell = 7, \\ \zeta(s-2)^{323}L(\text{sym}^3f, s-2)^{392}W_8(s)H_8(s) & \text{if } \ell = 8, \end{cases}$$

where

$$\begin{aligned}
 W_1(s) &= L(f, s-2)L(s, \chi)L(f \otimes \chi, s), \\
 W_2(s) &= L(f, s-2)^2 L(sym^2 f, s-2)L(s, \chi)^2 L(f \otimes \chi, s)^2 L(sym^2 f \otimes \chi, s), \\
 W_3(s) &= L(f, s-2)^5 L(sym^2 f, s-2)^3 L(s, \chi)^4 L(f \otimes \chi, s)^5 L(sym^3 f \otimes \chi, s)L(sym^2 f \otimes \chi, s)^3, \\
 W_4(s) &= L(f, s-2)^{12} L(sym^2 f, s-2)^9 L(sym^4 f, s-2)L(s, \chi)^9 L(f \otimes \chi, s)^{12} L(sym^3 f \otimes \chi, s)^4 \\
 &\quad \times L(sym^2 f \otimes \chi, s)^9 L(sym^4 f \otimes \chi, s), \\
 W_5(s) &= L(f, s-2)^{30} L(sym^2 f, s-2)^{25} L(sym^4 f, s-2)^5 L(sym^5 f, s-2)L(s, \chi)^{21} L(f \otimes \chi, s)^{30} \\
 &\quad \times L(sym^3 f \otimes \chi, s)^{14} L(sym^2 f \otimes \chi, s)^{25} L(sym^4 f \otimes \chi, s)^5 L(sym^5 f \otimes \chi, s), \\
 W_6(s) &= L(f, s-2)^{96} L(sym^2 f, s-2)^{69} L(sym^4 f, s-2)^{20} L(sym^5 f, s-2)^6 L(sym^6 f, s-2) \\
 &\quad \times L(s, \chi)^{51} L(f \otimes \chi, s)^{96} L(sym^2 f \otimes \chi, s)^{69} L(sym^3 f \otimes \chi, s)^{44} \\
 &\quad \times L(sym^4 f \otimes \chi, s)^{20} L(sym^5 f \otimes \chi, s)^6 L(sym^6 f \otimes \chi, s), \\
 W_7(s) &= L(f, s-2)^{196} L(sym^2 f, s-2)^{189} L(sym^4 f, s-2)^{70} L(sym^5 f, s-2)^{27} L(sym^6 f, s-2)^7 \\
 &\quad \times L(sym^7 f, s-2)L(s, \chi)^{127} L(f \otimes \chi, s)^{196} L(sym^2 f \otimes \chi, s)^{189} L(sym^3 f \otimes \chi, s)^{133} \\
 &\quad \times L(sym^4 f \otimes \chi, s)^{70} L(sym^5 f \otimes \chi, s)^{27} L(sym^6 f \otimes \chi, s)^7 L(sym^7 f \otimes \chi, s), \\
 W_8(s) &= L(f, s-2)^{512} L(sym^2 f, s-2)^{518} L(sym^4 f, s-2)^{230} L(sym^5 f, s-2)^{104} \\
 &\quad \times L(sym^6 f, s-2)^{35} L(sym^7 f, s-2)^8 L(sym^8 f, s-2)L(s, \chi)^{323} L(f \otimes \chi, s)^{512} \\
 &\quad \times L(sym^2 f \otimes \chi, s)^{518} L(sym^3 f \otimes \chi, s)^{392} L(sym^4 f \otimes \chi, s)^{230} L(sym^5 f \otimes \chi, s)^{104} \\
 &\quad \times L(sym^6 f \otimes \chi, s)^{35} L(sym^7 f \otimes \chi, s)^8 L(sym^8 f \otimes \chi, s),
 \end{aligned}$$

and $H_\ell(s)$ converges absolutely and uniformly in the right half-plane for $\operatorname{Re}(s) \geq \frac{5}{2} + \epsilon$ with $H_\ell(s) \neq 0$ for $\operatorname{Re}(s) = 3$.

Lemma 4. For any $\epsilon > 0$, we have

$$T_\ell(s) = \begin{cases} \zeta(s)\widetilde{W}_1(s)\widetilde{H}_1(s) & \text{if } \ell = 1, \\ \zeta(s)^2\widetilde{W}_2(s)\widetilde{H}_2(s) & \text{if } \ell = 2, \\ \zeta(s)^4 L(sym^3 f, s)\widetilde{W}_3(s)\widetilde{H}_3(s) & \text{if } \ell = 3, \\ \zeta(s)^9 L(sym^3 f, s)^4\widetilde{W}_4(s)\widetilde{H}_4(s) & \text{if } \ell = 4, \\ \zeta(s)^{21} L(sym^3 f, s)^{14}\widetilde{W}_5(s)\widetilde{H}_5(s) & \text{if } \ell = 5, \\ \zeta(s)^{51} L(sym^3 f, s)^{44}\widetilde{W}_6(s)\widetilde{H}_6(s) & \text{if } \ell = 6, \\ \zeta(s)^{127} L(sym^3 f, s)^{133}\widetilde{W}_7(s)\widetilde{H}_7(s) & \text{if } \ell = 7, \\ \zeta(s)^{323} L(sym^3 f, s)^{392}\widetilde{W}_8(s)\widetilde{H}_8(s) & \text{if } \ell = 8, \end{cases}$$

where

$$\begin{aligned}
 \widetilde{W}_1(s) &= L(f, s)L(s-2, \chi)L(f \otimes \chi, s-2), \\
 \widetilde{W}_2(s) &= L(f, s)^2 L(sym^2 f, s)L(s-2, \chi)^2 L(f \otimes \chi, s-2)^2 L(sym^2 f \otimes \chi, s-2), \\
 \widetilde{W}_3(s) &= L(f, s)^5 L(sym^2 f, s)^3 L(s-2, \chi)^4 L(f \otimes \chi, s-2)^5 L(sym^3 f \otimes \chi, s-2) \\
 &\quad \times L(sym^2 f \otimes \chi, s-2)^3, \\
 \widetilde{W}_4(s) &= L(f, s)^{12} L(sym^2 f, s)^9 L(sym^4 f, s)L(s-2, \chi)^9 L(f \otimes \chi, s-2)^{12} L(sym^3 f \otimes \chi, s-2)^4 \\
 &\quad \times L(sym^2 f \otimes \chi, s-2)^9 L(sym^4 f \otimes \chi, s-2), \\
 \widetilde{W}_5(s) &= L(f, s)^{30} L(sym^2 f, s)^{25} L(sym^4 f, s)^5 L(sym^5 f, s)L(s-2, \chi)^{21} L(f \otimes \chi, s-2)^{30} \\
 &\quad \times L(sym^3 f \otimes \chi, s-2)^{14} L(sym^2 f \otimes \chi, s-2)^{25} L(sym^4 f \otimes \chi, s-2)^5 \\
 &\quad \times L(sym^5 f \otimes \chi, s-2), \\
 \widetilde{W}_6(s) &= L(f, s)^{96} L(sym^2 f, s)^{69} L(sym^4 f, s)^{20} L(sym^5 f, s)^6 L(sym^6 f, s) \\
 &\quad \times L(s-2, \chi)^{51} L(f \otimes \chi, s-2)^{96} L(sym^2 f \otimes \chi, s-2)^{69} L(sym^3 f \otimes \chi, s-2)^{44} \\
 &\quad \times L(sym^4 f \otimes \chi, s-2)^{20} L(sym^5 f \otimes \chi, s-2)^6 L(sym^6 f \otimes \chi, s-2), \\
 \widetilde{W}_7(s) &= L(f, s)^{196} L(sym^2 f, s)^{189} L(sym^4 f, s)^{70} L(sym^5 f, s)^{27} L(sym^6 f, s)^7 L(sym^7 f, s) \\
 &\quad \times L(s-2, \chi)^{127} L(f \otimes \chi, s-2)^{196} L(sym^2 f \otimes \chi, s-2)^{189} L(sym^3 f \otimes \chi, s-2)^{133} \\
 &\quad \times L(sym^4 f \otimes \chi, s-2)^{70} L(sym^5 f \otimes \chi, s-2)^{27} L(sym^6 f \otimes \chi, s-2)^7 L(sym^7 f \otimes \chi, s-2), \\
 \widetilde{W}_8(s) &= L(f, s)^{512} L(sym^2 f, s)^{518} L(sym^4 f, s)^{230} L(sym^5 f, s)^{104} L(sym^6 f, s)^{35} L(sym^7 f, s)^8 \\
 &\quad \times L(sym^8 f, s)L(s-2, \chi)^{323} L(f \otimes \chi, s-2)^{512} L(sym^2 f \otimes \chi, s-2)^{518} \\
 &\quad \times L(sym^3 f \otimes \chi, s-2)^{392} L(sym^4 f \otimes \chi, s-2)^{230} L(sym^5 f \otimes \chi, s-2)^{104} \\
 &\quad \times L(sym^6 f \otimes \chi, s-2)^{35} L(sym^7 f \otimes \chi, s-2)^8 L(sym^8 f \otimes \chi, s-2),
 \end{aligned}$$

and $\widetilde{H}_\ell(s)$ converges absolutely and uniformly in the right half-plane for $\operatorname{Re}(s) \geq \frac{5}{2} + \epsilon$ with $\widetilde{H}_\ell(s) \neq 0$ for $\operatorname{Re}(s) = 3$.

3.1. Convexity/Subconvexity Bounds and Integral Moments

In this section, we state results related to the convexity/subconvexity bounds and integral moments of L -functions.

We recall results concerning the subconvexity bound, the twelfth moment of the Riemann zeta function, and the fourth moment of Dirichlet L -functions. Proofs may be found in [9], [1, Theorem 5], and [20], respectively.

Lemma 5. *Let $\zeta(s)$ be the Riemann zeta function. Then, for any $\epsilon > 0$, we have*

$$\zeta(\gamma + it) \ll_\epsilon (1 + |t|)^{\max\{\frac{13}{42}(1-\gamma), 0\} + \epsilon}$$

uniformly for $\frac{1}{2} \leq \gamma \leq 1$ and $|t| \geq 1$,

$$\int_1^T \left| \zeta\left(\frac{1}{2} + it, f\right) \right|^{12} dt \ll T^{2+\epsilon},$$

and

$$\int_1^T \left| L\left(\frac{1}{2} + it, \chi\right) \right|^4 dt \ll T^{1+\epsilon}$$

uniformly for $T \geq 1$.

For the L -functions $L(f, s)$, $L(s, \chi)$, and $L(\text{sym}^2 f, s)$, we recall the following subconvexity bounds; see [8, Corollary], [10], and [15], respectively, for the proofs.

Lemma 6. *For any $\epsilon > 0$, we have*

$$\begin{aligned} L(f, \gamma + it) &\ll (1 + |t|)^{\max\{\frac{2}{3}(1-\gamma), 0\} + \epsilon}, \\ L(\gamma + it, \chi) &\ll (1 + |t|)^{\max\{\frac{1}{3}(1-\gamma), 0\} + \epsilon}, \end{aligned}$$

and

$$L(\text{sym}^2 f, \gamma + it) \ll (1 + |t|)^{\max\{\frac{6}{5}(1-\gamma), 0\} + \epsilon}$$

uniformly for $\frac{1}{2} \leq \gamma \leq 1$ and $|t| \geq 1$.

Lemma 7 ([12]). *Let χ be a primitive character of modulus q and $\mathfrak{L}_{m,n}^d(s, \chi)$ be a general L -function of degree $2A$. For any $\epsilon > 0$, we have*

$$\mathfrak{L}_{m,n}^d(\gamma + it, \chi) \ll (q(1 + |t|))^{\max\{A(1-\gamma), 0\} + \epsilon}$$

uniformly for $-\epsilon \leq \gamma \leq 1 + \epsilon$ and

$$\int_T^{2T} |\mathfrak{L}_{m,n}^d(\gamma + it, \chi)|^2 dt \ll (qT)^{2A(1-\gamma) + \epsilon}$$

uniformly for $\frac{1}{2} \leq \gamma \leq 1 + \epsilon$ and $T \geq 1$.

Remark 1. From Fomenko's paper [6], we learn that $L(\text{sym}^3 f, s)$ has an analytic continuation to the half-plane for $\text{Re}(s) > \frac{1}{2}$ except for a simple pole at $s = 1$.

4. Proofs of Theorems 1 and 2

Proof of Theorem 1. To estimate the integrals, we frequently use Lemmas 5-7 and the Cauchy-Schwarz inequality. For $\ell = 2$, consider the sum $\sum_{n \leq x} a_{\mathbb{K}}^2(n)r(n)$. From Lemma 3 and Perron's formula, we have

$$\sum_{n \leq x} a_{\mathbb{K}}^2(n)r(n) = \frac{1}{2\pi i} \int_{3+\epsilon-iT}^{3+\epsilon+iT} R_2(s) \frac{x^s}{s} ds + \mathcal{O}\left(\frac{x^{3+\epsilon}}{T}\right),$$

where $1 \leq T \leq x$, for some T to be chosen later. Using Cauchy's residue theorem, we get

$$\begin{aligned} \sum_{n \leq x} a_{\mathbb{K}}^2(n)r(n) &= \text{Res}_{s=3} \left\{ R_2(s) \frac{x^s}{s} \right\} \\ &+ \frac{1}{2\pi i} \left\{ \int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT} + \int_{3+\epsilon-iT}^{\frac{5}{2}+\epsilon-iT} + \int_{\frac{5}{2}+\epsilon+iT}^{3+\epsilon+iT} \right\} R_2(s) \frac{x^s}{s} ds \\ &+ \mathcal{O}\left(\frac{x^{3+\epsilon}}{T}\right). \end{aligned}$$

Since $R_2(s)$ has a pole at $s = 3$ of order 2 coming out from $\zeta(s-2)^2$, we find

$$16 \text{Res}_{s=3} \left\{ \frac{R_2(s)x^s}{s} \right\} = x^3 P_2(\log x),$$

where $P_2(\log x)$ is a polynomial of degree 1 in $\log x$. By using Lemma 5, Lemma 6, and Lemma 7, we get

$$\begin{aligned} \left| \left\{ \int_{3+\epsilon-iT}^{\frac{5}{2}+\epsilon-iT} + \int_{\frac{5}{2}+\epsilon+iT}^{3+\epsilon+iT} \right\} R_2(s) \frac{x^s}{s} ds \right| &\ll \int_{\frac{1}{2}+\epsilon}^{1+\epsilon} \frac{|R_2(\gamma+iT)| x^{\gamma+2}}{T} d\gamma \\ &\ll \max_{\frac{1}{2}+\epsilon \leq \gamma \leq 1+\epsilon} x^{\gamma+2} T^{(\frac{13}{21} + \frac{4}{3} + \frac{6}{5})(1-\gamma)-1+\epsilon} \\ &\ll \frac{x^{3+\epsilon}}{T} + x^{\frac{5}{2}+\epsilon} T^{\frac{331}{210}-1+\epsilon} \end{aligned}$$

and

$$\left| \int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT} R_2(s) \frac{x^s}{s} ds \right| \ll x^{(\frac{5}{2}+\epsilon)} T^{\frac{331}{210}-\frac{13}{14}+\epsilon}.$$

Therefore,

$$16 \sum_{n \leq x} a_{\mathbb{K}}^2(n)r(n) = x^3 P_2(\log x) + \mathcal{O}\left(\frac{x^{3+\epsilon}}{T}\right) + \mathcal{O}\left(x^{\frac{5}{2}+\epsilon} T^{\frac{331}{210}-\frac{13}{14}+\epsilon}\right).$$

Now, consider $\sum_{n \leq x} a_{\mathbb{K}}^2(n)t(n)$. Since $T_2(s)$ (as in Lemma 4) is analytic in the obtained region, applying Perron's formula and Cauchy's residue theorem, we obtain

$$\sum_{n \leq x} a_{\mathbb{K}}^2(n)t(n) = \frac{1}{2\pi i} \left\{ \int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT} + \int_{3+\epsilon-iT}^{\frac{5}{2}+\epsilon-iT} + \int_{\frac{5}{2}+\epsilon+iT}^{3+\epsilon+iT} \right\} T_2(s) \frac{x^s}{s} ds + \mathcal{O}\left(\frac{x^{3+\epsilon}}{T}\right).$$

Also, we find

$$\left| \left\{ \int_{3+\epsilon-iT}^{\frac{5}{2}+\epsilon-iT} + \int_{\frac{5}{2}+\epsilon+iT}^{3+\epsilon+iT} \right\} T_2(s) \frac{x^s}{s} ds \right| \ll \frac{x^{3+\epsilon}}{T} + x^{\frac{5}{2}+\epsilon} T^{\frac{25}{12}-1+\epsilon}$$

and

$$\left| \int_{\frac{5}{2}+\epsilon-\iota T}^{\frac{5}{2}+\epsilon+\iota T} T_2(s) \frac{x^s}{s} ds \right| \ll x^{(\frac{5}{2}+\epsilon)} T^{\frac{5}{4}+\epsilon}.$$

Therefore,

$$4 \sum_{n \leq x} a_{\mathbb{K}}^2(n) t(n) \ll \frac{x^{3+\epsilon}}{T} + x^{\frac{5}{2}+\epsilon} T^{\frac{5}{4}+\epsilon}.$$

From Equation (12), we obtain

$$S_2(x) = x^3 P_2(\log x) + \mathcal{O}\left(\frac{x^{3+\epsilon}}{T}\right) + \mathcal{O}\left(x^{\frac{5}{2}+\epsilon} T^{\frac{5}{4}+\epsilon}\right).$$

To balance the error terms, we choose $T = x^{\frac{2}{9}}$, which implies

$$S_2(x) = x^3 P_2(\log x) + \mathcal{O}(x^{\frac{25}{9}+\epsilon}).$$

For $\ell = 3$: Consider the sum $\sum_{n \leq x} a_{\mathbb{K}}^3(n) r(n)$. From Lemma 3 and Perron's formula, we have

$$\sum_{n \leq x} a_{\mathbb{K}}^3(n) r(n) = \frac{1}{2\pi\iota} \int_{3+\epsilon-\iota T}^{3+\epsilon+\iota T} R_3(s) \frac{x^s}{s} ds + \mathcal{O}\left(\frac{x^{3+\epsilon}}{T}\right),$$

where $1 \leq T \leq x$, for some T to be chosen later. Using Cauchy's residue theorem, we get

$$\begin{aligned} \sum_{n \leq x} a_{\mathbb{K}}^3(n) r(n) &= \text{Res}_{s=3} \left\{ R_3(s) \frac{x^s}{s} \right\} \\ &+ \frac{1}{2\pi\iota} \left\{ \int_{\frac{5}{2}+\epsilon-\iota T}^{\frac{5}{2}+\epsilon+\iota T} + \int_{3+\epsilon-\iota T}^{\frac{5}{2}+\epsilon-\iota T} + \int_{\frac{5}{2}+\epsilon+\iota T}^{3+\epsilon+\iota T} \right\} R_3(s) \frac{x^s}{s} ds \\ &+ \mathcal{O}\left(\frac{x^{3+\epsilon}}{T}\right). \end{aligned}$$

Since $R_3(s)$ has a pole at $s = 3$ of order 5 coming out from $\zeta(s-2)^4$ and $L(\text{sym}^3 f, s-2)$ (see Remark 1), we find

$$16 \text{Res}_{s=3} \left\{ \frac{R_3(s) x^s}{s} \right\} = x^3 P_3(\log x),$$

where $P_3(\log x)$ is a polynomial of degree 4 in $\log x$. Again, by using Lemmas 5-7,

we get

$$\begin{aligned} \left| \left\{ \int_{3+\epsilon-\iota T}^{\frac{5}{2}+\epsilon-\iota T} + \int_{\frac{5}{2}+\epsilon+\iota T}^{3+\epsilon+\iota T} \right\} R_3(s) \frac{x^s}{s} ds \right| &\ll \int_{\frac{1}{2}+\epsilon}^{1+\epsilon} \frac{|R_3(\gamma + \iota T)| x^{\gamma+2}}{T} d\gamma \\ &\ll \max_{\frac{1}{2}+\epsilon \leq \gamma \leq 1+\epsilon} x^{\gamma+2} T^{\left(\frac{26}{21} + \frac{10}{3} + \frac{18}{5} + 2\right)(1-\gamma) - 1 + \epsilon} \\ &\ll \frac{x^{3+\epsilon}}{T} + x^{\frac{5}{2}+\epsilon} T^{\frac{753}{210} - 1 + \epsilon} \end{aligned}$$

and

$$\left| \int_{\frac{5}{2}+\epsilon-\iota T}^{\frac{5}{2}+\epsilon+\iota T} R_3(s) \frac{x^s}{s} ds \right| \ll x^{(\frac{5}{2}+\epsilon)} T^{\frac{534}{105} - \frac{13}{14} + \epsilon}.$$

Therefore,

$$16 \sum_{n \leq x} a_{\mathbb{K}}^3(n) r(n) = x^3 P_3(\log x) + \mathcal{O}\left(\frac{x^{3+\epsilon}}{T}\right) + \mathcal{O}\left(x^{\frac{5}{2}+\epsilon} T^{\frac{534}{105} - \frac{13}{14} + \epsilon}\right).$$

Now, consider $\sum_{n \leq x} a_{\mathbb{K}_3}^3(n) t(n)$. Since $T_3(s)$ is analytic in the obtained region, we apply Perron's formula and Cauchy's residue theorem to obtain

$$\sum_{n \leq x} a_{\mathbb{K}}^3(n) t(n) = \frac{1}{2\pi\iota} \left\{ \int_{\frac{5}{2}+\epsilon-\iota T}^{\frac{5}{2}+\epsilon+\iota T} + \int_{3+\epsilon-\iota T}^{\frac{5}{2}+\epsilon-\iota T} + \int_{\frac{5}{2}+\epsilon+\iota T}^{3+\epsilon+\iota T} \right\} T_3(s) \frac{x^s}{s} ds + \mathcal{O}\left(\frac{x^{3+\epsilon}}{T}\right).$$

On using Lemmas 5-7, we deduce

$$\left| \left\{ \int_{3+\epsilon-\iota T}^{\frac{5}{2}+\epsilon-\iota T} + \int_{\frac{5}{2}+\epsilon+\iota T}^{3+\epsilon+\iota T} \right\} T_3(s) \frac{x^s}{s} ds \right| \ll \frac{x^{3+\epsilon}}{T} + x^{\frac{5}{2}+\epsilon} T^{\frac{77}{12} - 1 + \epsilon}$$

and applying the Cauchy-Schwarz inequality, we get

$$\left| \int_{\frac{5}{2}+\epsilon-\iota T}^{\frac{5}{2}+\epsilon+\iota T} T_3(s) \frac{x^s}{s} ds \right| \ll x^{(\frac{5}{2}+\epsilon)} T^{\frac{23}{4} + \epsilon}.$$

Hence,

$$4 \sum_{n \leq x} a_{\mathbb{K}}^3(n) t(n) \ll \frac{x^{3+\epsilon}}{T} + x^{\frac{5}{2}+\epsilon} T^{\frac{23}{4} + \epsilon}.$$

From Equation (12), we have

$$S_3(x) = x^3 P_3(\log x) + \mathcal{O}\left(\frac{x^{3+\epsilon}}{T}\right) + \mathcal{O}\left(x^{\frac{5}{2}+\epsilon} T^{\frac{23}{4} + \epsilon}\right).$$

We balance the error terms by choosing $T = x^{\frac{2}{27}}$, which further provides the desired asymptotic formula

$$S_3(x) = x^3 P_3(\log x) + \mathcal{O}(x^{\frac{79}{27}+\epsilon}).$$

This completes the proof. \square

Proof of Theorem 2. We only give the proof when $\ell \geq 4$ is an even integer, and the other case follows similarly. Consider the sum $\sum_{n \leq x} a_{\mathbb{K}}^{\ell}(n)r(n)$. From Lemma 1, we have

$$\begin{aligned} R_{\ell}(s) &= \zeta(s-2)^{a_{0,\ell}} L(\text{sym}^3 f, s-2)^{a_{3,\ell}} \prod_{\substack{1 \leq t_1 \leq \ell \\ t_1 \neq 3}} L(\text{sym}^{t_1} f, s-2)^{a_{t_1,\ell}} \\ &\quad \times \prod_{0 \leq t_2 \leq \ell} L(\text{sym}^{t_2} f \otimes \chi, s)^{a_{t_2,\ell}} H_{\ell}(s). \end{aligned}$$

From [14, Lemma 2.4], we learn that the degree of

$$L(\text{sym}^3 f, s-2)^{a_{3,\ell}} \prod_{\substack{1 \leq t_1 \leq \ell \\ t_1 \neq 3}} L(\text{sym}^{t_1} f, s-2)^{a_{t_1,\ell}}$$

is $(3^{\ell} - a_{0,\ell})$. From Lemma 1 and Perron's formula, we have

$$\sum_{n \leq x} a_{\mathbb{K}}^{\ell}(n)r(n) = \frac{1}{2\pi i} \int_{3+\epsilon-iT}^{3+\epsilon+iT} R_{\ell}(s) \frac{x^s}{s} ds + \mathcal{O}\left(\frac{x^{3+\epsilon}}{T}\right).$$

Using Cauchy's residue theorem, we get

$$\begin{aligned} \sum_{n \leq x} a_{\mathbb{K}}^{\ell}(n)r(n) &= \text{Res}_{s=3} \left\{ R_{\ell}(s) \frac{x^s}{s} \right\} \\ &\quad + \frac{1}{2\pi i} \left\{ \int_{\frac{5}{2}+\epsilon-iT}^{\frac{5}{2}+\epsilon+iT} + \int_{3+\epsilon-iT}^{\frac{5}{2}+\epsilon-iT} + \int_{\frac{5}{2}+\epsilon+iT}^{3+\epsilon+iT} \right\} R_{\ell}(s) \frac{x^s}{s} ds \\ &\quad + \mathcal{O}\left(\frac{x^{3+\epsilon}}{T}\right). \end{aligned}$$

From Remark 1, the order of the pole at $s = 3$ in $R_{\ell}(s)$ is $(a_{0,\ell} + a_{3,\ell})$; therefore, we have

$$16 \text{Res}_{s=3} R_{\ell}(s) \frac{x^s}{s} = x^3 P_{\ell}(\log x),$$

where $P_{\ell}(\log x)$ is a polynomial in $\log x$ of degree $(a_{0,\ell} + a_{3,\ell}) - 1$. We use results from Subsection 3.1 to estimate the integrals appearing in the proof of this theorem,

from which one can easily find

$$\begin{aligned} \left| \left\{ \int_{3+\epsilon-\iota T}^{\frac{5}{2}+\epsilon-\iota T} + \int_{\frac{5}{2}+\epsilon+\iota T}^{3+\epsilon+\iota T} \right\} R_\ell(s) \frac{x^s}{s} ds \right| &\ll \int_{\frac{1}{2}+\epsilon}^{1+\epsilon} \frac{|R_\ell(\gamma + \iota T)| x^{\gamma+2}}{T} d\gamma \\ &\ll \max_{\frac{1}{2}+\epsilon \leq \gamma \leq 1+\epsilon} x^{\gamma+2} T^{\left(\frac{13}{42}(a_{0,\ell}) + \frac{3^\ell}{2} - \frac{a_{0,\ell}}{2} \right) (1-\gamma) - 1 + \epsilon} \\ &\ll \frac{x^{3+\epsilon}}{T} + x^{\frac{5}{2}+\epsilon} T^{\left(\frac{13a_{0,\ell}+21(3^\ell-a_{0,\ell})}{84} \right) - 1 + \epsilon} \end{aligned}$$

and

$$\left| \int_{\frac{5}{2}+\epsilon-\iota T}^{\frac{5}{2}+\epsilon+\iota T} R_\ell(s) \frac{x^s}{s} ds \right| \ll x^{\left(\frac{5}{2}+\epsilon\right)} T^{-\frac{13}{14} + \left(\frac{13a_{0,\ell}+21(3^\ell-a_{0,\ell})}{84} \right) + \epsilon}.$$

Therefore,

$$16 \sum_{n \leq x} a_{\mathbb{K}}^\ell(n) r(n) = x^3 P_\ell(\log x) + \mathcal{O}\left(\frac{x^{3+\epsilon}}{T}\right) + \mathcal{O}\left(x^{\frac{5}{2}+\epsilon} T^{-\frac{13}{14} + \left(\frac{13a_{0,\ell}+21(3^\ell-a_{0,\ell})}{84} \right) + \epsilon}\right).$$

Now, consider the sum $\sum_{n \leq x} a_{\mathbb{K}}^\ell(n) t(n)$. Since $T_\ell(s)$ is analytic (as in Lemma 2) in the obtained region, we apply Perron's formula and Cauchy's residue theorem to get

$$\sum_{n \leq x} a_{\mathbb{K}}^\ell(n) t(n) = \frac{1}{2\pi\iota} \left\{ \int_{\frac{5}{2}+\epsilon-\iota T}^{\frac{5}{2}+\epsilon+\iota T} + \int_{3+\epsilon-\iota T}^{\frac{5}{2}+\epsilon-\iota T} + \int_{\frac{5}{2}+\epsilon+\iota T}^{3+\epsilon+\iota T} \right\} T_\ell(s) \frac{x^s}{s} ds + \mathcal{O}\left(\frac{x^{3+\epsilon}}{T}\right).$$

Note that the degree of

$$\prod_{1 \leq t_2 \leq \ell} L(\text{sym}^{t_2} f \otimes \chi, s-2)^{a_{t_2,\ell}}$$

is $(3^\ell - a_{0,\ell})$. Therefore, we obtain

$$\left| \left\{ \int_{3+\epsilon-\iota T}^{\frac{5}{2}+\epsilon-\iota T} + \int_{\frac{5}{2}+\epsilon+\iota T}^{3+\epsilon+\iota T} \right\} T_\ell(s) \frac{x^s}{s} ds \right| \ll \frac{x^{3+\epsilon}}{T} + x^{\frac{5}{2}+\epsilon} T^{\left(\frac{4a_{0,\ell}+6(3^\ell-a_{0,\ell})}{24} \right) - 1 + \epsilon}$$

and applying the Cauchy-Schwarz inequality, we get

$$\left| \int_{\frac{5}{2}+\epsilon-\iota T}^{\frac{5}{2}+\epsilon+\iota T} T_\ell(s) \frac{x^s}{s} ds \right| \ll x^{\left(\frac{5}{2}+\epsilon\right)} T^{\frac{3^\ell}{4} - 1 + \epsilon}.$$

Hence,

$$4 \sum_{n \leq x} a_{\mathbb{K}}^{\ell}(n) t(n) \ll \frac{x^{3+\epsilon}}{T} + x^{\frac{5}{2}+\epsilon} T^{\frac{3\ell}{4}-1+\epsilon}.$$

From Equation (12), we have

$$S_{\ell}(x) = x^3 P_{\ell}(\log x) + \mathcal{O}\left(\frac{x^{3+\epsilon}}{T}\right) + \mathcal{O}\left(x^{\frac{5}{2}+\epsilon} T^{\frac{3\ell}{4}-1+\epsilon}\right).$$

We choose $T = x^{\frac{2}{3\ell}}$ to get

$$S_{\ell}(x) = x^3 P_{\ell}(\log x) + \mathcal{O}\left(x^{3-\frac{2}{3\ell}+\epsilon}\right),$$

where $P_{\ell}(\log x)$ is a polynomial in $\log x$ of degree $a_{0,\ell} + a_{3,\ell} - 1$. This completes the proof. \square

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