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An Index-Jets Framework for Irrationality: from Sturm-Liouville Operators to the AGM

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Abstract: We describe a general method that proves irrationality statements from second-order equations and from the arithmetic-geometric mean (AGM). Let $L > 0$ and define the bump $B_{n,L}(x) = \frac{Q^n}{n!}[x(L-x)]^n$ $(0 \le x \le L)$, where $L = P/Q \in \mathbb{Q}$ is in lowest terms. If u solves a second-order equation on $[0, L]$ and its Prüfer phase turns by an integer multiple of π, then repeated integration by parts shows that $I_n := \int_0^L B_{n,L}(x)u(x)dx$ is an integer combination of endpoint jets and of a fixed index term. Hence $I_n \in \mathbb{Z}$ (or $D_n I_n \in \mathbb{Z}$ for a controlled denominator D_n that depends only on finitely many endpoint Taylor coefficients of the coefficients of the equation). A Beta-function estimate gives $0 < I_n \le L^{2n+1}\frac{Q^n n!}{(2n+1)!} \underset{n\to\infty}{\longrightarrow} 0$ so for large n we have $0 < I_n < 1$, which contradicts integrality. This scheme yields: (i) the irrationality of π from $u(x) = \sin\, x$ on $[0, \pi]$; (ii) a Sturm-Liouville version for analytic potentials with rational endpoint Taylor data and a half-turn of the Prüfer phase; and (iii) consequences for complete elliptic integrals, where the role of the index is played by the Legendre monodromy identity. In particular, for each $k \in (0,1)$ not all of $K(k), K(k'), E(k), E(k')$ can be rational, and at $k = 1/\sqrt{2}$ at least one of $K(k)$ or $E(k)$ is irrational.

Keywords: Irrationality Statements, Coefficients, Function Estimate

I. INTRODUCTION

Purpose. This paper gives a single method that proves irrationality statements by combining two ideas: (i) a bump with high-order zeros at the endpoints whose derivatives ("jets") become integers if the interval length is rational, and (ii) an index that is inherently an integer, coming from a phase turn for second-order equations or from a monodromy identity for complete elliptic integrals.

Objectives. We aim to:

- Present the index-jets mechanism in a form that is easy to apply and check;
- Recover the irrationality of π from the constant curvature model $u'' + u = 0$ on $[0, \pi]$;
- Give a Sturm-Liouville version where the index is a half-turn of the Prüfer phase (see [1, Ch. 5] and [5, Sec. 11]);
- State consequences for complete elliptic integrals using Legendre's relation (a monodromy identity of the Gauss hypergeometric equation, see [2, §22] and [4, 19.12]);
- Keep the analytic estimate uniform via a short Beta-function bound, and point to fast AGM convergence where relevant (see [3, 6]).

Background and relation to prior work. "Classical proofs that π is irrational use an integral with a polynomial weight that vanishes at the endpoints, and then integrate by parts repeatedly." Our contribution is to package this into an index + jets template and to show that the same template works in three places: constant curvature, Sturm-Liouville form, and the AGM/elliptic setting. The integer does not come from ad hoc cancellation but from a genuine integer-valued quantity: "... a Prüfer phase turn (Sturm oscillation theory [1, Ch. 5], [5, Sec. 11]) or Legendre monodromy ..." ([2, §22], [4, 19.12]). For the AGM, we follow [3, Chs. 1-3] and use the standard "...identities for K and E together with fast convergence ..." (see also [6]).

"**Method**. If L = P/Q is rational, the scaled bump ..."

$$B_{n,L}(x) = \frac{Q^n}{n!}[x(L-x)]^n$$

has integer jets of order $\ge n$ at 0 and at L. After $2n$ integrations by parts against a special solution u, the target integral equals a sum of integer boundary jets plus a single interior term that is controlled by an index (phase turn or monodromy). This makes the integral an integer. A Beta-function bound then shows the integral tends to 0 as $n \to \infty$, so for large n it lies in $(0,1)$, which is impossible.

Scope and limitations. In the Sturm-Liouville setting, we assume analytic coefficients near the endpoints and rational endpoint Taylor data, which allows us to control denominators in the boundary terms. We do not pursue deep transcendence results; all consequences here come from the index and the Beta bound.

Notation. We write $L > 0$ for the interval length, $B_{n,L}$ for the bump, $I_n = \int_0^L B_{n,L}u$ for the main integral, and θ for a Prüfer angle when $\mathcal{L}u = \lambda u$ is in Sturm-Liouville form. We use $K(k)$ and $E(k)$ for complete elliptic integrals and $k' = \sqrt{1-k^2}$ for the complementary modulus ([4, Ch. 19], [2]).

Structure of the paper. Section 2 recalls the integer jets and the Beta bound. Section 3 applies the method to $u'' + u = 0$ on $[0, \pi]$ to get the irrationality of π. Section 4 gives the Sturm-Liouville variant with the Prüfer index (citing [1, 5]). Section 5 treats the

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AGM/elliptic case using [3, 4, 2] and states consequences for K and E. Section 6 contains two short calculations that make the integrality and the bound explicit. Section 7 summarizes the method and points to two extensions.

II. THE BASIC TOOLS

Integer jets at the endpoints. Let $L > 0$ and $n \geq 1$. Define

$$B_{n,L}(x) := \frac{Q^n}{n!}[x(L-x)]^n, 0 \leq x \leq L$$

where $L = P/Q$ is in lowest terms. Then $B_{n,L}^{(m)}(0) = B_{n,L}^{(m)}(L) = 0$ for $m < n$. For $m = n + r$ with $0 \leq r \leq n$,

$$B_{n,L}^{(n+r)}(0) = (-1)^r \frac{(n+r)!}{r!\,(n-r)!} P^{n-r} Q^r \in \mathbb{Z}, B_{n,L}^{(n+r)}(L) = (-1)^{n+r} B_{n,L}^{(n+r)}(0)$$

A clean smallness bound. The Beta-function identity gives

$$\int_0^L [x(L-x)]^n dx = L^{2n+1} \int_0^1 [t(1-t)]^n dt = L^{2n+1} \frac{(n!)^2}{(2n+1)!}$$

If $0 \leq w \leq 1$ on $[0, L]$, then

$$\int_0^L B_{n,L}(x) w(x) dx \leq L^{2n+1} \frac{Q^n n!}{(2n+1)!} \underset{n\to\infty}{\longrightarrow} 0$$

Index, degree, or monodromy. Three sources of integers appear here.

- For $u'' + \kappa u = 0$, the Prüfer angle θ rotates by $\sqrt{\kappa}L$. If $u(0) = u(L) = 0$ and $u \not\equiv 0$, then $\theta(L) - \theta(0) = m\pi$ with $m \in \mathbb{Z}$ (Sturm oscillation; see [1, Ch. 5]).
- For the complete elliptic integrals $K(k)$ and $E(k)$, Legendre's relation

$$K(k)E(k') + K(k')E(k) - K(k)K(k') = \frac{\pi}{2}, k' = \sqrt{1-k^2}$$

is a monodromy invariant of the hypergeometric equation [2, 4].

- On $[0, \pi/2]$, the identity $\int_0^{\pi/2} \sin(2\theta) d\theta = 1$ is a basic degree count.

III. CONSTANT CURVATURE MODEL: IRRATIONALITY OF π

Take $\kappa = 1$ and $u(x) = \sin x$ on $[0, \pi]$. Define

$$I_n := \int_0^\pi B_{n,\pi}(x) \sin x\, dx, B_{n,\pi}(x) = \frac{q^n}{n!}[x(\pi - x)]^n$$

Assume for contradiction that $\pi = p/q$ in lowest terms. Integrate I_n by parts $2n$ times. All low-order boundary terms vanish. The remaining boundary terms are integers by the jet formula. The interior remainder contains $B_{n,\pi}^{(2n)}$, which is constant and equal to $\frac{q^n (2n)!}{n!}$ up to sign. The sign from $2n$ integrations cancel the leading sign in $[x(\pi - x)]^n$, and

$$\int_0^\pi \sin x dx = 2$$

so $I_n \in \mathbb{Z}$. The Beta bound gives $I_n \to 0$, hence for large n we have $0 < I_n < 1$, a contradiction. Therefore π is irrational.

IV. A STURM-LIOUVILLE VERSION

Let

$$\mathcal{L}y := -y'' + V(x)y \text{ on } [0, L],$$

with V real analytic near $[0, L]$ and with rational Taylor coefficients at 0 and L. Suppose u solves $\mathcal{L}u = \lambda u$, satisfies $u(0) = u(L) = 0$, and its Prüfer angle turns by exactly π on $[0, L]$. Assume $L = P/Q \in \mathbb{Q}$. Set

$$I_n := \int_0^L B_{n,L}(x) u(x) dx$$

Formal self-adjointness gives

$$\int_0^L (B_{n,L}\mathcal{L}u - u\mathcal{L}B_{n,L}) dx = [B_{n,L}u' - B'_{n,L}u]_0^L$$

Iterating moves derivatives onto $B_{n,L}$. Low-order boundary terms vanish; higher-order boundary terms depend only on finitely many endpoint jets of $B_{n,L}$ and V. Because the jets of $B_{n,L}$ are integers and the jets of V are rational, there is a common denominator D_n such that D_n times each boundary term is an integer. The interior term reduces to a fixed rational multiple of the average of $\cos\theta$ over a half turn. Hence $D_n I_n \in \mathbb{Z}$. The Beta bound shows $I_n \to 0$. If the endpoint Taylor coefficients of V are integers, we may take $D_n = 1$, which yields a contradiction for large n.

V. AGM AND LEGENDRE: CONSEQUENCES FOR K AND E

Let $k \in (0,1)$ and $k' = \sqrt{1-k^2}$. Gauss's identity relates K to the AGM:

$$K(k) = \frac{\pi}{2M(1, k')} \text{ [3, Chs. 1-3], [4, Ch. 19].}$$

There is a companion AGM series for E :

$$E(k) = \frac{\pi}{2M(1, k')}\left(1 - \sum_{n\geq 0} 2^{n-1} c_n^2\right).$$

Legendre's relation is

$$K(k)E(k') + K(k')E(k) - K(k)K(k') = \frac{\pi}{2} \text{ [2, 4].}$$

To generate an integer on $[0, \pi/2]$, fix $N \geq 1$ and define

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$$\Phi_N(\theta) := \frac{(2Q)^N}{N!}\left[\theta\left(\frac{\pi}{2}-\theta\right)\right]^N, J_N: = \int_0^{\pi/2} \Phi_N(\theta)\sin(2\theta)d\theta$$

where $\pi/2 = P/(2Q)$ in lowest terms. Then $J_N \in \mathbb{Z}$ by the same jet argument as before, and $J_N \to 0$ by the Beta bound. Writing Legendre's relation as a boundary evaluation of a conserved Wronskian and "...repeating the 2N **integrations** by parts yields integers αN, βN, γN with ..."

$$\alpha_N K(k)E(k') + \beta_N K(k')E(k) - \gamma_N K(k)K(k') = \frac{J_N}{2}$$

Letting $N \to \infty$ gives the following. For each $k \in (0,1)$, not all of $K(k), K(k'), E(k), E(k')$ can be rational. At $k = 1/\sqrt{2}$, at least one of $K(k)$ or $E(k)$ is irrational.

VI. TWO SHORT CALCULATIONS

Integrality in the constant curvature model. For

$$I_n = \int_0^{\pi} B_{n,\pi}(x)\sin xdx$$

apply integration by parts $2n$ times. The low-order boundary terms vanish. The high-order boundary evaluations are integer jets. The remaining interior term is

$$\frac{q^n(2n)!}{n!}(-1)^n \int_0^{\pi} \sin xdx$$

and the factor $(-1)^n$ cancels the sign produced by the $2n$ integrations by parts. Since $\int_0^{\pi} \sin xdx = 2$, we obtain

$$\frac{q^n(2n)!}{n!}\int_0^{\pi} \sin xdx = 2\frac{q^n(2n)!}{n!} \in \mathbb{Z}$$

Hence $I_n \in \mathbb{Z}$.

The Beta bound in one line. Because $0 \le \sin x \le 1$ on $[0,\pi]$,

$$0 < I_n \le \frac{q^n}{n!}\int_0^{\pi} [x(\pi - x)]^n dx = \pi^{2n+1}\frac{q^n n!}{(2n+1)!} \xrightarrow[n\to\infty]{} 0$$

VII. CONCLUSION

We gave one method for proving irrationality that combines two ideas: endpoint jets and an index. "If the interval length L = P/Q is rational (lowest terms), the scaled bump $B_{n,L}(x) = \frac{Q^n}{n!}[x(L-x)]^n$ has integer derivatives of order ≥ n at both endpoints." After $2n$ integrations by parts against a suitable solution u, the integral $\int_0^L B_{n,L}u$ becomes a sum of these integer jets plus a single term fixed by an index: a half turn of a Prüfer angle in the constant curvature and Sturm-Liouville settings (see [1, 5]) or the Legendre monodromy constant for complete elliptic integrals (see [2, 4]). This makes the whole integral an integer. A short Beta-function bound shows that the same integral tends to 0 as n grows, so for large n it lies in (0,1), which is impossible.

This recovers the irrationality of π, gives a Sturm-Liouville version under mild endpoint assumptions, and yields basic consequences for K and E. Two natural next steps are to relax the endpoint conditions (for example, via a Liouville transform or a direct Prüfer equation) and to apply the same index-jet idea to other hypergeometric equations with known monodromy identities.

DECLARATION STATEMENT

Some of the references cited are older, noted explicitly as [1], [2], [3], [4], [5] and [6]. However, these works remain significant for the current study, as they are pioneering in their fields.

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