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Hybrid Mean Value of General Quartic Gauss Sums and Three-Term Exponential Sums

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***Abstract:** This paper explores the hybrid power mean of three-term exponential sums, incorporating weights derived from general quartic Gauss sums. By employing the theory of Dirichlet characters in conjunction with fundamental properties of classical Gauss sums, we establish several significant results. Furthermore, we determine the corresponding weight function for these three-term exponential sums, with particular applications in coding theory.*

***Keywords:** Dirichlet Character, Quartic Gauss Sums, Three-Term Exponential Sum.*
***MSC:** 11L05.*

I. INTRODUCTION

Let $q \geq 3$ be an integer and χ a Dirichlet character modulo q. For any positive integer m, a general l^{th}-Gauss sum $G\,(m, l, \chi; q)$ is defined by

$$G(m, l, x; q) = \sum_{b=1}^{q} x(b) e\left(\frac{mb^1}{q}\right)$$

When q is prime, k and t *are* positive integers, for integers m, n and s, a generalized three-term exponential sum is defined by

$$C(m, n, s, k, t, x; q) = \sum_{a=1}^{q-1} x(a) e\left(\frac{ma^k + sa^t + na}{q}\right)$$

where $e(g) = e^{2i\pi g}$.

Many researchers have actively contributed to this field. Zhang and Han [1] studied the sixth power mean of the two-term exponential sums,

$$\sum_{n=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{a^3 + na}{p}\right) \right| = 5p^4 - 8p^3 - p^2.$$

Du and Li [2] studied the fourth power mean of generalized three-term exponential sums and gave the formula:

$$\sum_{s=1}^{p} \sum_{n=1}^{p} |C(1, s, n, k, 2, \chi; p)|^2 = 2p^4 - 5p^3 + 3p^2.$$

Yu and Zhang [6], [5] estimated the trigonometric sums and the properties of the congruence equations. They also studied the sixth power mean value of the generalized three-term exponential sums and gave an exact computational formula:

$$\sum_{m=1}^{p} \sum_{n=1}^{p} \sum_{\chi \bmod p} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{a^k + ma^2 + na}{p}\right) \right|^6 = p^2(p-1)^2(6p^2 - 21p + 19).$$

Huaning LIU and Wanmei LI [7] also gave the identity for any integer $p \geq 3$,

$$\sum_{s=1}^{p} \sum_{n=1}^{p} |C(m, s, n, k, t, \chi; p)|^4 = 2p^4 - 7p^3 + 8p^2 - 3p.$$

The primary aim of this article is to investigate the mean value of three-term exponential sums and general quartic Gauss sums by employing estimates of Dirichlet characters and certain properties of Gauss sums,

$$\sum_{\chi \bmod q} \sum_{m=1}^{q} \sum_{n=1}^{q} \left| \sum_{u=1}^{q-1} \chi(u) e\left(\frac{\eta u^4}{q}\right) \right|^2 \left| \sum_{s=1}^{q-1} \chi(s) e\left(\frac{s^k + ms^2 + ns}{q}\right) \right|^2.$$

II. PRELIMINARY LEMMAS

In this section, we present certain properties of Gauss sums [3, 4] along with a set of lemmas that play a crucial role in establishing the main theorems.

Lemma 2.1 *For any integer $m \geq 1$, we have the formula*

$$G(1; m) = \frac{1}{2}\sqrt{m}(1+i)\left(1 + e^{\frac{-\pi i m}{2}}\right) = \begin{cases} \sqrt{m} & \text{if } m \equiv 1 \pmod 4; \\ 0 & \text{if } m \equiv 2 \pmod 4; \\ i\sqrt{m} & \text{if } m \equiv 3 \pmod 4; \\ (1+i)\sqrt{m} & \text{if } m \equiv 0 \pmod 4. \end{cases}$$

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Proof: See Theorem 9.16 of [3].

Lemma 2.2 *Let q be an odd prime, k, m, and n be positive integers. Then we have,*

$$\sum_{\chi(-1)=1}\sum_{m=1}^{q}\sum_{n=1}^{q}\left(\sum_{s=1}^{q-1}\chi(s)e\Big(\frac{s^k+ms^2+ns}{q}\Big)\right)^2=q^2(q-1)^3/2,$$

where χ varies over Dirichlet characters (mod q).

Proof. Since $q\equiv 3 \pmod 4$ for any integer t with $(t, q) = 1$. From the orthogonality property of characters (mod q),

$$\sum_{\chi(-1)=1}\chi(s)=\begin{cases}\frac{q-1}{2} & \text{if } s\equiv\pm1 \pmod q;\\ 0 & \text{otherwise.}\end{cases}$$

Now

$$\begin{aligned}
&\sum_{\chi(-1)=1}\sum_{m=1}^{q}\sum_{n=1}^{q}\left(\sum_{s=1}^{q-1}\chi(s)e\Big(\frac{s^k+ms^2+ns}{q}\Big)\right)^2\\
&=\sum_{\chi(-1)=1}\sum_{m=1}^{q}\sum_{n=1}^{q}\sum_{s=1}^{q-1}\sum_{t=1}^{q-1}\chi(s)\chi(\bar t)e\left(\frac{s^k-t^k+m(s^2-t^2)+n(s-t)}{q}\right)\\
&=\sum_{\chi(-1)=1}\sum_{m=1}^{q}\sum_{n=1}^{q}\sum_{s=1}^{q-1}\sum_{t=1}^{q-1}\chi(s\bar t)e\left(\frac{s^k-t^k+m(s^2-t^2)+n(s-t)}{q}\right)\\
&=\sum_{\chi(-1)=1}\sum_{s=1}^{q-1}\sum_{t=1}^{q-1}\chi(s\bar t)e\left(\frac{s^k-t^k}{q}\right)\times\sum_{m=1}^{q}\sum_{n=1}^{q}e\left(\frac{m(s^2-t^2)+n(s-t)}{q}\right)
\end{aligned}$$

$$=\sum_{s=1}^{q-1}\sum_{t=1}^{q-1}\sum_{\chi(-1)=1}\chi(s)e\left(\frac{t^k(s^k-1)}{q}\right)\times\sum_{\substack{m=1\\ s^2\equiv1 \bmod q\\ s\equiv 1 \bmod q}}^{q}\sum_{n=1}^{q}e\left(\frac{mt^2(s^2-1)+nt(s-1)}{q}\right)$$

$$=q^2\sum_{s=1}^{q-1}\sum_{t=1}^{q-1}\sum_{\chi(-1)=1}\chi(s)e\left(\frac{t^k(s^k-1)}{q}\right)$$

$$=q^2\sum_{s=1}^{q-1}\sum_{\chi(-1)=1}\chi(s)\sum_{t=1}^{q-1}e\left(\frac{t^k(s^k-1)}{q}\right)$$

$$=q^2(q-1)/2\sum_{\substack{s=1\\ s\equiv1 \bmod q\\ s^2\equiv1 \bmod q\\ s^k\equiv1 \bmod q}}^{q-1}\sum_{t=1}^{q-1}1=q^2(q-1)/2\sum_{\substack{s=1\\ s\equiv1 \bmod q\\ s^2\equiv1 \bmod q\\ s^k\equiv1 \bmod q}}^{q-1}(q-1)$$

$$=q^2(q-1)^2/2\sum_{\substack{s=1\\ s\equiv1 \bmod q\\ s^2\equiv1 \bmod q\\ s^k\equiv1 \bmod q}}^{q-1}1=q^2(q-1)^3/2.$$

Lemma 2.3 Let q be an odd prime with $q\equiv 3 \pmod 4$, k, m, n positive integers. Then we have

$$\sum_{\chi(-1)=1}\left(\sum_{u=1}^{q-1}\chi(u)\Big(\frac{u^4-1}{q}\Big)\right)\left(\sum_{s=1}^{q-1}\chi(s)e\Big(\frac{s^k+ms^2+ns}{q}\Big)\right)^2=0,$$

where χ varies over non-principal Dirichlet character (mod q).

Proof. As for a non-principal even character χ and an integer $k>0$, $\chi=\chi$

implies

$$\sum_{\chi(-1)=1}\left(\sum_{u=1}^{q-1}\chi(u)\left(\frac{u^4-1}{q}\right)\right)\left(\sum_{m=1}^{q}\sum_{n=1}^{q}\left(\sum_{s=1}^{q-1}\chi(s)e\left(\frac{s^k+ms^2+ns}{q}\right)\right)^2\right)$$

$$=\sum_{\chi(-1)=1}\left(\sum_{u=1}^{q-1}\overline{\chi}(u)\left(\frac{u^4-1}{q}\right)\right)\left(\sum_{m=1}^{q}\sum_{n=1}^{q}\left(\sum_{s=1}^{q-1}\overline{\chi}(s)e\left(\frac{s^k+ms^2+ns}{q}\right)\right)^2\right) \quad (2.1)$$

Note that,

$\left(\frac{-1}{q}\right)=-1$, so

$$\begin{aligned}\sum_{u=1}^{q-1}\overline{\chi}(u)\left(\frac{u^4-1}{q}\right) &= \sum_{u=1}^{q-1}\chi(\overline{u})\left(\frac{u^4-1}{q}\right)\\ &= \sum_{u=1}^{q-1}\chi(u)\left(\frac{\overline{u}^4-1}{q}\right)\\ &= \sum_{u=1}^{q-1}\chi(u)\left(\frac{1-u^4}{q}\right) = -\sum_{u=1}^{q-1}\chi(u)\left(\frac{u^4-1}{q}\right), \end{aligned} \quad (2.2)$$

and

$$\sum_{m=1}^{q}\sum_{n=1}^{q}\left|\sum_{s=1}^{q-1}\overline{\chi}(s)e\left(\frac{s^k+ms^2+ns}{q}\right)\right|^2 = \sum_{m=1}^{q}\sum_{n=1}^{q}\left|\sum_{s=1}^{q-1}\chi(s)e\left(\frac{s^k+ms^2+ns}{q}\right)\right|^2 \quad (2.3)$$

Form the equations (2.1), (2.2) and (2.3) we get,

$$\sum_{\chi(-1)=1}\sum_{m=1}^{q}\sum_{n=1}^{q}\sum_{u=1}^{q-1}\chi(u)\left(\frac{u^4-1}{q}\right)\left|\sum_{s=1}^{q-1}\chi(s)e\left(\frac{s^k+ms^2+ns}{q}\right)\right|^2=0.$$

Lemma 2.4 Let q be an odd prime and k, m, n positive integers. Then we have

$$\sum_{m=1}^{q}\sum_{n=1}^{q}\left|\sum_{s=1}^{q-1}e\left(\frac{s^k+ms^2+ns}{q}\right)\right|^2=q^2(q-1)^2.$$

Proof. Now,

$$\begin{aligned}\sum_{m=1}^{q}\sum_{n=1}^{q}\left|\sum_{s=1}^{q-1}e\left(\frac{s^k+ms^2+ns}{q}\right)\right|^2 &= \sum_{m=1}^{q}\sum_{n=1}^{q}\sum_{s=1}^{q-1}\sum_{t=1}^{q-1}e\left(\frac{s^k+t^k+m(s^2+t^2)+n(s+t)}{q}\right)\\ &= \sum_{m=1}^{q}\sum_{n=1}^{q}\sum_{s=1}^{q-1}\sum_{t=1}^{q-1}e\left(\frac{s^k+t^k}{q}\right)e\left(\frac{m(s^2+t^2)}{q}\right)e\left(\frac{n(s+t)}{q}\right)\\ &= \sum_{m=1}^{q}\sum_{n=1}^{q}\sum_{s=1}^{q-1}\sum_{\substack{t=1\\ s^k+t^k\equiv 0 \mod q\\ s^2+t^2\equiv 0 \mod q\\ s+t\equiv 0 \mod q}}^{q-1} 1\end{aligned}$$

$$= \sum_{m=1}^{q}\sum_{n=1}^{q}\sum_{s=1}^{q-1}(q-1)$$

$$= \sum_{m=1}^{q}\sum_{n=1}^{q}(q-1)^2 = q^2(q-1)^2.$$

Lemma 2.5 Let q be an odd prime with q ≡ 3 (mod 4), η be a positive integer with
(η, q) = 1. Then, for any non-principal even character χ mod q, we get

$$\mid G(\eta,4,\chi;q)\mid^2 + \mid G(\eta,4,\overline{\chi};q)\mid^2 = 4q.$$

Proof. Here χ is a non-principal even character mod q, then we can write.

$$\begin{aligned}
\mid G(\eta,4,\chi;q)\mid^2 &= \sum_{u=1}^{q-1}\sum_{v=1}^{q-1}\chi(u)\overline{\chi}(v)e\left(\frac{\eta(u^4-v^4)}{q}\right)\\
&= \sum_{u=1}^{q-1}\sum_{v=1}^{q-1}\chi(u)e\left(\frac{\eta v^4(u^4-1)}{q}\right)\\
&= \sum_{u=1}^{q-1}\chi(u)\sum_{v=1}^{q-1}e\left(\frac{\eta v^4(u^4-1)}{q}\right)\\
&= 2(q-1)+\sum_{u=2}^{q-2}\chi(u)\left[\sum_{v=1}^{q-1}e\left(\frac{\eta v^4(u^4-1)}{q}\right)\right]\\
&= 2q+\left(\frac{\eta}{q}\right)G(1,q)\sum_{u=1}^{q-1}\chi(u)\left(\frac{u^4-1}{q}\right)
\end{aligned}$$

Therefore,

$$\mid G(\eta,4,\chi;q)\mid^2 = 2q+i\left(\frac{\eta}{q}\right)\sqrt{q}\sum_{u=1}^{q-1}\chi(u)\left(\frac{u^4-1}{q}\right). \tag{2.4}$$

Similarly, we find the identity for $\overline{\chi}$

$$\mid G(\eta,4,\overline{\chi};q)\mid^2 = 2q+i\left(\frac{\eta}{q}\right)\sqrt{q}\sum_{u=1}^{q-1}\overline{\chi}(u)\left(\frac{u^4-1}{q}\right)$$

If $q \equiv 3 \pmod 4$, then we have $\left(\frac{-1}{q}\right) = -1$. Using this, we get,

$$\sum_{u=1}^{q-1}\overline{\chi}(u)\left(\frac{u^4-1}{q}\right) = \sum_{u=1}^{q-1}\chi(\overline{u})\left(\frac{u^4-1}{q}\right) = \sum_{u=1}^{q-1}\chi(u)\left(\frac{\overline{u}^4-1}{q}\right) = \sum_{u=1}^{q-1}\chi(u)\left(\frac{1-u^4}{q}\right)$$
$$= -\sum_{u=1}^{q-1}\chi(u)\left(\frac{u^4-1}{q}\right).$$

So,

$$\mid G(\eta,4,\overline{\chi};q)\mid^2 = 2q-i\left(\frac{\eta}{q}\right)\sqrt{q}\sum_{u=1}^{q-1}\chi(s)\left(\frac{u^4-1}{q}\right). \tag{2.5}$$

Combining (2.4) and (2.5), we have

$$\mid G(\eta,4,\chi;q)\mid^2 + \mid G(\eta,4,\overline{\chi};q)\mid^2 = 4q.$$

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III. MAIN RESULTS

In this section, we give the complete proof of the main results by using the previous lemmas.

Theorem 3.1 Let q be an odd prime with q ≡ 3 (mod 4), k, m, n positive integers. Then we have

$$\sum_{m=1}^{q}\sum_{n=1}^{q}\sum_{\chi \mod q}\left|\sum_{u=1}^{q-1}\chi(u)e\left(\frac{\eta u^4}{q}\right)\right|^2\left|\sum_{s=1}^{q-1}\chi(s)e\left(\frac{s^k+ms^2+ns}{q}\right)\right|^2 = q^2(q-1)^2(q^2+1).$$

where χ varies over Dirichlet characters (mod q).

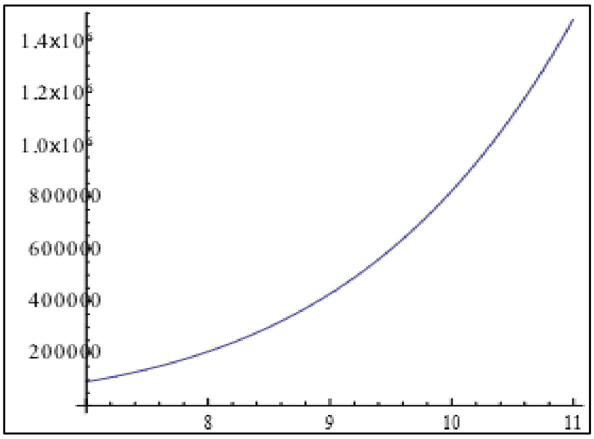
Proof. As for χ, an even character mod q, then $\overline{\chi}$ is also an even character mod q,

$$\sum_{m=1}^{q}\sum_{n=1}^{q}\sum_{\chi \mod q}\left|\sum_{u=1}^{q-1}\chi(u)e\left(\frac{\eta u^4}{q}\right)\right|^2\left|\sum_{s=1}^{q-1}\chi(s)e\left(\frac{s^k+ms^2+ns}{q}\right)\right|^2$$
$$=\sum_{m=1}^{q}\sum_{n=1}^{q}\sum_{\substack{\chi(-1)=1\\ \chi\neq\chi_0}}\left|\sum_{u=1}^{q-1}\chi(u)e\left(\frac{\eta u^4}{q}\right)\right|^2\left|\sum_{s=1}^{q-1}\chi(s)e\left(\frac{s^k+ms^2+ns}{q}\right)\right|^2$$
$$+\sum_{m=1}^{q}\sum_{n=1}^{q}\left|\sum_{u=1}^{q-1}e\left(\frac{\eta u^4}{q}\right)\right|^2\left|\sum_{s=1}^{q-1}\chi_0(s)e\left(\frac{s^k+ms^2+ns}{q}\right)\right|^2.$$

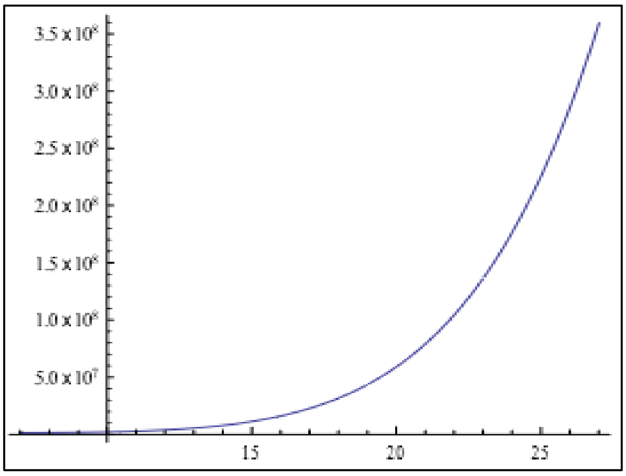
So,

$$\sum_{m=1}^{q}\sum_{n=1}^{q}\sum_{\chi \mod q}\left|\sum_{u=1}^{q-1}\chi(u)e\left(\frac{\eta u^4}{q}\right)\right|^2\left|\sum_{s=1}^{q-1}\chi(s)e\left(\frac{s^k+ms^2+ns}{q}\right)\right|^2$$
$$=\sum_{m=1}^{q}\sum_{n=1}^{q}\sum_{\substack{\chi(-1)=1\\ \chi\neq\chi_0}}\left(2q+i\left(\frac{\eta}{q}\right)\sqrt{q}\sum_{u=1}^{q-1}\chi(u)\left(\frac{u^4-1}{q}\right)\right)\left(\sum_{s=1}^{q-1}\chi(s)e\left(\frac{s^k+ms^2+ns}{q}\right)\right)^2$$
$$+\sum_{m=1}^{q}\sum_{n=1}^{q}\left|\sum_{u=1}^{q-1}e\left(\frac{\eta u^4}{q}\right)\right|^2\left|\sum_{s=1}^{q-1}\chi_0(s)e\left(\frac{s^k+ms^2+ns}{q}\right)\right|^2$$
$$=\sum_{\chi(-1)=1}2q\sum_{m=1}^{q}\sum_{n=1}^{q}\left(\sum_{s=1}^{q-1}\chi(s)e\left(\frac{s^k+ms^2+ns}{q}\right)\right)^2$$
$$+\sum_{m=1}^{q}\sum_{n=1}^{q}\sum_{\chi(-1)=1}\left(i\left(\frac{\eta}{p}\right)\sqrt{q}\sum_{u=1}^{q-1}\chi(u)\left(\frac{u^4-1}{q}\right)\right)\left(\sum_{s=1}^{q-1}\chi(s)e\left(\frac{s^k+ms^2+}{q}\right.\right.$$
$$+\sum_{m=1}^{q}\sum_{n=1}^{q}\left|\sum_{u=1}^{q-1}e\left(\frac{\eta u^4}{q}\right)\right|^2\left|\sum_{s=1}^{q-1}\chi_0(s)e\left(\frac{s^k+ms^2+ns}{q}\right)\right|^2$$
$$=2q\sum_{\chi(-1)=1}\sum_{m=1}^{q}\sum_{n=1}^{q}\left(\sum_{s=1}^{q-1}\chi(s)e\left(\frac{s^k+ms^2+ns}{q}\right)\right)^2$$
$$+i\left(\frac{\eta}{q}\right)\sqrt{q}\sum_{\chi(-1)=1}\sum_{m=1}^{q}\sum_{n=1}^{q}\left(\sum_{u=1}^{q-1}\chi(u)\left(\frac{u^4-1}{q}\right)\right)\left(\sum_{s=1}^{q-1}\chi(s)e\left(\frac{s^k+ms^2}{q}\right.\right.$$
$$+\left|\sum_{u=1}^{q-1}e\left(\frac{\eta u^4}{q}\right)\right|^2\sum_{m=1}^{q}\sum_{n=1}^{q}\left|\sum_{s=1}^{q-1}\chi_0(s)e\left(\frac{s^k+ms^2+ns}{q}\right)\right.$$

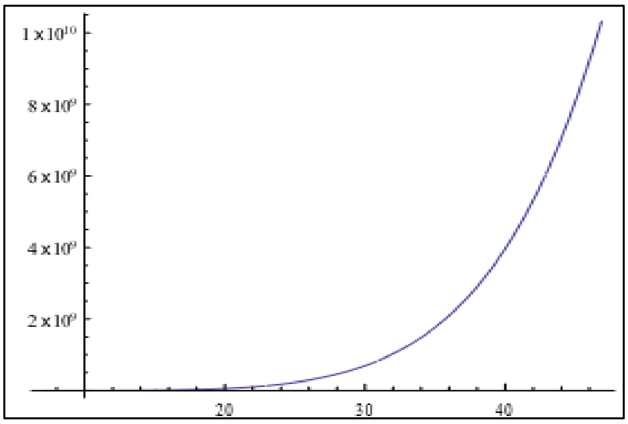
$$= 2q^3(q-1)^3/2 + \left|\sum_{u=0}^{q-1} e\left(\frac{\eta u^4}{q}\right) - 1\right|^2 q^2(q-1)^2$$
$$= q^3(q-1)^3 + (q+1)q^2(q-1)^2$$
$$= (q-1)^2q^2\big(q(q-1) + (q+1)\big)$$
$$= q^2(q-1)^2(q^2 - q + q + 1)$$
$$= q^2(q-1)^2(q^2+1).$$



[Fig.1: $q = 7, 11$]



[Fig.2: $q = 7, 11, 19, 23$]



[Fig.3: $q = 7, 11, 19, 23, 31, 43, 47$]

These plots represent the dynamics of the hybrid power involving the three-term exponential sum and the general quartic Gauss sum. The results indicate that the function is consistently positive, never zero, and exhibits an increasing trend for varying values of q. Additionally, across different segments of q, the functions' graph displays a clear and striking behaviour along the q—axis.

Theorem 3.2 Let q be an odd prime and χ be a Dirichlet character over mod q, then.

for any integer η with (η, q) = 1, we have

$$\sum_{\chi \bmod q}\sum_{m=1}^{q}\sum_{n=1}^{q}\left|G(\eta,4,\chi;q)\right|^2\left|\sum_{s=1}^{q-1}\chi(s)e\left(\frac{s^k+ms^2+ns}{q}\right)\right|^2 = q^3(q-1)^3.$$

Proof. Since χ is an even character over *mod q,* then χ *is* also an even character over *mod q.*

$$\sum_{\substack{\chi(-1)=1\\ \chi\neq\chi_0}}\sum_{m=1}^{q}\sum_{n=1}^{q}\left|G(\eta,4,\chi;q)\right|^2\left|\sum_{s=1}^{q-1}\chi(s)e\left(\frac{s^k+ms^2+ns}{q}\right)\right|^2$$

$$= \sum_{\substack{\chi(-1)=1\\ \chi\neq\chi_0}}\left|G(\eta,4,\overline{\chi};q)\right|^2\sum_{m=1}^{q}\sum_{n=1}^{q}\left|\sum_{a=1}^{q-1}\overline{\chi}(s)e\left(\frac{s^k+ms^2+ns}{q}\right)\right|^2$$

$$= \sum_{\substack{\chi(-1)=1\\ \chi\neq\chi_0}}\left|G(\eta,4,\overline{\chi};q)\right|^2\sum_{m=1}^{q}\sum_{n=1}^{q}\left|\sum_{s=1}^{q-1}\chi(s)e\left(\frac{s^k+ms^2+ns}{q}\right)\right|^2$$

$$= \frac{1}{2}\sum_{\substack{\chi(-1)=1\\ \chi\neq\chi_0}}\left(\left|G(\eta,4,\overline{\chi};q)\right|^2 + \left|G(\eta,4,\chi;q)\right|^2\right)\times$$

$$\sum_{m=1}^{q}\sum_{n=1}^{q}\left|\sum_{s=1}^{q-1}\chi(s)e\left(\frac{s^k+ms^2+ns}{q}\right)\right|^2$$

$$= \frac{1}{2}4q\sum_{\substack{\chi(-1)=1\\ \chi\neq\chi_0}}\sum_{m=1}^{q}\sum_{n=1}^{q}\left|\sum_{s=1}^{q-1}\chi(s)e\left(\frac{s^k+ms^2+ns}{q}\right)\right|^2$$

$$= 2q\left(q^2(q-1)^3/2\right) = q^3(q-1)^3.$$

IV. WEIGHT DISTRIBUTION OF CODEWORD C(Q)

Let p be prime and $q = p^r$ for $r \geq 2$. We denote the finite field with q elements by F_q. Any $m, s, n \in F_q,$ the three-term of the exponential sum

$$C(m,s,n;q) = \sum_{a\in F_q^*} e(Tr(m,s,a))$$

where $e(g) = e^{\frac{2\pi i g}{p}}$ and $Tr(m,s,a) = tr(n$

The code $c(q) \subseteq F_p^{q-1}$ is defined as the

$\varphi(m,s) = [tr(m$

Thus, $c(q)$ is a code of length $q-1$.

Theorem 4.1 *For codeword* ϕ *(m, s)* $\in$ *c(q), the* ϕ *(m, s) is given,*

$$w(\varphi(m,s)) = \frac{1}{p}[(q-1) - \sum_{\delta=1}^{p-1} C(\delta m, \delta s, q)]$$

Proof. For all $m, s \in F_q$ and all $a \in F^*$, $e\,(Tr\,(m, s, a))$ is a p-th root of unity and equal to 1 or not according to $Tr\,(m, s, a) = 0$

$$\sum_{\delta=0}^{p-1} e(\delta Tr(m,s,a)) = \begin{cases} p & \text{if } Tr(m,s,a)=0; \\ 0 & \text{if } Tr(m,s,a)\neq 0. \end{cases}$$

Therefore,

$$\begin{aligned} \{a \in F_q^*; Tr(m,s,a)=0\} &= \frac{1}{p}\sum_{a\in F_q^*}\sum_{\delta=0}^{p-1} e(\delta Tr(m,s,a)), \\ &= \frac{1}{p}\sum_{\delta=0}^{p-1} C(\delta m, \delta n, q). \end{aligned}$$

Since $w\,(\phi\,(m, s))$ is the number of non-zero entries of the vector $\phi\,(m, s)$ we have,

$$\begin{aligned} w(\varphi(m,s)) &= (q-1) - \frac{1}{p}\sum_{\delta=0}^{p-1} C(\delta m, \delta s, q) \\ &= (q-1) - \frac{1}{p}(q-1)\sum_{\delta=1}^{p-1} C(\delta m, \delta s, q) \\ &= \frac{1}{p}[p(q-1) - (q-1)\sum_{\delta=1}^{p-1} C(\delta m, \delta s, q)] \\ &= \frac{1}{p}(q-1)[p - \sum_{\delta=1}^{p-1} C(\delta m, \delta s, q)]. \end{aligned}$$

V. CONCLUSION

In this paper, we evaluated the hybrid power mean of a three-term exponential sum with weights given by general quartic Gauss sums. Through the use of Dirichlet characters and classical Gauss sum properties, we obtained new identities and explicit evaluations. We also computed the associated weight function, demonstrating its relevance in coding theory, particularly in analyzing code structures.

ACKNOWLEDGMENT

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DECLARATION STATEMENT

Some of the references cited are older, noted explicitly as [1], [3], [4], and [6]. However, these works remain significant for the current study, as they are pioneering in their fields.

After aggregating input from all authors, I must verify the accuracy of the following information as the article's author.

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- **Author's Contributions:** The authorship of this article is contributed equally to all participating individuals.

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