Philosophical Magazine Series 3

ISSN: 1941-5966 (Print) 1941-5974 (Online) Journal homepage: http://www.tandfonline.com/loi/tphm14

XXXI. On the refraction of differently-coloured rays in crystals, with one and two axes of double refraction

M. Rudberg

To cite this article: M. Rudberg (1832) XXXI. On the refraction of differently-coloured rays in crystals, with one and two axes of double refraction , Philosophical Magazine Series 3, 1:2, 136-146, DOI: 10.1080/14786443208647853

To link to this article: http://dx.doi.org/10.1080/14786443208647853

Published online: 01 Jun 2009.

sets of reduced observations, which are of the highest value in reference to this important branch of meteorology. These observations will be published in successive Numbers of this Journal; and while they will enable me to compare my own theoretical view with observations, they will be received by the scientific meteorologist as data of inestimable value in fixing the principles of this new science.

It appears from the first of the preceding tables, that the mean temperature of Nicolaieff for four successive years, from 1827—1830, at 10^h A.M. and 10^h P.M. is 7°·52 Reaumur, or 48°·92 Fahr. When we correct this result by + 0·122, the quantity by which the mean of 10^h and 10^h differs from that of the 24 hours, we obtain,

	Fahr.
Corrected mean temp. of Nicolaieff	49°·04
Add for elevation of 20 toises..........................	·36
Mean temp. of level of sea	49°·40
Mean temp. calculated by formula T = 86·3 sin. D − 3°½ D, the dist. from the Asiatic Pole being = 39°·27	51 ·33
Difference between the observation and the formula	+1°·93
The mean temperature of the year 1827 at Nicolaieff was fully	52°¼

So that the formula gives a result within the varying limits of observations for different years, and differing very little from the mean result. D. B.]

XXXI. *On the Refraction of differently-coloured Rays in Crystals, with one and two Axes of Double Refraction.* By M. Rudberg.

[Continued from page 6.]

Section II. *Refraction in Crystals with two Optical Axes.*

THE crystals of this kind, which I have been able to procure, were arragonite, colourless topaz, and the topaz of Schneckenstein. I have not, however, been able to make use of the last of these, of which I have large and fine specimens, because throughout their interior there are cleavage planes, which being always parallel to the external planes reflect the sun's rays in so confused a manner, that the spectrum is not distinct. I have consequently been able to make experiments only with arragonite, and white or colourless topaz.

Before giving an account of these experiments, I shall briefly explain the theory of double refraction in crystals with two axes, because it is only by this elegant theory of Fresnel that we can find the directions in which the prisms must be cut. Fresnel founded his theory on two hypotheses, viz. 1. That in doubly refracting crystals the elasticity of the vibrating medium is different in different directions; and 2. That the vibrations of the light polarized are at the same time perpendicular to the direction of its propagation and to the plane of polarization.

He supposes that in every crystallized substance there are three directions perpendicular to each other, called axes of elasticity, according to which the elasticity may in general be different. If the elasticity is the same in all these three directions, the crystal belongs to the regular system, and has no double refraction. If it is equal in two directions, the crystal refracts doubly, and has *one* optical axis; and if the elasticity is unequal in all the three directions, the crystal has *two* optical axes. From this difference of elasticity there results for light a different velocity, which ought necessarily, in general, to become unequal for the two rays into which the light becomes divided itself, and whose planes of polarization are perpendicular to each other. There are in crystals with two optical axes only two directions; that of the axes themselves, in which the two rays are propagated with the same velocity. Consequently, in order to appreciate the velocity of the two rays in any direction, we must determine their planes of polarization, which is done by the following considerations. The plane in which the two optical axes are situated contains also two of the axes of crystallization, one of which bisects the acute, and the other the obtuse angle of the optical axes. If we conceive, then, two planes passing through the direction in which we wish to have the velocity of the two rays, and respectively through each of the optical axes, the plane which bisects the angle formed by these two planes will be the plane of polarization of one of the rays, that of the other being perpendicular to this plane, and passing through the given direction.

It follows from this, that if the light comes in a direction perpendicular to one of the axes of crystallization, one of the rays ought to have its plane of polarization perpendicular to this axis. The velocity with which these vibrations are propagated, depending only on the elasticity in the direction of this axis, it is evident that it remains the same whatever be the direction of the ray in the plane perpendicular to the axis. The other ray, on the contrary, whose plane of polarization passes through the axes, and consequently changes with its direction, will have different velocities in different directions, because its vibrations being always made in the plane of the other two axes of crystallization, may become successively parallel to both of these axes, and consequently undergo every change of velocity of propagation which the difference of elasticity in these two directions admits.

If therefore we cut a prism in such a manner that its edge is parallel to one of the axes of crystallization, that of the two rays whose plane of polarization is perpendicular to the axis ought to have a constant velocity, and follow in its refraction the law of Descartes (Snellius). The velocity of the other ray depends on its direction in reference to the other two axes of crystallization. Having thus cut three prisms, each of which had its edge respectively parallel to one of the axes of crystallization, and determining in each prism the index of refraction of the ray whose velocity remains invariable, we shall have *the three elements* on which the double refraction of the crystal depends.

The exposition of the results of the mathematical theory of Fresnel will illustrate still better what we have said. Calling, in the spirit of the system of emanation, v', v'' the velocities of the two rays, ε', ε'' the angles which the two optical axes form with the common direction of the rays, we shall have the velocity of one of these by the equation

$$v'^2 = A + B \,.\, \sin^2 \tfrac{1}{2} (\varepsilon' - \varepsilon''),$$

and that of the other by the equation

$$v''^2 = A + B \,.\, \sin \tfrac{1}{2} (\varepsilon' + \varepsilon''),$$

in which A and B are constants.

It has already been remarked, that two of the axes of crystallization are situated in the same plane as the optical axes, and that the third is perpendicular to this plane. I shall in the sequel call the axis of crystallization which bisects the acute angle of the optical axes the *axis A*, that which bisects the obtuse angle the *axis B*, and that which is perpendicular to the plane of the optical axes the *axis C.* From the preceding observations we conclude,

1. *If the edge of the prism is parallel to the axis A*, and if the two rays are consequently refracted in a plane perpendicular to this axis, we shall always have, if the angles ε' and ε'' are reckoned from the axis A, $\varepsilon' + \varepsilon'' = 180°$, and therefore

$$v'^2 = A + B \,.\, \cos^2 \varepsilon'', \text{ and } v''^2 = A + B.$$

This last velocity is constant, and is that of the ray whose plane of polarization is perpendicular to the axis A.

The velocity of the other ray depends on the value of the angle ε'', which may vary from 90° to 90° $- \frac{1}{2}\,\alpha$, calling α the acute angle of the optical axes. The value of the square of this velocity will thus vary

between A and $A + B \sin^2 \frac{1}{2}\,\alpha$.

2. *If the edge of the prism is parallel to the axis B,* we have always $\varepsilon' = \varepsilon''$, and consequently

$$v'^2 = \text{A and } v''^2 = \text{A} + \text{B} \sin^2 \varepsilon''.$$

The velocity v' is in this prism constant, and belongs to the ray which is polarized in a plane perpendicular to the axis B.

The velocity of the other ray depends on the value of ε'' between the limits $\frac{1}{2}\alpha$ and 90°. Hence the square of the velocity may vary

$$\text{between A} + \text{B and A} + \text{B} \sin^2 \tfrac{1}{2}\alpha.$$

3. *If the edge of the prism is parallel to the axis C,* we shall always have $\varepsilon' = \varepsilon'' + \alpha$; hence

$$v'^2 = \text{A} + \text{B} \,.\, \sin^2 \tfrac{1}{2}\alpha, \text{ and}$$
$$v''^2 = \text{A} + \text{B} \,.\, \sin^2 (\varepsilon'' + \tfrac{1}{2}\alpha).$$

In this prism the velocity v' is constant, and belongs to the ray whose plane of polarization is perpendicular to the axis C.

As the angle ε'' may have different values from $90° - \frac{1}{2}\alpha$ to $-\frac{1}{2}\alpha$, the square of the velocity of the other ray will vary

$$\text{between A and A} + \text{B}.$$

If in the three prisms, cut as now described, we observe the deviation of the ray, whose velocity remains constant independently of the direction, and if we calculate the index of refraction, we shall have the values of three quantities A, B and z. Calling n' the index in the prism whose edge is parallel to A, n'' that in the prism whose edge is parallel to B, and n''' that in the prism whose edge is parallel to C, we shall have, the velocity of light in air being taken as unity,

$$n'^2 = \text{A} + \text{B},\ n''^2 = \text{A},\ n'''^2 = \text{A} + \text{B} \,.\, \sin^2 \tfrac{1}{2}\alpha,$$

and consequently

$$\text{A} = n''^2,\ \text{B} = n'^2 - n'''^2, \text{ and } \sin^2 \tfrac{1}{2}\alpha = \frac{n''^2 - n'''^2}{n'^2 - n'''^2}.$$

I come now to an account of my experiments.

Arragonite.—Out of a crystal of this mineral from Bohemia, I cut three prisms:

1. *The prism A* having the edges of its refracting angles parallel to the axis A of the pyramidal crystal. A, No. 1, and A, No. 2, are two different refracting angles.

2. *The prisms B* had their edges parallel to the axis B; the two are marked B, No. 1. and B, No. 2.

3. *The prisms C* had their edges parallel to the axis C; two of them thus cut are named C, No. 1. and C, No. 2. The light which moves with a constant velocity may be known by its passing through a plate of tourmaline, having its axis parallel to the edge of the prism.

The prism A, No. 1.—Refracting angle 66° 43′ 17″. Temp. + 19° cent. In the spectrum, where the deviations were the greatest, the ray F was brought to a minimum of deviation, and in the other spectrum the ray H.

The prism A, No. 2.—Refracting angle 51° 48′ 31″. Temp. + 18° cent. The rays F in the two spectra were brought to a minimum of deviation.

The following are the indices of refraction for the spectrum, whose plane of polarization is perpendicular to A.

Fixed lines.	Prism A, No. 1.	Prism A, No. 2.	Diff.
H	1·54226	1·54225	= 0·00001
G	1·53880	1·53885	— 0·00005
F	1·53480	1·53478	+ 0·00002
E	1·53264	1·53265	— 0·00001
D	1·53015	1·53011	+ 0·00004
C	1·52818	1·52822	— 0·00004
B	1·52747	1·52751	— 0·00004

These differences are evidently errors of observation, and the invariability of the velocity of the ray polarized perpendicularly to the axis A is consequently well established. With respect to the other ray, its velocity cannot be constant according to theory; and this is proved by observation, as is shown by the two following measures in the spectrum whose plane of polarization passes through A.

	Prism A, No. 1.	Prism A, No. 2.	Diff.
H	1·70996	1·70590	0·00406
F	1·69502	1·69128	0·00374

Prism B, No. 1.—Refracting angle 36° 13′ 30″. Temp. + 18°. In the spectrum with the greatest deviations, the ray F was reduced to a minimum of deviation, and in the other the ray H. The same was the case in

Prism B, No. 2.—Refracting angle 40° 12′ 3″. Temp. + 18°. The following were the indices of refraction.

	Prism B, No. 1.	Prism B, No. 2.	Diff.
H	1·71019	1·71004	0·00015
G	1·70325	1·70311	0·00014
F	1·69520	1·69510	0·00010
E	1·69091	1·69078	0·00013
D	1·68595	1·68583	0·00012
C	1·68206	1·68200	0·00006
B	1·68066	1·68057	0·00009

The differences here, though greater than in prism A, owing to the difficulty of cutting a face exactly perpendicular to the plane of the optical axes, are yet sufficient to prove the invariability of the velocity of the ray polarized perpendicularly to B. That the velocity of the rays in the other spectrum varies, is proved by the two following observations.

	Prism B, No. 1.	Prism B, No. 2.	Diff.
H	1·54242	1·54277	0·00035
G	1·53493	1·53529	0·00036.

Prism C, No. 1.—Refracting angle 29° 43′ 21″. Temp. + 17°. The ray H was in both spectra brought to a minimum deviation.

Prism C, No. 2.—Refracting angle 41° 34′ 32″. Temp. + 16°. In the spectrum of greatest deviation the ray F, and in the least the ray H, was brought to a minimum deviation.

The following were the refractive indices in the spectrum, whose plane of polarization was perpendicular to the axis C.

	Prism C, No. 1.	Prism C, No. 2.
H	1·70512	1·70505
G	1·69830	1·69843
F	1·69049	1·69058
E	1·68634	1·68635
D	1·68157	1·68156
C	1·67777	1·67781
B	1·67632	1·67630

The indices vary in the other spectrum, as the following observations show.

	Prism C, No. 1.	Prism C, No. 2.
H	1·55043	1·56158
F	1·54265	1·55331.

All these observations incontestably confirm the fundamental theorem of Fresnel, *that the velocity of one ray is invariable as long as its plane of polarization remains the same.*

The following means of the two systems of indices for the three spectra, whose planes of polarization are perpendicular to the three axes of crystallization, exhibit the elements of refraction of arragonite.

	Axis A.	Axis B.	Axis C.
H	1·54226	1·71011	1·70509
G	1·53882	1·70318	1·69836
F	1·53479	1·69515	1·69053
E	1·53264	1·69084	1·68634
D	1·53013	1·68589	1·68157
C	1·52820	1·68203	1·67779
B	1·52749	1·68061	1·67631

Calling n', n'' and n''' the indices for the spectra polarized perpendicularly to the axes A, B and C, and calculating the ratios $\frac{n'''}{n'}$ and $\frac{n'''}{n''}$, we shall find

	Ratio $\frac{n'''}{n'}$	Ratio $\frac{n'''}{n''}$
H	1·10883	1·00294
G	1·10681	1·00284
F	1·10449	1·00273
E	1·10322	1·00267
D	1·10154	1·00257
C	1·10066	1·00253
B	1·10024	1·00256

Hence every colour in arragonite has a double refraction as much greater as it is more refrangible. This result agrees with that for rock crystal and Iceland spar; and we may therefore conclude in general, that

*Each colour has its individual double refraction as much greater as its own refrangibility is greater**.

By means of the preceding values of the indices n', n'' and n''', we may calculate the angle of inclination α of the optical axes by the formula $\sin^2 \frac{1}{2} \alpha = \frac{n'''^2 - n''^2}{n'''^2 - n'^2}$ · as in the following table.

H	20°	25′	6″
G	20	12	6
F	20	0	50
E	19	53	0
D	19	37	8
C	19	33	14
B	19	44	40

Hence we see that in arragonite *the inclination of the optical axes diminishes continually from the violet to the red light.*

Dr. Brewster gives for the true inclination of the optical axes 18° 18′, calculated from the observed apparent inclination. But as he has not given the value of this apparent inclination, nor the index which he made use of to calculate the true inclination, it is impossible to compare his result with that of my experiments†.

Having several times measured the apparent inclination of the axes by means of a plate with parallel faces cut perpendicularly to the axis A, I found it a little more than 32°. To make a comparison with this value, we must calculate the *apparent inclinations from the true inclinations* as given in the above table. This is easily done; since we can now determine the velocity of light in the direction even of an optical axis. If we insert in the formulæ, given at the beginning of this section (p. 138), $\varepsilon'' = o$, and $\varepsilon' = \alpha$, we obtain

$$v'^2 = v''^2 = n'''^2 - (n'''^2 - n'^2) \sin^2 \tfrac{1}{2} \alpha$$

in which $v' = v'' = n'$.

* See our last Number, p. 6. † See the next Article.

The ray which in emerging from the plate deviates according to the law of Descartes (Snellius), takes a direction without, making with the normal of the plate an angle $\frac{1}{2}i$, which is calculated by the formula $\sin \frac{1}{2} i = n'' \sin \frac{1}{2} \alpha$.

The following are the values of i for the different colours.

Apparent Inclination of the Optical Axes.

H . .	35° 10′ 54″
G . .	34 39 30
F . .	34 10 0
E . .	33 51 10
D . .	33 17 46
C . .	33 6 24
B . .	33 24 22

The mean inclination is about 34°, which differs about 2° from the observed inclination. Notwithstanding the difficulty of taking this angle with precision, the difference of 2° is still too great. I cannot tell the cause, unless the two rays, which, within the plate, pass along the same optical axes, separate at their egress.

The preceding experiments having demonstrated that the ratio of the indices of refraction varies in the three spectra with the colours, the true ratio between the elasticities of the vibrating medium along the three axes of crystallization cannot be determined. If we take the elasticity of the vibrating medium in air to be unity, the elasticity along the axis A will be $= \frac{1}{n'^2}$, along $B = \frac{1}{n'''^2}$, and along $C = \frac{1}{n''^2}$; since the velocities being $\frac{1}{n'}$, $\frac{1}{n'''}$, and $\frac{1}{n''}$ in the system of undulations are as the square roots of the elasticity. But when the ratios $\frac{n'^2}{n'''^2}$ and $\frac{n'^2}{n''^2}$ change with the colours, they do not express exactly the ratios of the elasticity along the three axes of crystallization. However, in taking the elasticity along the axis A as unity, and calculating the above ratios for one of the middle rays of the spectrum, such as F, we shall have the following elasticities for Arragonite.

A	B	C
1	0·81975	0·82424

And for calcareous spar,

Along Axis.	Perpendicular to Axis.
1	0·79874

Colourless Topaz.—The prisms of this mineral were cut in the same manner as those of arragonite. As the two spectra always cover one another, I used a plate of tourmaline to separate them in the manner already described for rock crystal.

Prism A, No. 1. — Refracting angle 30° 15′ 29″. Temp. +19°. In the spectrum of greatest deviation, the ray F was reduced to the minimum deviation, and in the other the ray H.

Prism A, No. 2.—Refracting angle 42° 40′ 16″.

The following were the indices observed in the spectrum perpendicular to A.

	Prism A, No. 1.	Prism A, No. 2.	Diff.
H . .	1·63506 . .	1·63490 . .	+0·00016
G . .	1·63123 . .	1·63140 . .	−0·00017
F . .	1·62652		
E . .	1·62408		
D . .	1·62109		
C . .	1·61880		
B . .	1·61791		

And in the spectrum polarized parallel to A.

	Prism A, No. 1.	Prism A, No. 2.	Diff.
H . .	1·62551 . .	1·62758 . .	0·00207
G . .	1·62156 . .	1·62374 . .	0·00010

Prism B.—Refracting angle 49° 3′ 8″. Temp. +19°. In both spectra the ray H was brought to the minimum deviation.

Prism C.—Refracting angle 38° 38′ 54″. Temp. 16°. In both spectra the ray H was brought to a minimum deviation.

The following are the indices for the spectrum polarized perpendicularly to the axes A, B and C.

	A.	B.	C.
H . .	1·63506 . .	1·62539 . .	1·62745
G . .	1·63123 . .	1·62154 . .	1·62365
F . .	1·62652 . .	1·61701 . .	1·61914
E . .	1·62408 . .	1·61452 . .	1·61668
D . .	1·62109 . .	1·61161 . .	1·61375
C . .	1·61880 . .	1·60935 . .	1·61144
B . .	1·61791 . .	1·60840 . .	1·61049

And the ratios $\frac{n'}{n''}$, $\frac{n'}{n'''}$ will be as follows:

	Ratio $\frac{n'}{n''}$	Ratio $\frac{n'}{n'''}$
H . .	1·00466 . .	1·00595
G . .	1·00467 . .	1·00597
F . .	1·00456 . .	1·00588
E . .	1·00458 . .	1·00592
D . .	1·00455 . .	1·00588
C . .	1·00459 . .	1·00587
B . .	1·00461 . .	1·00591

These ratios differ so little from each other, that one would be led to regard the differences as only errors of observation. They, however, appear to increase a little from the violet to the red, and consequently do not contradict the result obtained for Iceland spar, rock crystal and Arragonite.

The inclinations of the optical axes calculated by the formula $\sin^2 \frac{1}{2} \alpha = \frac{n''^2 - n'''^2}{n'^2 - n'''^2}$ are as follows:

Inclination of the Optical Axes.

H . .	54° 54′ 0″
G . .	55 34 24
F . .	56 37 24
E . .	56 40 30
D . .	56 37 30
C . .	56 3 0
B . .	55 51 58

Abstracting the irregularities in these values towards the red extremity of the spectrum, it appears that *the inclination of the optical axes goes on diminishing with the refrangibility of the rays*, whilst the contrary takes place in Arragonite.

With regard to the value of the inclination, Dr. Brewster has found it = 65°, and M. Biot = 64° 14′. This difference of more than 8°, appears to indicate errors in the determination of the indices, unless the inclination in different specimens of colourless topaz is different, as Dr. Brewster found it to be for different kinds of Brazil topaz. It is to be observed, that all the prisms with which I made the preceding observations, came from the same topaz. Having after this only thin plates, I could not, on account of the great extent of the elliptical rings, measure the inclination of the axes with precision.

Taking for topaz as for Arragonite the elasticity along the axis A as unity, the following will be the elasticities along the other axes.

A.	B.	C.
1	1·01186	1·00922

In his memoir on Double Refraction (*Mém. de l'Institut*, tom. vii.) Fresnel has given from his experiments on diffraction made with colourless topaz, the ratio between the least and greatest velocity. He found it 0·9938. My experiments make the mean result $\frac{n'''}{n'} = \frac{1}{1{\cdot}00591} = 0{\cdot}99412$, which exceeds the former by 0·0003. Assuming the ratio 0·9938, and the inclination of the optical axes = 65°, the ratio $\frac{n''}{n'}$ may be found by the equation,

$$\frac{n''^2}{n'^2} = \frac{n'''^2}{n'^2} + \left(1 - \frac{n'''^2}{n'^2}\right) \sin^2 \tfrac{1}{2} \alpha.$$

We thus obtain 0·99560. My experiments give $\frac{n''}{n'} = 0{\cdot}99542$, which is 0·00018 in defect. These differences, however, evidently arise, on the one hand, from the difficulty of determining these ratios by experiments on refraction with prisms differently cut, with a precision comparable to that which we obtain by means of experiments on diffraction; and on the other hand from the probable inaccuracy in the value of the inclination of the optical axes found by the observation of the coloured rings.

XXXII. *Observations on the preceding Memoir.* By SIR DAVID BREWSTER, *K.H. LL.D. F.R.S. V.P.R.S. Ed.*

NOTWITHSTANDING the great accuracy and value of the preceding observations, and the importance of the deductions which the author has drawn from them, yet we are constrained to state, that almost all the general principles at which M. Rudberg has arrived have been previously established by English philosophers, though not by observations made by means of the fixed lines in the spectrum.

The variation of the inclination of the optic axes with the different colours of the spectrum, and the increase of that angle with the refrangibility of the colour in some crystals, such as Arragonite, and its decrease with the refrangibility in other crystals, such as topaz, is the discovery of Sir John F. W. Herschel, and one of the most important that has been made on the subject of double refraction; and yet the name of Sir John Herschel is not once mentioned. Sir John indeed did not examine *Arragonite* and *topaz*, but he found the very same phænomena in *sulphate of barytes* and *Rochelle salts;* and as I had myself discovered that all those crystals in which the inclination of the optic axes increases with the refrangibility, have the *red* ends of their systems of rings *inwards*, or towards the axis A; while those in which this inclination decreases with the refrangibility have the *red* ends of their rings outwards, or towards the axis B, and had determined that *Arragonite* had the red ends of its rings inwards, and *topaz* the red ends outwards*; the variation of the inclination of the optic axes in these two minerals, and its inverse character, were both known before M. Rudberg's experiments. To M. Rudberg, however, there remains the merit of having given the values of these angles, and that too in reference to fixed points of the spectrum.

It is impossible to overlook the great difference between theory and observation in the inclination of the optic axes of Arragonite and topaz as given by M. Rudberg. His observa-

* Art. *Optics*, in the *Edinburgh Encyclopædia*, vol. xv. p. 596.