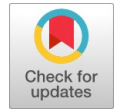


# Complex Dynamics of an Aquatic Tri-Trophic Food Chain System with Holling Type IV Functional Response



Partha Ghosh

**Abstract:** Complex dynamics of a modified Hastings-Powell (HP) model (phytoplankton-zooplankton-fish) with Holling type IV functional response are investigated in this article. Boundedness of the system has been established. A detailed study of the boundary equilibrium points and their local stability has been carried out. The condition of uniform persistence of the system has also been derived. Dynamical complexities and subsequent changes in the states of the system have been portrayed using numerical simulation. Modified HP model with Holling type IV functional response gives rise to a similar type of chaotic dynamics (inverted 'teacup attractor') as observed in the original HP model with Holling type II functional response. Chaotic or stable dynamics are also numerically verified using Sil'nikov eigenvalue analysis.

**Keywords:** Holling type IV, Boundedness, Uniform persistence, Chaos.

**Mathematics Subject Classification 2010:** 92 XX

**Nomenclature:**

HP: Hastings-Powell

## I. INTRODUCTION

Tri-trophic food chain systems with Holling-type functional response terms (type II and type III) have been investigated extensively over the years. Models with Holling type II functional response give rise to chaotic fluctuation in the system. Hastings and Powell [1] introduced a tri-trophic food chain model with Holling type II functional response term, which is widely accepted for explaining the chaotic movement of aquatic systems. There are many subsequent investigations of the type II system. Stollenwerk et. al. [2] analysed the Holling type II system in both deterministic and stochastic environments. Myint and Wang [3] studied the Holling type II system with prey protection zone. In their pioneering work, Freedman and Waltman [4] derived the condition of persistence for these models. Holling type III systems are generally stable in nature. Francis et.al. [5] carried out a detailed stability analysis of a four-species Holling type III system. One can find many articles analysing different aspects of these systems.

Both Holling type II and Holling type III functional responses are monotonic in nature. However, many experimental and observational evidence found in natural ecosystems indicates the existence of a non-monotonic type response in community interactions. Also, dynamical complexities like bifurcation, chaos, etc., observed in these types of ecological systems can be explained more accurately with a non-monotonic type functional response than with traditional monotonic response terms.

Andrews [6] first suggested a non-monotonic "Holling type IV" or "Monod-Haldane" functional response term as,

$$f(x) = \frac{px}{a + bx + cx^2}$$

A simplified "Monod-Haldane" function is of the following form,

$$f(x) = \frac{px}{a + x^2}$$

Examples of non-monotonic type interactions can be found in many natural ecosystems, such as 'inhibition' in microbial dynamics, 'group defence' in population dynamics, and others. If the concentration of the nutrient reaches a certain high level, it may induce an inhibitory effect on the specific growth rate of the species. For example, one may investigate the process of waste decomposition or water purification by microorganisms. 'Monod-Haldane' functional response is like the Michaelis-Menten type function for low concentrations, but includes the inhibitory effect at high concentrations. Sarwardi et.al. [7] studied the effects of gestation delay and predator harvesting in the model system using non-monotonic functional response terms. Ghosh et.al. [8] investigated the bifurcation behaviour and stability of a three-species system with a non-monotonic functional response.

Sokol and Howell proposed a simplified 'Monod-Haldane' function [9] during their experiments on the uptake of phenol by pure culture of *Pseudomonas putida* growing on phenol in continuous culture. They observed that the Holling type IV functional response fit their experimental data significantly better than other forms of functional response.

In population dynamics, group defence [10] describes the phenomenon of decrease or prevention of predation due to the increased ability of the prey to defend better or disguise themselves when their numbers are large enough. Wolves can regularly attack a lone ox successfully. Small herds of Musk ox grazing together are attacked by wolves, but the success rate of the attacks is very low. On the other hand, successful attacks on larger herds are generally not observed.

Daphnia can consume 'Filamentous algae' at low concentrations; However, they can survive the attacks at high concentration by

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jamming their filtering apparatus [11]. In general, it can be said that when the per capita rate of predation decreases at sufficiently high prey densities, the Holling type IV functional response can be used with good effect. Examples of Holling type IV functional response can also be found in many other aspects of ecological study, as follows. Liu and Huang [12] analyzed a Holling type IV system with optimal harvesting and proved that maximum sustainable total yield and optimal economic profit do not exist with independent harvesting mode. Chai et al. [13] studied a stochastic Holling type IV system and provided sufficient conditions for persistence and extinction of populations. Alessandro et.al. [14] investigated a predator-prey system with generalised Holling type IV functional response and studied the interplay between the functional response and both strong and weak 'Allee effects' in the prey population.

In the present article, a deterministic tri-trophic food chain system with a simplified 'Holling type IV' functional response term has been investigated. The present model may be considered to represent an aquatic plankton food chain system comprising phytoplankton (*Coscinodiscus*), zooplankton (*Daphnia*), and fish (salmon) populations. The aim is to study some critical properties like stability, boundedness, permanence, etc. Stability, especially global stability of the system indicates that the system is ecologically well organized and robust in nature. Boundedness of the system implies that the system is well-behaved and the population biomass does not explode at any future time. Geometrically speaking, the solutions will always remain in the positive quadrant.

From an ecological point of view, uniform persistence or permanence is a fundamental property of a model system, and several researchers have studied it [4]. Uniform persistence implies the survival of all populations in the system. In the case of uniformly persistent systems, strictly positive solutions are eventually uniformly bounded away from the boundary. The weak persistence property of a system only guarantees that the extinction of a particular species is not inevitable. But it may still happen that solutions approach arbitrarily close to the boundary for an extended period. In that case, any significant perturbation may drive the solution to the boundary, resulting in the extinction of the species. The idea of uniform persistence or permanence avoids this problem, and it is generally regarded as a more robust concept than weak persistence.

One of the most crucial aspects of the dynamical complexities of a three-species system is the occurrence of chaotic oscillation. After the pioneering work of Hastings and Powell (HP) [1], chaos in model system has been studied by many researchers. The systems which are highly sensitive to initial conditions are prone to be chaotic. For these systems, a minute change in the initial conditions may cause a significantly diverging outcome. This behaviour of the system is defined as deterministic chaos or simply chaos. Chaotic systems do not have any definite pattern in their trajectories; instead, they always exhibit erratic oscillation. The chaotic behaviour of the present system has been investigated in detail through numerical simulation.

The paper is organised as follows: In section II, a brief description of the model's formulation is provided. Boundedness of the system has been established in Section

III. A detailed study of boundary equilibrium points and their local stability has been carried out in Section IV. The condition of uniform persistence of the system has been derived in Section V. Existence and uniqueness of the interior equilibrium point  $E^*(x^*, y^*, z^*)$  have been investigated in section VI. A detailed numerical simulation is carried out in Section VII. A general discussion about the whole analysis is given in the concluding remarks in Section VIII.

## II. THE MATHEMATICAL MODEL

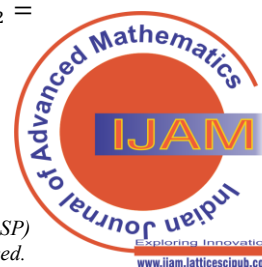
The present model system (modified Hastings-Powell model [1] with Holling type IV functional response) describes a tri-trophic food chain model composed of a prey, a predator and a super predator whose population densities are denoted by  $X, Y$  and  $Z$  respectively. Behaviours of the entire community are assumed to arise from the coupling of these interacting species, where  $Z$  preys on  $Y$  only and  $Y$  preys on  $X$ . The 'Holling type IV' or 'Monod-Haldane' functional response has been chosen as the predator response function for the species ( $X, Y$ ) and also for the species ( $Y, Z$ ). With these assumptions, the following tri-trophic food chain model has been introduced, subject to the initial conditions  $X, Y, Z > 0$ .

$$\begin{aligned}\frac{dX}{dt} &= RX\left(1 - \frac{X}{K}\right) - \frac{C_1 A_1 XY}{B_1 + X^2} \\ \frac{dY}{dt} &= \frac{A_1 XY}{B_1 + X^2} - \frac{A_2 YZ}{B_2 + Y^2} - D_1 Y \\ \frac{dZ}{dt} &= \frac{C_2 A_2 YZ}{B_2 + Y^2} - D_2 Z\end{aligned} \quad \dots (1)$$

where  $R$  is the intrinsic growth rate of the prey population and  $K$  is its environmental carrying capacity.  $A_1$  and  $A_2$  are the maximal growth rates of the predator and the super predator, respectively.  $B_1$  and  $B_2$  are the half-saturation constants,  $C_1^{-1}$  and  $C_2$  are conversion rates for prey to predator for the species  $Y$  and  $Z$  respectively.  $D_1$  and  $D_2$  These are the death rates of predators and super predators, respectively. All the parameters are assumed to be positive. To reduce the number of parameters and determine which combination controls the system's behaviour, it has been nondimensionalized in the following way. Let  $x = \frac{X}{K}, y = \frac{C_1 Y}{K}, z = \frac{C_2 Z}{C_2 K}$  and  $t = RT$ . Then system (1) takes the form (after simplification)

$$\begin{aligned}\frac{dx}{dt} &= x(1 - x) - \frac{a_1 xy}{1 + b_1 x^2} = xF(x, y) \\ \frac{dy}{dt} &= \frac{a_1 xy}{1 + b_1 x^2} - \frac{a_2 yz}{1 + b_2 y^2} - d_1 y = yG(x, y, z) \\ \frac{dz}{dt} &= \frac{a_2 yz}{1 + b_2 y^2} - d_2 z = zH(x, y, z) \\ x > 0, y > 0, z > 0\end{aligned} \quad \dots (2)$$

where  $a_1 = \frac{A_1 K}{R B_1}, b_1 = \frac{K^2}{B_1}, a_2 = \frac{C_2 A_2 K}{C_1 B_2 R}, b_2 = \frac{K^2}{B_2 C_1^2}, d_1 = \frac{D_1}{R}$  and  $d_2 = \frac{D_2}{R}$ .



### III. BOUNDEDNESS OF THE SYSTEM

Boundedness of a system implies that the system is biologically well organized. In the following theorem, the boundedness of the system (2) is established.

**Theorem 3.1** All solutions of the system (2) which start in  $\mathbb{R}_+^3$  are uniformly bounded.

Proof. Let  $(x(t), y(t), z(t))$  be any solution of the system with favourable initial conditions.

Since  $\frac{dx}{dt} \leq x(1-x)$ , we have  $\limsup x(t) \leq 1$ .

Let  $W = x + y + z$ .

$$\begin{aligned} \text{Then } \frac{dW}{dt} &\leq x - d_1 y - d_2 z \\ &\leq 2 - \delta W, \text{ where } \delta = \min\{1, d_1, d_2\}. \end{aligned}$$

$$\text{Therefore, } \frac{dW}{dt} + \delta W \leq 2.$$

So following Birkoff and Rota [15] we obtain,

$$0 \leq W(x, y, z) \leq \frac{2}{\delta} + \frac{W(x(0), y(0), z(0))}{e^{\delta t}}$$

$$\text{For } t \rightarrow \infty, \text{ we have } 0 \leq W \leq \frac{2}{\delta}$$

Thus, all solutions of the system (2) remain in the following region.

$$\hat{B} = \left\{ (x, y, z) : 0 \leq W \leq \frac{2}{\delta} + \epsilon, \text{ for any } \epsilon > 0 \right\}$$

This proves the theorem.

### IV. BOUNDARY EQUILIBRIUM POINTS: EXISTENCE AND STABILITY

The system (2) always has two boundary equilibrium points, namely  $E_0(0,0,0)$  and  $E_1(1,0,0)$ . The planner equilibrium points are given by  $E_2(\hat{x}, \hat{y}, 0)$  and  $E_3(\hat{x}', \hat{y}', 0)$ .

#### A. Equilibrium Points $E_0(0, 0, 0)$ and $E_1(1, 0, 0)$

The variational matrices  $V(E_0)$  and  $V(E_1)$  are respectively given by,

$$V(E_0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -d_1 & 0 \\ 0 & 0 & -d_2 \end{bmatrix},$$

$$V(E_1) = \begin{bmatrix} -1 & -\frac{a_1}{1+b_1} & 0 \\ 0 & \frac{a_1}{1+b_1} - d_1 & 0 \\ 0 & 0 & -d_2 \end{bmatrix}$$

Two of the eigenvalues of  $V(E_0)$  are negative, and one is positive, so the point at the origin has both stable and unstable manifolds. For  $V(E_1)$ , whenever  $\frac{a_1}{1+b_1} > d_1$ , two of the eigenvalues are negative and one is positive, and as before, both stable and unstable manifolds exist for  $E_1(1,0,0)$ . Consequently both  $E_0$  and  $E_1$  become saddle points.

#### B. Equilibrium Points $E_2(\hat{x}, \hat{y}, 0)$ and $E_3(\hat{x}', \hat{y}', 0)$

Since  $X \leq K$  (where  $K$  is the environmental carrying capacity of prey), it follows at once from the relation  $x = \frac{X}{K}$  that  $x \leq 1$ . We can see that  $x$  is given by the roots of the equation

$$b_1 d_1 x^2 - a_1 x - d_1 = 0 \text{ and } y \text{ is given by, } y = \frac{x(1-x)}{d_1}$$

From the expression of  $y$  we can say that the existence of  $x$  automatically implies the existence of  $y$ . The equilibrium points  $E_2(\hat{x}, \hat{y}, 0)$  and  $E_3(\hat{x}', \hat{y}', 0)$  exist if and only if  $a_1^2 > 4b_1 d_1^2$  and is given by,

$$\hat{x} = \frac{a_1 + (a_1^2 - 4b_1 d_1^2)^{\frac{1}{2}}}{2b_1 d_1}$$

$$\hat{y} = \frac{1}{d_1} \left( \frac{a_1 + (a_1^2 - 4b_1 d_1^2)^{\frac{1}{2}}}{2b_1 d_1} \right) \left( 1 - \frac{a_1 + (a_1^2 - 4b_1 d_1^2)^{\frac{1}{2}}}{2b_1 d_1} \right)$$

$$\text{and } \hat{x}' = \frac{a_1 - (a_1^2 - 4b_1 d_1^2)^{\frac{1}{2}}}{2b_1 d_1}$$

$$\hat{y}' = \frac{1}{d_1} \left( \frac{a_1 - (a_1^2 - 4b_1 d_1^2)^{\frac{1}{2}}}{2b_1 d_1} \right) \left( 1 - \frac{a_1 - (a_1^2 - 4b_1 d_1^2)^{\frac{1}{2}}}{2b_1 d_1} \right)$$

So, we have the following result,

**Theorem 4.1** Equilibrium points  $E_2(\hat{x}, \hat{y}, 0)$  and  $E_3(\hat{x}', \hat{y}', 0)$  exist if and only if  $x < 1$  and  $a_1^2 > 4b_1 d_1^2$ .

The variational matrix  $V(E_2)$  at  $E_2(\hat{x}, \hat{y}, 0)$  is given by,

$$V(E_2) = \begin{bmatrix} v_{11} & v_{12} & 0 \\ v_{21} & 0 & v_{23} \\ 0 & 0 & v_{33} \end{bmatrix}$$

$$\text{Where } v_{11} = 1 - 2\hat{x} - \frac{a_1 \hat{y}}{1 + b_1 \hat{x}^2} + \frac{2a_1 b_1 \hat{x}^2 \hat{y}}{(1 + b_1 \hat{x}^2)^2}, v_{12} = -\frac{a_1 \hat{x}}{1 + b_1 \hat{x}^2}$$

$$\begin{aligned} v_{21} &= \frac{a_1 \hat{y}}{1 + b_1 \hat{x}^2} - \frac{2a_1 b_1 \hat{x}^2 \hat{y}}{(1 + b_1 \hat{x}^2)^2}, v_{23} = -\frac{a_2 \hat{y}}{1 + b_2 \hat{y}^2}, v_{33} \\ &= \frac{a_2 \hat{y}}{1 + b_2 \hat{y}^2} - d_2 \end{aligned}$$

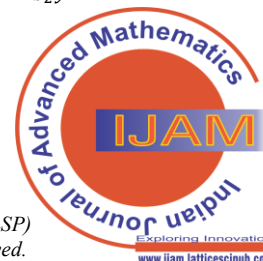
The characteristic equation of  $V(E_2)$  is given by,  $(\lambda^2 + B\lambda + C) \left( \lambda + d_2 - \frac{a_2 \hat{y}}{1 + b_2 \hat{y}^2} \right) = 0$ .

$$\text{Where } B = -v_{11} \text{ and } C = -v_{12} v_{21}.$$

The eigenvalues are given by,  $\lambda_{1,2} = \frac{-B \pm (B^2 - 4C)^{\frac{1}{2}}}{2}$ ,  $\lambda_3 = -d_2 + \frac{a_2 \hat{y}}{1 + b_2 \hat{y}^2}$ .

After a bit of calculation, it is found that  $C = -v_{12} v_{21}$  is always negative. In contrast, cap  $B$  is positive (or, negative) whenever  $\frac{4b_1 d_1^2}{8b_1 d_1 - 3a_1} < \frac{a_1}{1+b_1}$  (or,  $\frac{4b_1 d_1^2}{8b_1 d_1 - 3a_1} > \frac{a_1}{1+b_1}$ ). Since  $C$  is negative, the roots are always real but their sign will depend on the sign of  $B$ . If  $B > 0$  (i.e.  $\frac{4b_1 d_1^2}{8b_1 d_1 - 3a_1} < \frac{a_1}{1+b_1}$ ), then  $\lambda_1 > 0$  and  $\lambda_2 < 0$ , which implies  $E_2$  becomes a saddle point in  $xy$  plane. If  $B < 0$  (i.e.  $\frac{4b_1 d_1^2}{8b_1 d_1 - 3a_1} > \frac{a_1}{1+b_1}$ ), then again  $\lambda_1 > 0$  and  $\lambda_2 < 0$ , which implies  $E_2$  becomes a saddle point in  $xy$  plane.  $E_2$  becomes locally asymptotically stable in  $z$  direction if and only if  $-d_2 + \frac{a_2 \hat{y}}{1 + b_2 \hat{y}^2} < 0$ . Therefore, we get the following theorem.

**Theorem 4.2**  $E_2$  is a saddle point in the  $xy$  plane.  $E_2$  becomes locally



asymptotically stable (or unstable) in  $z$  direction if and only if  $-d_2 + \frac{a_2 \hat{y}}{1+b_2 \hat{y}^2} < ( \text{ or } > ) 0$ .

Proceeding similarly to above, we can obtain the following result for  $E_3(\hat{x}', \hat{y}', 0)$ .

**Theorem 4.3** (a)  $E_3$  is locally asymptotically stable in the  $xy$  plane if and only if  $\frac{4b_1 d_1^2}{8b_1 d_1 - 3a_1} > \frac{a_1}{1+b_1}$  and unstable if  $\frac{4b_1 d_1^2}{8b_1 d_1 - 3a_1} < \frac{a_1}{1+b_1}$ .

(b)  $E_3$  is locally asymptotically stable (unstable) in the  $z$  direction if and only if  $-d_2 + \frac{a_2 \hat{y}'}{1+b_2 \hat{y}'^2} < ( \text{ or } > ) 0$ .

## V. UNIFORM PERSISTENCE

Uniform persistence or permanence is a stronger concept than weak persistence or persistence. Permanence of the present system is established in the following theorem: whenever there exists a finite number of limit cycles in  $xy$  Plane.

**Theorem 5.1** Let  $\frac{a_1}{1+b_1} > d_1, \frac{4b_1 d_1^2}{8b_1 d_1 - 3a_1} > \frac{a_1}{1+b_1}$  and there exists a finite number (say  $n$ ) of periodic solutions  $x = \phi_i(t), y = \psi_i(t) (i = 1, 2, \dots, n)$  in the  $xy$  plane. Then the system is uniformly persistent, provided for each periodic solution of period  $\hat{T}$ ,

$$-d_2 + \frac{1}{\hat{T}} \int_0^{\hat{T}} \frac{a_2 \psi_i(t)}{1+b_2 \psi_i^2(t)} dt > 0, (i = 1, 2, \dots, n)$$

Proof. Let  $\theta(\hat{X})$  be the orbit through the point  $\hat{X} = (x, y, z)$  and  $\Omega(\hat{X})$  be the omega limit set of the orbit through  $\hat{X}$ . We note that  $\Omega(\hat{X})$  is bounded.

First, we shall show that  $E_0$  does not belong to  $\Omega(\hat{X})$ . If  $E_0 \in \Omega(\hat{X})$ , by the Butler-McGehee lemma [4], there exists a point  $P$  in  $\Omega(\hat{X}) \cap W^s(E_0)$ . Where  $W^s(E_0)$  the stable manifold of  $E_0$ .

Since  $\theta(p)$  lies in  $\Omega(\hat{X})$  and  $W^s(E_0)$  is the  $yz$  plane, we found that  $\theta(p)$  is unbounded, which is a contradiction.

Next  $E_1$  does not belong to  $\Omega(\hat{X})$ , for then since  $E_1$  is a saddle point from the condition  $\frac{a_1}{1+b_1} > d_1$ , by the Butler-McGehee lemma [4], there exists a point  $P$  in  $\Omega(\hat{X}) \cap W^s(E_1)$ .  $W^s(E_1)$  is the  $x$  axis implies that an unbounded orbit lies  $\Omega(\hat{X})$ , which is a contradiction to the boundedness of the system.

Since  $E_2$  is always a saddle point, proceeding as above, we conclude that  $E_2$  does not belong to  $\Omega(\hat{X})$ .

Now we only need to consider the equilibrium point  $E_3$ . We shall show that no periodic orbit in the  $xy$  plane or  $E_3$  belongs to  $\Omega(\hat{X})$ . According to condition (b) of the theorem, eigenvalues  $\lambda'_1, \lambda'_2$  of  $V(E_3)$  have a negative real part. Let  $\gamma_i (i = 1, 2, \dots, n)$  denotes the closed orbit of the periodic solution  $(\phi_i(t), \psi_i(t))$  in  $xy$  plane such that  $\gamma_i$  lies inside  $\gamma_{i-1}$ . The variational matrix  $V_i(\phi_i(t), \psi_i(t), 0)$  corresponding to  $\gamma_i$  is given by,

$$V(E_3) = \begin{bmatrix} F(\phi_i(t), \psi_i(t)) & \phi_i(t)F_y(\phi_i(t), \psi_i(t)) & 0 \\ \psi_i(t)G_x(\phi_i(t), 0) & G(\phi_i(t), 0) & \psi_i(t)G_z(\phi_i(t), 0) \\ 0 & 0 & H(\psi_i(t)) \end{bmatrix}$$

where,  $H(\psi_i(t)) = -d_2 + \frac{a_2 \psi_i(t)}{1+b_2 \psi_i^2(t)}$

Computing the fundamental matrix of the linear periodic system  $M' = V_i(t)M, M(0) = I$  we find that its Floquet multiplier in the  $z$  direction is  $e_i^\eta(t)$ . Then, following the approach of Kumar and Freedman [16], we conclude that no  $\gamma_i$  lies in  $\Omega(\hat{X})$ . Thus  $\Omega(\hat{X})$  lies in the positive octant, and the system is persistent.

Finally, since only the closed orbits and the equilibria form the omega limit set of the solutions on the boundary of  $\mathbb{R}_+^3$ , and the system (2) is dissipative, then by the main theorem in [17], the system (2) is uniformly persistent.

## VI. THE INTERIOR EQUILIBRIUM POINT: EXISTENCE AND UNIQUENESS

In this section, the conditions for the existence and uniqueness of the interior equilibrium point  $(x^*, y^*, z^*)$  of the system (2) has been derived. We can easily see that there exist two values of  $y^*$ .

$$y_1^* = \frac{a_2 + (a_2^2 - 4b_2 d_2^2)^{\frac{1}{2}}}{2b_2 d_2}, y_2^* = \frac{a_2 - (a_2^2 - 4b_2 d_2^2)^{\frac{1}{2}}}{2b_2 d_2}$$

If  $a_2^2 = 4b_2 d_2^2$  then the two values are equal and given by  $y^* = \frac{a_2}{2b_2 d_2}$ .

$x^*$  is the positive solution of the equation  $x^3 - x^2 + \frac{1}{b_1}x + \frac{a_1 a_2 - 2b_2 d_2}{2b_1 b_2 d_2} = 0$  and  $z^*$  is given by,  $z^* = \frac{x^*(1-x^*) - d_1 y^*}{d_2}$

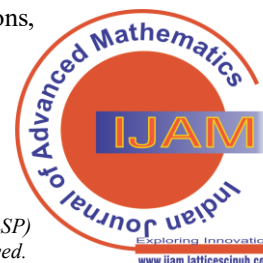
We note that  $x^*$  is less than 1, and by Descartes' rule of signs, there always exists at least one positive root of the equation. Also by the relation between roots and coefficients of cubic equation we find that if  $\left(\frac{a_1 a_2 - 2b_2 d_2}{2b_1 b_2 d_2} + \frac{1}{3b_1} - \frac{2}{27}\right)^2 + 4\left(\frac{27b_1 - 1}{27b_1^2}\right)^3 > 0$  then two of the roots of the equation become imaginary and consequently unique positive root exists for the equation. So, we have the following theorem.

**Theorem 6.1** The unique positive equilibrium of the system (2) exists if and only if  $a_2^2 = 4b_2 d_2^2$  and  $\left(\frac{a_1 a_2 - 2b_2 d_2}{2b_1 b_2 d_2} + \frac{1}{3b_1} - \frac{2}{27}\right)^2 + 4\left(\frac{27b_1 - 1}{27b_1^2}\right)^3 > 0$ .

## VII. NUMERICAL SIMULATION

### A. Transition from Stable to Chaotic Dynamics

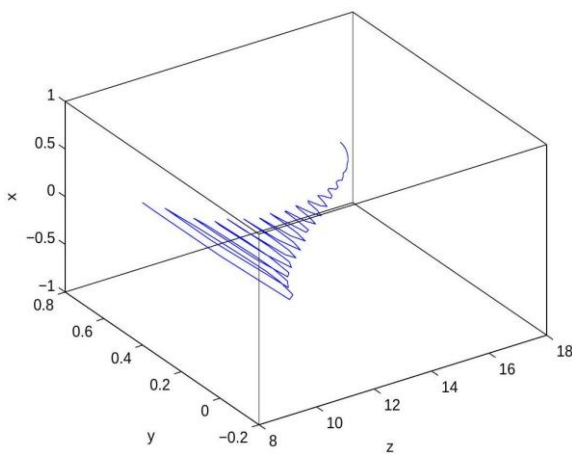
The present system has been studied by varying the parameter  $b_1$  (half saturation constant) gradually keeping the values of the other five parameters fixed as taken in the original HP model [1].  $a_1 = 5, a_2 = 0.1, b_2 = 2, d_1 = 0.4, d_2 = .01$ . The initial values are taken as  $x(0) = 0.75, y(0) = 0.15, z(0) = 9.75$ . For these values of parameters and initial conditions, the interior equilibrium point  $E^*(x^*, y^*, z^*)$  is given by,



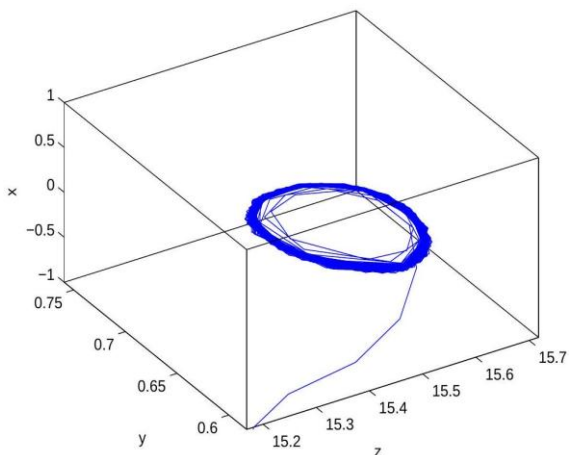
$x^* = 0.782544, y^* = 0.10208, z^* = 12.93363$ .

It has been observed that for  $b_1 \in (0, 1.5)$ , the system (2) exhibits stable behaviour (see Fig. 1(a)). Now, as the value of  $b_1$  is gradually increased, the system transits through different phases and eventually turns chaotic. It has also been observed that dynamics is susceptible to small changes in system parameters, and system dynamics can change significantly within a very short interval. Limit cycle oscillation can be observed in the system for  $b_1 \in (1.58, 1.7)$  (Fig. 1(b)). Further period doubling occurs (Fig. 1(c)) for  $b_1 = 1.78$ . Finally, chaotic oscillations appear in the system along with the evolution of the famous inverted teacup attractor for  $b_1 \in (1.83, 3.2)$ . Onset of chaotic fluctuation can be observed in Fig. 1(d).

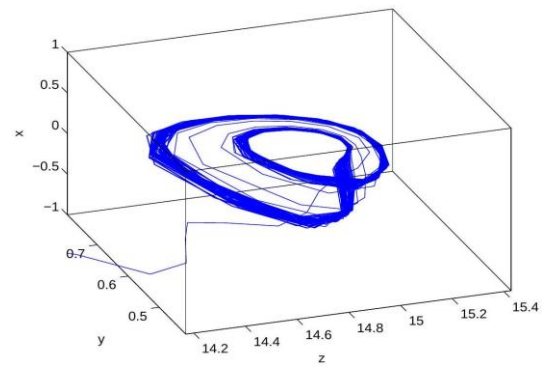
As observed in the original HP model, trajectories start in the "handle" of the "teacup," move to the wide part, spiral along the teacup to the narrow end, enter the handle again, and continue with the same pattern. Also, all the trajectories in the "handle" of the "teacup" are very close together, which contributes to the sensitive dependence of future dynamics on the current state. Fig. 2(a) shows the chaotic time series for  $b_1 = 2.2$  and Fig. 2(b) is the corresponding chaotic attractor (inverted teacup). When the value of  $b_1$  is increased further, the shape of the chaotic attractor changes, but the dynamics of the system remain chaotic. Also, no species extinction is observed throughout the interval.



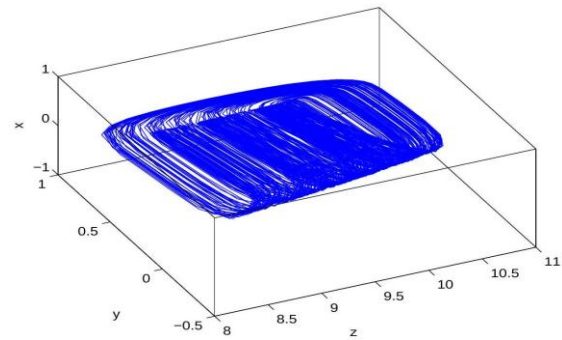
[Fig.1(a) Stable Phase Portrait for  $b_1 = 1.5$ ]



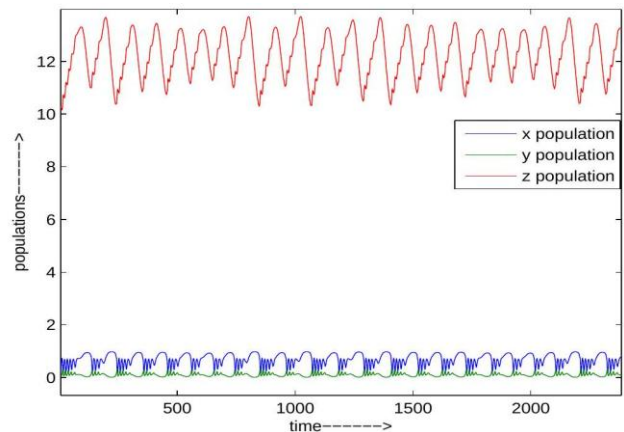
[Fig.1(b) Limit Cycle for  $b_1 = 1.68$ ]



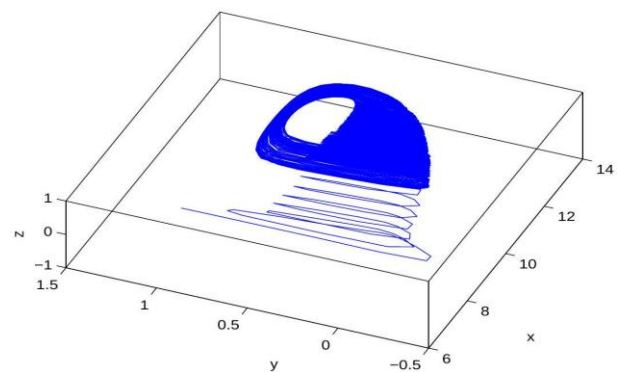
[Fig.1(c) Period Doubling for  $b_1 = 1.78$ ]



[Fig.1(d) Onset of Chaotic Fluctuation for  $b_1 = 2$ ]



[Fig.2(a) Chaotic Time Series for  $b_1 = 2.2$ ]



[Fig.2(b) Chaotic Attractor for  $b_1 = 2.2$ ]

## B. Sil'nikov Chaos

If the equilibrium point of the system is a saddle focus, and the eigenvalues  $\bar{\gamma}$  and  $\bar{\alpha} \pm i\bar{\beta}$  of the corresponding Jacobian matrix satisfies the following Sil'nikov inequality [18]

$$\bar{\beta} \neq 0, \bar{\gamma}\bar{\alpha} < 0, \text{ and } |\bar{\gamma}| > |\bar{\alpha}| \geq 0$$

then the system may have Sil'nikov chaos in some neighbourhood of the equilibrium point.

To establish the existence of chaotic oscillation in the system according to Sil'nikov criteria, the eigenvalues have been calculated for the specified set of system parameter values using MATD. It has been observed that Sil'nikov conditions are not satisfied in the region  $b_1 \in (0, 1.65)$ , but they are satisfied with  $b_1 \geq 1.65$ . For example, when  $b_1 = 1.2$  (in the stable region) the eigenvalues are  $-0.0385$  and  $-0.1623 \pm i0.4075$ . But for  $b_1 = 2.2$  (in the chaotic region) the eigenvalues are  $-0.6114$  and  $0.0651 \pm i0.797$ . There are several articles containing the analysis in this regard [19].

## VIII. CONCLUSION

Complex dynamics of a deterministic three-species food chain model with a simplified Holling type IV functional response is investigated in this article. As mentioned in the beginning, the present model system may be considered to represent an aquatic plankton food chain system comprising phytoplankton (*Coscinodiscus*), zooplankton (*Daphnia*), and fish (salmon) populations. More specifically speaking, it may be envisaged as the aquaculture of salmonids (salmon and steelhead). Aquaculture is the process of farming and harvesting salmonids under controlled conditions for both commercial and recreational purposes. The Atlantic salmon, Chinook salmon, rainbow trout and brown trout are the most commonly farmed salmonids for recreational and subsistence fishing in America and Europe. Stability of these types of aquacultures can be affected by various factors like 'uncontrolled harvesting', 'poaching', 'uncontrolled recreational and subsistence fishing' or 'escape into wild habitats during storms', etc. These external factors may drive the stable system to a chaotic one.

In the present investigation, it has been found that the system is uniformly bounded and permanent. Existence of equilibrium points and their stability conditions have been derived. Interestingly, it has also been observed that the system exhibits chaotic oscillation within a specific range of system parameters. It is observed that the system transits through different states, from Stable to limit cycle behaviour, then to period doubling, and finally to the chaotic attractor. The existence of chaos in the system has also been established through Sil'nikov eigenvalue analysis. Although chaotic fluctuation is observed in the present system, it is interesting to note that no population extinction occurs. So, uniform persistence of the system is validated by the numerical findings.

It should be noted that although chaos is predicted in many model systems, its example in the natural world, particularly in terrestrial ecosystems, is highly elusive. Also, uniformly persistent systems are not, in general, chaotic in nature. But this is not the case for aquatic systems, as they are more prone to chaos. The results obtained here through analytical and numerical simulation can be used as a tool to determine the tipping point between the persistence and extinction of a population. As a matter of fact, many ecological species are in a threatened condition and may be stabilised by suitable management. Further investigation is necessary for proper monitoring of the ecological system.

## DECLARATION STATEMENT

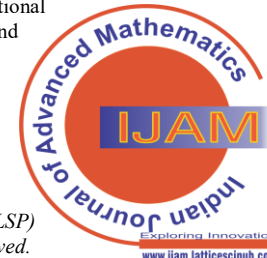
Some of the references used in the present article are pretty old. For example, Reference no. [1], [4], [6] [9], [16], [17] and [18]. But readers will find that these articles are extremely important for the results developed in the present study. Most of them are actually pioneering works in their respective fields. They have also been cited extensively by researchers in several articles within the fields of mathematical biology and ecology.

I must verify the accuracy of the following information as the article's author.

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- **Funding Support:** This article has not been funded by any organizations or agencies. This independence ensures that the research is conducted with objectivity and without any external influence.
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- **Data Access Statement and Material Availability:** The adequate resources of this article are publicly accessible.
- **Author's Contributions:** The authorship of this article is contributed solely.

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