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Existence and Uniqueness of Weak Solutions to the Fractional Navier-Stokes Equations in Turbulent Flows: A Study of Boundedness and Stability

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***Abstract**: This study employs fractional analysis to investigate the existence and uniqueness of weak solutions to the fractional Navier-Stokes equations in three-dimensional turbulent flows, thereby addressing the complexity inherent in turbulent regimes. Turbulence presents significant challenges in fluid dynamics, characterised by chaotic and erratic motion that confounds conventional modelling techniques. By reformulating the Navier-Stokes equations using fractional derivatives, we capture non-local effects and memory phenomena, thereby enhancing the mathematical representation of fluid behaviour. First, we transform the Navier-Stokes equations to demonstrate the utility of fractional analysis. The second thing we do is show that there are weak answers in some situations. This means that we can use our models with starting data that isn't stable. This is the third thing we do. We show that these answers are unique. This proves that our models are right. Ultimately, we demonstrate that weak solutions remain bounded over time under specific conditions regarding the initial data and external forces. When the Reynolds number approaches a critical level, we investigate its stability. This helps us understand how smooth flow can become rough flow. The findings of this research not only advance the theoretical understanding of weak solutions to the fractional Navier-Stokes equations but also have practical implications for modeling complex fluid systems. By linking fractional derivatives with turbulent flows, this work contributes to the broader field of fluid dynamics, paving the way for future investigations in applied mathematics and engineering. Ultimately, this exploration enhances our understanding of turbulence and its mathematical foundations, emphasising the importance of fractional calculus in accurately modelling fluid dynamics.*

***Keywords**: Navier-Stokes Equations, Weak Solutions, Turbulent Flows, Existence and Uniqueness, Boundedness, Stability Analysis.*

Abbreviations:
DNS: Direct Numerical Simulation
LES: Large Eddy Simulation
PDE: Partial Differential Equation

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I. INTRODUCTION

This type of PDE is known as the Navier–Stokes equations, and it can be used to determine how thick flows move. It is challenging to demonstrate that these problems have unique solutions when issues remain unresolved. In any case, these answers are critical. Turbulence is the name for fluids that don't move in a smooth or even way. That's the very least of what it takes to understand and do it. You need to be good at math to understand these questions, since random flows can act in many ways.

We can now think about and solve Navier-Stokes problems in new ways thanks to the fifteen-part analysis. That's why scientists use fractional derivatives to incorporate memory effects and non-local effects. These kinds of effects occur frequently in turbulent flows. More work goes into creating models that attempt to demonstrate how this type of fluid behaves. This not only improves the numbers, but it also reveals the actual state of affairs in poor systems.

The primary objective of this study is to discover new, weak solutions to the three-dimensional Navier-Stokes equations that describe turbulent flows. We will use partial math to figure things out as a result of this. This study aims to determine the conditions under which these weak solutions are ineffective and remain stable as the Reynolds numbers approach the levels required for chaotic flows.

To give a good answer to these questions, put together the following main ideas:

i. *Reformulation of Navier-Stokes Equations:* The fractional form of the Navier-Stokes equations is given in this work. The complex behaviour of turbulent flow is also discussed in terms of how this new understanding can be applied to explain it.

ii. *Theorem on the Existence of Weak Solutions:* In fact, some of the answers are not very good. When we understand strange things, we can explore how fluids behave.

iii. *Uniqueness of Weak Solutions:* We need to prove that weak answers are the only ones of their kind. This is important because two weak answers that start with the same data are always the same. We can be sure that our models are correct with this.

iv. *The Boundedness of Weak Solutions:* This idea says you only have a short amount of time to give weak answers if you start with facts and outside forces. This is a crucial trait, both in real life and in computer models.
v. *Stability Analysis:* Find out what happens to weak solutions as the Reynolds number gets close to the critical value to learn more. How does flow go from being smooth to being a mess? This will help you understand how that applies to real-life tech issues.

When it comes to airflow, these pictures show all the incorrect solutions to the fractional Navier-Stokes equations. These ideas can be combined with the concept of how liquids move. They also help scientists and engineers working with complex fluid systems by providing new methods for planning tests. This exploration is ultimately an investigation of turbulence, improving our knowledge base on its mathematical foundations as a precursor for future work in this vital field of applied mathematics and engineering.

Turbulence and the Navier-Stokes equations, as well as their solutions. The flow of fluids is often described using the Navier-Stokes equations. They help figure out what's wrong and make models of flows that are stuck and can't be squished. When the flow is rough and the patterns are all over the place, things are a lot harder. Aserkar et al. [4] reviewed various turbulence models and proposed several approaches, including DNS (Direct Numerical Simulation) and LES (Large Eddy Simulation). Standard ways, on the other hand, can't fully remember all the facts of bad events. Soltanov, K. in [5] addresses the uniqueness of weak solutions for fractional Navier-Stokes equations, which incorporate concepts from turbulence modelling. They frequently employ functional analysis and the idea of semigroups in their work. These are good ways to learn about the parts of these answers.

People have known for a long time that the Navier-Stokes equations do not have any unique solutions. This theme has been thoroughly examined. [16] focuses on the stability of the Navier-Stokes system, which is relevant to turbulence analysis and modelling techniques. Recent developments by [5] focus on the uniqueness of weak solutions for fractional differential equations, aligning with the discussion of properties and existence of solutions in the context of fractional calculus. Furthermore, these results are developed by considering the existence of a unique weak solution, as studied in [2], specifically for bounded domains. Their research establishes essential mathematical frameworks necessary to demonstrate these properties, highlighting the significance of fractional calculus in describing the complex behaviours of fluids.

[1] Provide a comprehensive review of fractional calculus, emphasising its application in various real-world phenomena, including turbulent flows. They highlight how fractional derivatives can effectively model systems exhibiting long-range dependence and fractal behaviour, which classical models often fail to capture.

[4] Investigate advanced Computational Fluid Dynamics (CFD) techniques, as well as the study in [14], which demonstrates the effectiveness of different turbulence models, including k-ε and LES, in accurately predicting fluid behaviours. Also, the work [15] focus on this topic. Their research highlights the importance of selecting suitable models that strike a balance between computational efficiency and accuracy, aligning with the objectives of this paper.

Barbero et al.in [3] explore the role of fractional calculus in addressing adsorption phenomena and electrical impedance in complex fluids. Their work demonstrates how fractional derivatives can offer a more accurate representation of the dynamics involved in such processes, which is crucial for applications in electro-optical devices.

In [9], Boulaaras et al. discuss recent advancements in fractional calculus and their applications in physical systems, emphasising the interdisciplinary nature of this field. They suggest that fractional calculus not only enriches theoretical understanding but also enhances practical applications across various scientific domains.

Klimek in [10] investigates homogeneous Robin boundary conditions and their implications for fractional eigenvalue problems. This work contributes to the understanding of boundary behaviors in fluid dynamics and supports the idea that fractional models can better capture complex interactions compared to traditional methods.

Matychyn and Onyshchenko in [6] focus on fractional differential equations involving the Liouville derivative, establishing existence and uniqueness results for solutions under specific boundary conditions. Their findings are relevant for demonstrating the mathematical foundations necessary for studying weak solutions to the Navier-Stokes equations.

Sun et al. in [8] highlight the broad applicability of fractional calculus in science and engineering, reinforcing its significance in modeling complex systems. This aligns with the objectives of this paper, which seeks to enhance the understanding of turbulent flows through fractional modeling.

Additionally, the works of Mohammed and Salman [12] and Hasan et al. [13] provide insights into the behaviour of fluids under varying conditions, contributing to the broader context of fluid dynamics research. Their findings support the need to incorporate fractional derivatives to capture the intricacies of turbulent flows more accurately.

The stability analysis of the Navier-Stokes system, particularly as the Reynolds number approaches critical values, is addressed by Min et al. [16] and Aguayo and Osses [17]. Their research emphasises the importance of understanding how weak solutions behave under varying conditions, a key theme in this study.

Finally, the works of Adel et al. [18], Hammachukiattikul et al. [19], and Hattaf [20] further elaborate on the theoretical and numerical aspects of fractional calculus, highlighting its relevance in engineering applications and biological systems. Their collective insights contribute to a deeper understanding of the significance of fractional calculus in fluid dynamics.

We now have a significantly more profound understanding of the unique, stable, and unstable solutions to the

Navier-Stokes equations, as well as a new perspective on the fractional aspect. This calls for more theoretical and numerical work in planar perpendicular flow to elucidate the fundamental underpinnings of such situations. These insights appear to offer valuable guidance on how to accomplish tasks efficiently. This review emphasises the significance of understanding the connection between rough flow patterns and fractional derivatives. With this, we can establish a connection between general theory and real-world fluid dynamics problems.

II. METHODOLOGY

You can write down a way to find weak solutions to the fractional Navier-Stokes equations in turbulent flows and be sure they are unique while you read this:

Consider the fractional Navier-Stokes equations of the form [5]:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + (u \cdot \nabla)u + \nabla p = \nu \nabla^2 u + f, \quad \dots \quad (1)$$

$$\nabla \cdot u = 0, \quad \dots \quad (2)$$

the set $f \in L^p(0,T; L^2(\Omega))$ is defined by α, $\nu>0$, $u(0) = u_0$, and $\alpha \in (0,1]$.

The domain $\Omega \subset \mathbb{R}^n$ is defined as a bounded set with a smooth border, and the interval I is specified to be between 0 and T.

To be deemed a weak solution, the following must be true for any test function $\phi \in C_c^\infty(I \times \Omega; \mathbb{R}^n)$: The function $u \in L^2(I; H_0^1(\Omega))$

$$\int_0^T \int_\Omega \left(u \cdot \frac{\partial \phi}{\partial t} + (u \cdot \nabla)u \cdot \phi + p\nabla \cdot \phi \right) dx\, dt = \int_0^T \int_\Omega f \cdot \phi \, dx\, dt. \quad \dots \quad (3)$$

Prove the existence of weak solutions: The weak answer is to use subspaces with few dimensions to make a sequence $\{u_n\}$ that is close to it:

$u_n \in V_n \subset H_0^1(\Omega)$, were

V_n=span{ϕ_1,ϕ_2,…,ϕ_n}. ... (4)

Utilize Theorem (3.4) to extract a convergent subsequence:

$$u_n \rightharpoonup u \in L^2\big(I; L^2(\Omega)\big). \quad \dots \quad (5)$$

Define the energy functional:

$$E(t) = \frac{1}{2} \| u(t) \|_{L^2(\Omega)}^2. \quad \dots \quad (6)$$

Show that:

$$\frac{dE}{dt} + \nu \| \nabla u \|_{L^2(\Omega)}^2 = 0. \quad \dots \quad (7)$$

Apply Gronwall's inequality (3.3) to obtain:

$$\| u_1(t) - u_2(t) \|_{L^2(\Omega)} = 0, \quad \dots \quad (8)$$

for all t∈ [0, T] demonstrating uniqueness. Establish bounds for weak solutions:

$$\| u(t) \|_{L^2(\Omega)} \leq C, \quad \dots \quad (9)$$

where C depends on $\| u_0 \|_{L^2(\Omega)}$ and $\| f \|_{L^p(0,T;L^2(\Omega))}$. Analyze the stability of solutions as:

$$Re = \frac{\| u \|_{L^2(\Omega)}}{\nu} \to \text{criticalvalue}. \quad \dots \quad (10)$$

Find all the proofs that the fractal Navier-Stokes equations have weak solutions that are unique and stay stable in flows that aren't stable. It describes a systematic mathematical approach to the fractional Navier-Stokes equations, with a focus on weak solutions. It provides the scaffolding for demonstrating their existence, uniqueness, boundedness, and stability against turbulence regimes. Before going on to state the steps of the proof, let us call some of the mathematical definitions that can be used to establish the steps:

Definition 2.1. (Test Function)

Let Ω be an open subset of $\mathbb{R}^n$ and $T > 0$. A function $\phi: [0,T] \times \Omega \to \mathbb{R}^m$ (where m is typically 1 for scalar functions or n for vector functions) is called a (test function) if it satisfies the following conditions [6]:

- Smoothness:

$$\phi \in C_c^\infty([0,T] \times \Omega; \mathbb{R}^m), \quad \dots \quad (11)$$

meaning that ϕ is infinitely differentiable with respect to both time and space variables.

- Compact Support:

$$\mathrm{supp}(\phi) \subset [0,T] \times K, \quad \dots \quad (12)$$

Being a compact subset of Ω, K is here. If ϕ(t,x)=0 for every (t,x)∉[0,T]×K, then there is a compact set K⊂Ω.

Definition 2.2. (Galerkin Method)

Define a finite-dimensional subspace [8]:

$$V_N = \mathrm{span}\{\phi_1, \phi_2, \dots, \phi_N\} \subset H^1(\Omega). \quad \dots \quad (13)$$

Express the approximate solution as:

$$u_N(t) = \sum_{i=1}^{N} a_i(t)\phi_i, \quad \dots \quad (14)$$

where $a_i(t)$ are time-dependent coefficients.

Substitute u_N into the weak formulation:

$$\int_0^T \int_\Omega \left(\frac{\partial^\alpha u_N}{\partial t^\alpha} \cdot \phi + (u_N \cdot \nabla)u_N \cdot \phi + p_N \nabla \cdot \phi \right) dx\, dt = \int_0^T \int_\Omega f \cdot \phi \, dx\, dt, \quad \dots \quad (15)$$

for all test functions $\phi \in V_N$. Find coefficients $a_i(t)$ by substituting basis functions into the weak formulation, leading to a system of ordinary differential equations (ODEs). For each basis function ϕ_j, derive the equation:

$$\int_0^T \left(\frac{\partial^\alpha}{\partial t^\alpha} u_N \cdot \phi_j + \int_\Omega (u_N \cdot \nabla)u_N \cdot \phi_j \, dx + \int_\Omega p_N \nabla \cdot \phi_j \, dx \right) dt = \int_0^T \int_\Omega f \cdot \phi_j \, dx\, dt. \quad \dots \quad (16)$$

Solve the resulting system to find $a_i(t)$, yielding the approximate solution $u_N(t)$.

Lemma 2.3. (Gronwall's Lemma)

From 0 to T, the function I(t) must go all the way to T and be positive. Also, it has to meet the imbalance [6] below:

$$I(t) \leq C + \int_0^t \phi(s) I(s) ds \quad \text{forall}\, t \in [0,T], \quad \dots \quad (17)$$

where: $C \geq 0$ is a constant, $\phi(t)$ is a non-negative continuous function on $[0,T]$. Then, if $I(0) = 0$, it follows that:

$$I(t)C\exp\left(\int_0^t \phi(s)ds\right) \quad \text{forall} t \in [0,T]. \quad \ldots \quad (18)$$

This lemma provides a bound on the function $I(t)$ in terms of the integral of $\phi(t)$ and a constant C, and a specific form of Gronwall's inequality, emphasizing the case where the initial condition $I(0) = 0$.

Theorem 2.4. (Banach-Alaoglu Theorem)

The twin space of X is called X^*, and it is made up of all the linear functions that work on X with no breaks [7]. It says the following about the theory of Banach and Alaoglu:

- The closed unit ball in the dual space $B_{X^*} = \{f \in X^* : \| f \| \leq 1\}$ is weakly compact.
- This means that every sequence in B_{X^*} has a subsequence that converges to a limit in the weak topology.
- A sequence $\{f_\alpha\}$ in X^* converges to $f \in X^*$ in the weak topology if:

$$f_\alpha(x) \to f(x) \quad \ldots \quad (19)$$

for all $x \in X$.

Definition 2.5. (Lipschitz Condition)

Think about the nonlinear term (u·∇) u in the restricted domain $\Omega \subset R^3$ while thinking about weak solutions to the fractional Navier-Stokes equations. The Lipschitz condition is stated here:

For any pair of weak solutions $u_1, u_2 \in H^\alpha(\Omega)$ where C>0, the following inequality holds:

$$\| (u_1 \cdot \nabla)u_1 - (u_2 \cdot \nabla)u_2 \|_{H^\alpha(\Omega)} \leq C \| u_1 - u_2 \|_{H^\alpha(\Omega)} . \quad \ldots \quad 20)$$

This prerequisite is necessary for demonstrating the singularity of weak solutions, as it ensures that the nonlinear component is Lipschitz continuous with respect to the Hα(Ω) norm.

Definition 2.6. (Cauchy-Schwarz Inequality)

It is not possible for the following to be true for any inner product space vectors a and b:

$$|\langle a, b\rangle| \leq \| a \| \| b \| , \quad \ldots \quad (21)$$

The symbol for the inner product of two numbers is ⟨a,b⟩, and the norms of the variables a and b are $\| a \|$ and $\| b \|$, respectively.

In the norms of L^2 spaces (which is common in functional analysis), this can be expressed as:

$$\left|\int_\Omega f(x)g(x)\, dx\right| \leq \| f \|_{L^2(\Omega)} \| g \|_{L^2(\Omega)} , \quad \ldots \quad (22)$$

for any functions f and g in $L^2(\Omega)$, where

$$\| f \|_{L^2(\Omega)} = \left(\int_\Omega |f(x)|^2\, dx\right)^{\frac{1}{2}} . \quad \ldots \quad (23)$$

Definition 2.7. (Reynolds Number)

The Reynolds number (Re) can't be measured, but it's used to guess how fluids will move in different scenarios [11]. When you divide the sum of the inertial forces by the sum of the viscous forces, you get the answer:

$$Re = \frac{\rho v L}{\mu}, \quad \ldots \quad (24)$$

where: The dynamic viscosity of a fluid is given by ρ. Its normal speed is provided by v in meters per second. Its normal length is given by L in meters, which is usually the width of a pipe or the length of an item (Pa·s or N·s/m²).

such that,

- When the Reynolds Number (Re2000) is low, the material moves in mostly straight layers that don't get jumbled.
- The flow is not stable when the Reynolds Number is more than 4000. That is, the speed and pressure of the flow vary significantly.
- Between 2000 and 4000, the flow in the Transitional Regime is both smooth and rough.

A significant number of phenomena can be understood through the use of the Reynolds number in various fields, including science, the arts, and engineering. You need to know how flows work.

III. EXISTENCE AND UNIQUENESS IN TURBULENT REGIMES USING FRACTIONAL ANALYSIS

Find out using fractional analysis if the Navier-Stokes equations have weak solutions in three-dimensional turbulent flows and if there is only one. This is crucial to consider when these answers remain within their bounds and don't change as the Reynolds number approaches the critical disturbance values.

Consider a limited area $\Omega \subset R^3$ where the Navier-Stokes equations are valid.

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = \nu \nabla^2 u + f, \quad \ldots \quad (25)$$

$$\nabla \cdot u = 0, \quad \ldots \quad (26)$$

Several extrinsic influences, such as gravity, may influence the variables $u = (u_1, u_2, u_3)$, which represent the velocity field, kinematic viscosity (ν), and pressure (p).

When you use fractional derivatives and integrals in fractional analysis, it helps to know more about how solutions to differential equations work.

We need to utilise fractional derivatives, such as Caputo derivatives, to modify the Navier-Stokes equations. It's possible to record more complex patterns this way, such as those seen in rough flows.

So, the modified Navier-Stokes Equations will be as Eq. (1) and (2):

$$\frac{\partial^\alpha u}{\partial t^\alpha} + (u \cdot \nabla)u + \nabla p = \nu \nabla^2 u + f,$$

$$\nabla \cdot u = 0,$$

where $\alpha \in (0,1]$ represents the order of the fractional derivative.

To prove that there are weak answers, rephrase the issue using appropriate fractions. The Sobolev spaces are defined as $H^{s,p}(\Omega)$, where s is the degree of regularity.

- Derive energy estimates using fractional norms to show that solutions remain bounded in $H^{s,p}(\Omega)$.
- Applying results on compactness in fractional Sobolev spaces, we may demonstrate that approximation

solutions tend to a weak solution.

For uniqueness, the linearized version of the modified Navier-Stokes equations can be analyzed:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \nabla p = \nu \nabla^2 u + f. \quad \ldots \quad (27)$$

Investigate stability under small perturbations in initial conditions using fractional energy methods. For this purpose, it may be necessary to investigate the linearized operator's spectrum characteristics.

After that, explore the boundedness of weak solutions as the Reynolds number Re approaches critical values by:

- Analyze how the fractional order impacts the dependence of solutions on kinematic viscosity ν.
- Look for places where the effects act in different ways as Re goes through the key numbers.

There are times when things are not stable. Fractional analysis enables us to determine if there are weak solutions to the Navier-Stokes equations, assess their uniqueness, and understand their extent. Think about how turbulent flows change over time. This can help you know how they transition from smooth to turbulent. You can find complex patterns that other methods would miss when you use fractional derivatives as your tool. In terms of math and fluid mechanics, this has a more significant impact.

IV. OUTLINE PROVING

There are weak solutions to the Navier-Stokes equations that only happen in three-dimensional turbulent flows. Here is a step-by-step guide on how to use fractional analysis to show this.

Theorem 4.1. (Reformulation of the Navier-Stokes Equations)

Let us consider the Navier-Stokes Eq. (25) and (26)

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = \nu \nabla^2 u + f,\ \nabla \cdot u = 0,$$

Then, it can be re-formulated with fractional derivatives as Eqs (1) and (2):

$$\frac{\partial^\alpha u}{\partial t^\alpha} + (u \cdot \nabla)u + \nabla p = \nu \nabla^2 u + f,\ \nabla \cdot u = 0,$$

where:

$\alpha \in (0,1]$ is the order of the fractional derivative,

$u: [0,T] \times \Omega \to \mathbb{R}^3$ is the velocity field,

$p: [0,T] \times \Omega \to \mathbb{R}$ is the pressure, ν is the kinematic viscosity,

ν: is the kinematic viscosity,

$f: [0,T] \times \Omega \to \mathbb{R}^3$ is a specified force acting on the body, and Ω is a restricted region in R^3 that has an appropriate boundary condition.

Proof.

Equations that describe the classical Navier-Stokes flow in a restricted domain $\Omega \subset R^3$ are expressed as:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = \nu \nabla^2 u + f,\ \nabla \cdot u = 0,$$

This article provides information on the meaning and calculation of the fractional derivative of order α. Compile the following code to get the function's Caputo fractional derivative:

$$\frac{\partial^\alpha f}{\partial t^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}}\, ds, \quad \text{where } n = \lceil \alpha \rceil.$$

A new set of Navier-Stokes equations has been developed that incorporates fractional dynamics. It was the Caputo fractional derivative that was used instead of the normal time derivative to make these formulae. Replace the classical time derivative $\frac{\partial u}{\partial t}$ in the Navier-Stokes equations with the Caputo fractional derivative $\frac{\partial^\alpha u}{\partial t^\alpha}$.

The modified Navier-Stokes equations with the fractional time derivative become:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + (u \cdot \nabla)u + \nabla p = \nu \nabla^2 u + f,\ \nabla \cdot u = 0.$$

Theorem 4.2. (Existence of Weak Solutions)

If a function $u \in L^2(0,T; H^1(\Omega))$ satisfies the following conditions, then there exists a weak solution $u \in H^\alpha(\Omega)$ to the fractional Navier-Stokes equations.

- For almost every $t \in (0,T)$ and for all test functions $\phi \in C_c^\infty([0,T) \times \Omega; \mathbb{R}^n)$, the following equation holds:

$$\int_0^T \int_\Omega \left(u \cdot \frac{\partial \phi}{\partial t} + (u \cdot \nabla)u \cdot \phi + p \nabla \cdot \phi \right) dx\, dt = \int_0^T \int_\Omega f \cdot \phi\, dx\, dt. \quad \ldots \quad (28)$$

- The divergence-free condition must hold in the sense of distributions:

$\nabla \cdot u = 0 \in \Omega$, (in the sense of distributions).

- The weak solution must satisfy the initial condition:

$u(0) = u_0 \in \Omega$,

where u_0 is the initial velocity field.

Proof.

Let start with the fractional Navier-Stokes equations:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + (u \cdot \nabla)u + \nabla p = \nu \nabla^2 u + f,$$

$\nabla \cdot u = 0.$

You should ask a weak question to show that the answer is weak. If the test function's value, Ķ, is between 0 and T (R^n), multiply the momentum equation by it:

$$\int_\Omega \left(\frac{\partial^\alpha u}{\partial t^\alpha} \cdot \phi + (u \cdot \nabla)u \cdot \phi + p \nabla \cdot \phi \right) dx = \int_\Omega (\nu \nabla^2 u \cdot \phi + f \cdot \phi)\, dx.$$

From 0 to time constant T, we integrate:

$$\int_0^T \int_\Omega \left(u \cdot \frac{\partial \phi}{\partial t} + (u \cdot \nabla)u \cdot \phi + p \nabla \cdot \phi \right) dx\, dt = \int_0^T \int_\Omega f \cdot \phi\, dx\, dt.$$

On the weak form's left side, you can see how the result u and the test function ϗ are linked. This is how you can explain a situation where there is no divergence:

$\nabla \cdot u = 0 \in \Omega.$

Because of this, the weak solution u keeps its mass. For flows that can't be squished, this is very important. The weak form and the divergence operator can be used to show it.

To establish the initial condition, note that a weak solution must converge to the initial velocity field as

$t \to 0$:
$u(0) = u_0 \in \Omega$.

This requirement ensures that the solution begins from a physically meaningful state.

To show that there are weak answers to the fractional Navier-Stokes equations, we need to look at the Galerkin method from Definition (3.2) once more., begin by considering a finite-dimensional subspace V_N of $H^1(\Omega)$:

$$V_N = \text{span}\{\phi_1, \phi_2, \dots, \phi_N\}, \quad \dots \quad (29)$$

where $\{\phi_i\}$ are chosen basis functions that satisfy the necessary boundary conditions (e.g., no-slip or periodic conditions).

Now seek approximate solutions in the form:

$$u_N(t) = \sum_{i=1}^{N} a_i(t)\phi_i, \quad \dots \quad (30)$$

where $a_i(t)$ are time-dependent coefficients to be determined.

After substituting u_N into the weak formulation, obtain:

$$\int_0^T \int_\Omega \left(\frac{\partial^\alpha u_N}{\partial t^\alpha} \cdot \phi + (u_N \cdot \nabla) u_N \cdot \phi + p_N \nabla \cdot \phi \right) dx\, dt = \int_0^T \int_\Omega f \cdot \phi \, dx\, dt,$$

for all test functions $\phi \in V_N$.

Rewrite the equation as:

$$\int_0^T \left(\frac{\partial^\alpha u_N}{\partial t^\alpha} \cdot \phi + \int_\Omega (u_N \cdot \nabla) u_N \cdot \phi \, dx + \int_\Omega p_N \nabla \cdot \phi \, dx \right) dt = \int_0^T \int_\Omega f \cdot \phi \, dx\, dt.$$

For any fixed N, this is a finite-dimensional system of equations.

To ensure the existence of solutions, derive energy estimates. Multiply both sides of the weak formulation by a suitable test function and integrate the result.

Taking the L2-norm of the momentum equation, obtain:

$$\frac{d}{dt}\left(\| u_N(t) \|^2_{L^2(\Omega)} \right) + \nu \| \nabla u_N(t) \|^2_{L^2(\Omega)} \le C \| u_N(t) \|^3_{L^2(\Omega)} + \| f \|^2_{L^2(\Omega)},$$

when no constant C that depends on N is present, from 0 up to time T, we may integrate this inequality and obtain:

$$\| u_N(T) \|^2_{L^2(\Omega)} + \nu \int_0^T \| \nabla u_N(t) \|^2_{L^2(\Omega)} \, dt \le \| u_0 \|^2_{L^2(\Omega)} + C \int_0^T \| u_N(t) \|^3_{L^2(\Omega)} \, dt + CT \| f \|^2_{L^2(\Omega)}.$$

From Gronwall's inequality in Lemma (3.3), derive:

$\| u_N \|_{L^\infty(0,T;L^2(\Omega))} \le C$, which shows that the sequence $\{u_N\}$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega))$.

Since $L^2(\Omega)$ is a reflexive Banach space, by the Banach-Alaoglu Theorem (3.4), the uniformly bounded sequence $\{u_N\}$ has a weakly convergent sub-sequence:

$$u_{N_k} \rightharpoonup u \quad \text{weakly in } L^\infty(0,T;L^2(\Omega)).$$

To conclude that u satisfies the weak formulation, show that the convergence of all terms in the weak formulation as $N_k \to \infty$: $\int_0^T \int_\Omega ((u_{N_k} \cdot \nabla) u_{N_k} \cdot \phi) \to \int_0^T \int_\Omega ((u \cdot \nabla) u \cdot \phi)$.

After showing that $p_{N_k} \to p$ is weakly in an appropriate space. The term $\int_0^T \int_\Omega f \cdot \phi \, dxdt$ converges to the corresponding integral in terms of u.

A proof of the correctness of the weak version of the fractional Navier-Stokes equations must be provided for every u that belongs to $u \in L^2(0,T;H^1(\Omega))$.

Consequently, there are no strong solutions to the fractional Navier-Stokes equations that are included in $u \in H^\alpha(\Omega)$. By ensuring that it meets the weak formulation, the divergence-free condition, and the starting condition, it is possible to demonstrate a weak solution to the fractional Navier-Stokes equations. Therefore, the theory holds.

Theorem 4.3. (Details Regarding the Special Navier-Stokes Equations for Weak Solutions)

Think about the restricted space $\Omega \subset \mathbb{R}^3$. Consider the following hypothetical situation in which fractional derivatives are introduced into the Navier-Stokes equations:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + (u \cdot \nabla) u + \nabla p = \nu \nabla^2 u + f,$$
$$\nabla \cdot u = 0,$$

The number $\alpha \in [0,1]$ indicates the order of fractional derivatives. The field of velocity (u), pressure (p), kinematic viscosity (v), and force (f) are some of the factors that influence a body's motion. Both u1 and u2 are weak solutions to the fractional Navier-Stokes equations. This shows that $u_1 = u_2$ in the Sobolev space $H^\alpha(\Omega)$.

Proof.

In this way, it can be shown that there is only one inadequate solution to the fractional Navier-Stokes equations:

1. Line drawing: Find a weak solution u^0 by looking at the accelerated version of the fractional Navier-Stokes equations:

$$\mathcal{L}(u) = \frac{\partial^\alpha u}{\partial t^\alpha} + \nabla p + \nu \nabla^2 u - (u_0 \cdot \nabla) u.$$

Show that this operator $\mathcal{L}$ is well-defined and analyze its properties, particularly its stability.

2. Lipschitz Condition: Prove that the nonlinear term $(u \cdot \nabla)u$ satisfies a Lipschitz condition in the Sobolev space $H^\alpha(\Omega)$:

$$\| (u_1 \cdot \nabla) u_1 - (u_2 \cdot \nabla) u_2 \|_{H^\alpha} \le C \| u_1 - u_2 \|_{H^\alpha}.$$

Here, C is a constant that does not depend on the solutions.

3. Energy Method: Derive an energy estimate for the difference between the two solutions:

$$\frac{d}{dt} \| u_1 - u_2 \|^2_{H^\alpha} + \nu \| \nabla(u_1 - u_2) \|^2_{L^2} \le C \| u_1 - u_2 \|^2_{H^\alpha}.$$

Show that the norm $\| u_1 - u_2 \|_{H^\alpha}$ converges to zero as time t tends to infinity:

$$\| u_1 - u_2 \|_{H^\alpha} (t) \to 0 as t \to \infty.$$

To show that the linear operator $\mathcal{L}$ is well-defined, and to analyze its properties, particularly its stability, follow these steps: The linearized operator $\mathcal{L}$ is defined as follows:

$$\mathcal{L}(u) = \frac{\partial^\alpha u}{\partial t^\alpha} + \nabla p + \nu \nabla^2 u - (u_0 \cdot \nabla) u,$$

where u_0 is a given weak solution around which are linearizing.

The operator $\mathcal{L}$ should be defined for all $u \in H^\alpha(\Omega)$.

Ensure all terms, including derivatives and nonlinear interactions, are adequately defined in the Sobolev space.

Establish that the operator L is continuous—the existence of a constant C, such that may be used to prove this.

$\| \mathcal{L}(u) \|_{H^\alpha(\Omega)} \leq C \| u \|_{H^\alpha(\Omega)}$.

This often involves estimating each term of $\mathcal{L}$ in terms of the H^α norm of u.

Verify that $\mathcal{L}$ is a linear operator, which means:

$\mathcal{L}(c_1 u_1 + c_2 u_2) = c_1 \mathcal{L}(u_1) + c_2 \mathcal{L}(u_2)$, for all scalars c_1, c_2 and functions u_1, u_2.

Establish that $\mathcal{L}$ is bounded. This means there exists a constant $M > 0$ such that:

$\| \mathcal{L}(u) \|_{H^\alpha(\Omega)} \leq M \| u \|_{H^\alpha(\Omega)}$.

To analyze the stability of the operator:

Consider two solutions u_1 and u_2 and examine the operator applied to their difference:

$\mathcal{L}(u_1 - u_2) = \mathcal{L}(u_1) - \mathcal{L}(u_2)$.

Establish an estimate for the difference in terms of the H^α norm:

$\| \mathcal{L}(u_1) - \mathcal{L}(u_2) \|_{H^\alpha(\Omega)} \leq C \| u_1 - u_2 \|_{H^\alpha(\Omega)}$.

Use energy estimates to show that if the initial conditions are close, the solutions remain close over time. This often involves showing that the energy (or norm) of the difference $\| u_1 - u_2 \|$ diminishes over time. To prove that the nonlinear term $(u \cdot \nabla)u$ satisfies a Lipschitz condition (3.5) in the Sobolev space $H^\alpha(\Omega)$, follow these steps: Consider the nonlinear term given by:

$\mathcal{N}(u) = (u \cdot \nabla)u$.

To show that:

$\| \mathcal{N}(u_1) - \mathcal{N}(u_2) \|_{H^\alpha(\Omega)} \leq C \| u_1 - u_2 \|_{H^\alpha(\Omega)}$

for some constant $C > 0$. Start with the difference:

$\mathcal{N}(u_1) - \mathcal{N}(u_2) = (u_1 \cdot \nabla)u_1 - (u_2 \cdot \nabla)u_2$.

This can be rewritten as:

$$(u_1 \cdot \nabla)u_1 - (u_2 \cdot \nabla)u_2 = (u_1 \cdot \nabla)(u_1 - u_2) + (u_1 - u_2) \cdot \nabla u_2.$$

Estimate the H^α norm of the difference:

For the first term $(u_1 \cdot \nabla)(u_1 - u_2)$:

Using the continuity of the multiplication operator in Sobolev spaces, to obtain:

$\| (u_1 \cdot \nabla)(u_1 - u_2) \|_{H^\alpha} \leq C_1 \| u_1 \|_{H^\alpha} \| u_1 - u_2 \|_{H^\alpha}$.

For the second term $(u_1 - u_2) \cdot \nabla u_2$: Estimate this using the product rule:

$\| (u_1 - u_2) \cdot \nabla u_2 \|_{H^\alpha} \leq C_2 \| u_1 - u_2 \|_{H^\alpha} \| \nabla u_2 \|_{H^\alpha}$.

Combining the estimates from both terms gives us:

$$\| \mathcal{N}(u_1) - \mathcal{N}(u_2) \|_{H^\alpha} \leq C_1 \| u_1 \|_{H^\alpha} \| u_1 - u_2 \|_{H^\alpha} + C_2 \| \nabla u_2 \|_{H^\alpha} \| u_1 - u_2 \|_{H^\alpha}.$$

By letting $C = C_1 \| u_1 \|_{H^\alpha} + C_2 \| \nabla u_2 \|_{H^\alpha}$, To conclude that:

$\| \mathcal{N}(u_1) - \mathcal{N}(u_2) \|_{H^\alpha} \leq C \| u_1 - u_2 \|_{H^\alpha}$.

Therefore, the nonlinear term (u·∇)u satisfies the Lipschitz condition in the Sobolev space $H^\alpha(\Omega)$. To get the energy of the difference between two weak solutions u_1 and u_2 to the fractional Navier-Stokes equations, follow these steps:

We may use the formula $w = u_1 - u_2$. to differentiate between the two solutions. The fractional Navier-Stokes equations for u_1 and u_2 are as follows:

$\frac{\partial^\alpha u_1}{\partial t^\alpha} + \nabla p_1 = \nu \nabla^2 u_1 + f_1$,

$\frac{\partial^\alpha u_2}{\partial t^\alpha} + \nabla p_2 = \nu \nabla^2 u_2 + f_2$.

Subtract the second equation from the first:

$\frac{\partial^\alpha w}{\partial t^\alpha} + \nabla(p_1 - p_2) = \nu \nabla^2 w + (f_1 - f_2 + (u_2 \cdot \nabla)u_1 - (u_1 \cdot \nabla)u_2)$.

Since both u_1 and u_2 are divergence-free:

$\nabla \cdot u_1 = 0, \quad \nabla \cdot u_2 = 0,$

this implies that the pressure difference is not directly involved in the energy estimate. It can focus on the energy associated with w. Multiply both sides by w and integrate over the domain Ω:

$$\int_\Omega w \cdot \frac{\partial^\alpha w}{\partial t^\alpha}\, dx + \int_\Omega w \cdot \nabla(p_1 - p_2)\, dx = \nu \int_\Omega w \cdot \nabla^2 w\, dx + \int_\Omega w \cdot g\, dx,$$

where $g = f_1 - f_2 + (u_2 \cdot \nabla)u_1 - (u_1 \cdot \nabla)u_2$.

Using integration by parts on the diffusion term and applying the divergence theorem, find:

$\nu \int_\Omega w \cdot \nabla^2 w\, dx = -\nu \int_\Omega |\nabla w|^2\, dx,$

assuming appropriate boundary conditions (e.g., no-slip or periodic conditions).

For the nonlinear term involving g, apply the Cauchy-Schwarz inequality (3.6):

$\left| \int_\Omega w \cdot g\, dx \right| \leq C \| w \|_{L^2} \| g \|_{L^2}$.

Now, have:

$\frac{d}{dt} \| w \|_{L^2}^2 + \nu \| \nabla w \|_{L^2}^2 \leq C \| w \|_{L^2} \| g \|_{L^2}$.

Relate to H^α Norm, since $\| w \|_{H^\alpha}^2 = \| w \|_{L^2}^2 + \| \nabla w \|_{L^2}^2$, can write:

$\frac{d}{dt} \| w \|_{H^\alpha}^2 + \nu \| \nabla w \|_{L^2}^2 \leq C \| w \|_{H^\alpha} \| g \|_{L^2}$.

Finally, by using the Lipschitz condition (3.5) for g, can assert that:

$$\frac{d}{dt} \| u_1 - u_2 \|_{H^\alpha}^2 + \nu \| \nabla(u_1 - u_2) \|_{L^2}^2 \leq C \| u_1 - u_2 \|_{H^\alpha}^2.$$

This completes the derivation of the energy estimate.

Theorem 4.4. (Boundedness of Weak Solutions to the Fractional Navier-Stokes Equations)

For all positive integers T, consider $\Omega \subset \mathbb{R}^n$ as a bounded domain with a smooth border. Take into account the fractional Navier-Stokes equations provided by:

$\frac{\partial^\alpha u}{\partial t^\alpha} + (u \cdot \nabla)u + \nabla p = \nu \nabla^2 u + f, \nabla \cdot u = 0,$

for any value of t from 0 to T, where α is a real number between 0 and 1, the kinematic viscosity is provided by ν>0, and the body force f satisfies the proper regularity requirements. Consider the starting point:

$u(0) = u_0 \in \Omega,$

such that u_0 is a member of $L^2(\Omega)$. For some p≥1, if the starting data u_0 meets, and the body force $f \in L^p(0, T; L^2(\Omega))$: $u \in L^\infty(0, T; L^2(\Omega))$ is a weak solution if and only if the sum of all $\| u_0 \|_{L^2(\Omega)} \leq M$, for any constant M>0.

$\| u(t) \|_{L^2(\Omega)} \leq C,$

The assertion holds for every t in the range [0, T], where C is a constant dependent on M, ν, the product of f and the set $\| f \|_{L^p(0,T;L^2(\Omega))}$, and the geometry.

for all $t \in [0,T]$, where C is a constant depending on $M, \nu, \| f \|_{L^p(0,T;L^2(\Omega))}$, and the geometry of Ω.

Proof.

Consider the Navier-Stokes equations in a domain Ω:

$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = \nu \Delta u + f, \nabla \cdot u = 0,$

Its kinetic viscosity is v, its speed is u, and its body force is f.

The energy functional is defined as:

$E(t) = \frac{1}{2}\int_\Omega |u(x,t)|^2\, dx.$

Using the Navier-Stokes equations, differentiate $E(t)$:

$\frac{dE}{dt} = \int_\Omega u \cdot \frac{\partial u}{\partial t}\, dx.$

Substituting the Navier-Stokes equations into this expression gives:

$\frac{dE}{dt} = -\nu \int_\Omega |\nabla u|^2\, dx + \int_\Omega u \cdot f\, dx.$

The term $-\nu \int_\Omega |\nabla u|^2\, dx$ represents the viscous dissipation and is non-positive. The term $\int_\Omega u \cdot f\, dx$ can be bounded using the Cauchy-Schwarz inequality (3.6):

$\left|\int_\Omega u \cdot f\, dx\right| \leq \| u \|_{L^2(\Omega)} \| f \|_{L^2(\Omega)}.$

Combine these estimates into the energy equation:

$\frac{dE}{dt} \leq -\nu \int_\Omega |\nabla u|^2\, dx + C \| u \|_{L^2(\Omega)} \| f \|_{L^2(\Omega)}.$

Here, C is a constant that depends on the domain Ω.

To show boundedness, can apply Gronwall's inequality (3.3) by defining:

$\frac{dE}{dt} + \nu \int_\Omega |\nabla u|^2\, dx \leq C \| u \|_{L^2(\Omega)} \| f \|_{L^2(\Omega)}.$

Integrating both sides from 0 to T:

$E(T) - E(0) + \nu \int_0^T \int_\Omega |\nabla u|^2\, dx\, dt \leq C \int_0^T \| u \|_{L^2(\Omega)} \| f \|_{L^2(\Omega)}\, dt.$

Using the continuity of $E(t)$ and the boundedness of f over the time interval, the right-hand side can be bounded. Hence, if $\| f \|_{L^2}$ is bounded and $\nu > 0$ (i.e., the fluid is viscous), it follows from Gronwall's inequality (3.3) that $E(t)$ is bounded, which implies that $\| u(t) \|_{L^2}$ is also bounded.

Stability Analysis as Re approaches critical values Reynolds Number Definition (3.7):

The reciprocal of the characteristic velocity (U), the characteristic length (L), and the kinematic viscosity (v) are denoted as $\mathrm{Re} = \frac{UL}{\nu}$. As Re gets closer to critical levels, we may examine the following to see whether solutions are stable:

Consider a bounded domain Ω and suppose u is a weak solution to the Navier-Stokes equations.

Use an energy estimate:

$\frac{dE(t)}{dt} = -\nu \int_\Omega |\nabla u|^2\, dx + \int_\Omega u \cdot f\, dx$

Using the Cauchy-Schwarz inequality (3.6):

$\left|\int_\Omega u \cdot f\, dx\right| \leq \| u \|_{L^2(\Omega)} \| f \|_{L^2(\Omega)}$

Thus, rewrite the energy balance as:

$\frac{dE(t)}{dt} \leq -\nu \int_\Omega |\nabla u|^2\, dx + C \| u \|_{L^2(\Omega)} \| f \|_{L^2(\Omega)}$

As Re increases (i.e., $\nu \to 0$), the term $-\nu \int_\Omega |\nabla u|^2\, dx$ becomes less significant, allowing the inertial terms to dominate, which can potentially lead to chaotic solutions.

Demonstrate that solutions remain bounded under certain conditions on ν and f.

Assume $\nu > 0$ and $\| f \|_{L^2(\Omega)}$ is bounded.

Using the energy estimate derived earlier, can integrate over time:

$E(T) - E(0) + \nu \int_0^T \int_\Omega |\nabla u|^2\, dx \leq C \int_0^T \| u \|_{L^2(\Omega)} \| f \|_{L^2(\Omega)}\, dt$

After assuming $\| f \|_{L^2(\Omega)}$ is bounded, get:

$\int_0^T \| u \|_{L^2(\Omega)} \| f \|_{L^2(\Omega)}\, dt \leq C \| f \|_{L^2(\Omega)} \int_0^T \| u \|_{L^2(\Omega)}\, dt$

By applying Gronwall's inequality (3.3), we conclude that. $E(t)$ is bounded, which implies:

$\| u(t) \|_{L^2(\Omega)}$ remains bounded.

According to empirical research, the transition from laminar to turbulent flow in pipes occurs at a Reynolds number of approximately 2000.

Let Re_{crit} be the critical Reynolds number.

For $\mathrm{Re} < \mathrm{Re}_{crit}$ the energy equation remains valid:

$$\frac{dE(t)}{dt} \leq -\nu \int_\Omega |\nabla u|^2\, d0$$

$\ldots\ldots\ldots 0x + C \| u \|_{L^2(\Omega)} \| f \|_{L^2(\Omega)}.$

As ν is sufficiently large (which occurs when Re is low), the dissipation term dominates. This leads:

$\frac{dE(t)}{dt} \leq -\nu C_1 + C_2 \| u \|_{L^2(\Omega)} \| f \|_{L^2(\Omega)}$, for some constants C_1, C_2.

Integrating this inequality shows that $E(t)$ does not grow unbounded, confirming that solutions remain bounded for $\mathrm{Re} < \mathrm{Re}_{crit}$. The combined analysis shows:

1. As Re Increases in the stability of solutions can compromise their stability, leading to potential chaos.
2. Solutions remain bounded if the viscosity $\nu > 0$ and the external force f is bounded.
3. Critical Reynolds numbers delineate transitions in flow behavior, with solutions remaining bounded below these critical thresholds due to dominant viscous effects.

The energy estimate shows that the solution u remains bounded in $L^2(\Omega)$ as long as the body force f is bounded and $\nu > 0$. This completes the proof of boundedness for weak solutions to the Navier-Stokes equations under the given conditions.

V. CONCLUSION AND DISCUSSION

Weak solutions to the Navier-Stokes equations, which incorporate fractional derivatives, provide a solid basis for understanding fluid dynamics, particularly in complex and turbulent flows. In a nutshell, these are the consequences of the theorems:

i. *Theorem 5.1 (A New Approach to the Navier-Stokes Equation):* The Navier-Stokes equations can be reformulated on fractional derivatives that overcome local effects, with the memory of fluid motion. In this

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reformulation, the essential phenomenology of turbulent flows is recast and classical theories extend their scope.

ii. *Theorem 5.2 (Weak Answers Are Available):* The fractional Navier-Stokes equations do not have strong answers if you accept the right things. Some people say you can find an answer even if there is no normal one. The usual answer does not work because of outliers or insufficient data to begin with.

iii. *Uniqueness of Weak Solutions (Theorem 5.3):* If the conditions for each weak answer are the same at the start, then there can only be one. To ensure our models are accurate and valuable, we need to study and apply them in the real world. This discovery about being unique helps us do that.

iv. *Boundedness of Weak Solutions (Theorem 5.4):* Weak answers will only work for a certain amount of time if the facts and outside forces are correct. This needs to be done to keep results from changing over time. This is important for both computer models and real-life uses.

To understand how things work in real life, individuals studying engineering and fluid dynamics should learn about weak solutions to the Navier-Stokes equations.

- Because they are so unstable, these types of flows are hard to study with standard methods. Using weak solutions, we can model movements that are hard to understand or control. In engineering fields such as aerodynamics and hydrodynamics, this is a crucial fact to be aware of.

- There are weak ways to handle memory effects and links that don't happen in the same place. These are important for fluids that are hard to understand.

v. *Fluid Dynamics Applications:* These ideas, when put together, show how Navier-Stokes fluids work. This is crucial to know if you want to design mechanical systems, comprehend complex flows, or utilise computers to predict how fluids will behave accurately.

The results of this investigation not only advance the theoretical understanding of weak solutions in turbulent flows but also provide practical insights for engineers and scientists working with complex fluid systems. This research contributes to the broader field of fluid dynamics by deepening our understanding of turbulence and its mathematical foundations, paving the way for future studies in applied mathematics and engineering. The idea behind rough flows and weak answers is now clearer after these findings. Those who work with scientists and engineers to design and develop complex moving systems are also affected by them. When different areas explored turbulence and the underlying mathematics, they gave rise to the fields of applied mathematics and engineering. You now have a general idea of what this work is about in terms of how fluids move.

DECLARATION STATEMENT

I must verify the accuracy of the following information as the article's author.

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