The Flexure of a Circular Plate. By J. H. Michell.

Received November 24th, 1901. Read December 12th, 1901.

1. The flexure of a thin circular plate, clamped along its edge and loaded in any manner, has been discussed by Clebsch in his *Theorie der Elasticität.** He obtains expressions in series for the deflection due to a concentrated load at any point of the plate and remarks on the complexity of the result. The process of inversion, described in a previous paper,† leads to the solution for an eccentric load by the inversion of that for a central load, and thus reduces the former to the simplicity of the latter. Mathematically, the problem is nothing more than the expression of "Green's function" for the differential equation $\nabla^4 w = 0$ within a circular boundary. The solution of the similar problem of the two-dimensional flow of viscous liquid within a circular boundary is merely noted. Rayleigh‡ has already treated this problem in another manner.

2. The deflection w of the portions of a plate§ free from load satisfies the differential equation $\nabla^4 w = 0$, and the conditions at the clamped edge are $w = 0$, $\partial w/\partial n = 0$, δn being an element of normal to the edge. Inverting with respect to the origin O of the polar coordinates (r, θ), we obtain a solution $w' = w/r^2$ in the inverse plane, satisfying $\nabla^4 w' = 0$ in that plane and also, clearly, satisfying the conditions $w' = 0$, $\partial w'/\partial n' = 0$ for a clamped edge, along the inverse of the clamped edge in the original plane.

In the neighbourhood of a concentrated load W, w takes the form $\kappa W r_1^2 \log r_1 + w_1$, where r_1 is the distance from the point of application C of the load, κ depends on the stiffness of the plate, and w_1 is regular. This need not be formally proved, as it follows from the known solution quoted below. Let O' be the inverse of C with respect to O, and let ρ, ρ_1 be the distances from O, O' respectively of the inverse of the point at distances r, r_1 from O, C respectively.

* St. Venant's edition, § 76.

† "The Inversion of Plane Stress," *Proc. Lond. Math. Soc.*, Vol. xxxiv., p. 134.

‡ *Phil. Mag.*, Vol. xxxvi., 1893, p. 354.

§ The plate is supposed isotropic in its plane.

Taking unit radius of inversion and writing $OO = c$, we have

$$\frac{r_1}{\rho_1} = \frac{c}{\rho}, \quad r\rho = 1;$$

and therefore the inverse of the form $\kappa W r_1^2 \log r_1$ is

$$\kappa W c^2 \rho_1^2 \log \rho_1 \frac{c}{\rho} \quad \text{or} \quad \kappa W c^2 \rho_1^2 \log \rho_1 + \kappa W c^2 \rho_1^2 \log \frac{c}{\rho}.$$

Thus there is a concentrated load Wc^2 at the point O'. The fact that O is also a singular point is immaterial, as we shall suppose it outside the plate.

3. The relation between the deflections w, w' at corresponding points P, P' may be written $w/OP = w'/OP'$. Remembering the smallness of w, w', this shows that the deflected position of the point P' is the inverse of that of P, or that the strained form of one plate is the geometrical inverse of that of the other. Now it is known that the lines of curvature of a surface invert into lines of curvature. Hence the lines of curvature or flexure of the two strained plates correspond. The quantitative relations between the stresses in the two plates are readily obtained. The couples on a polar element of a plate can be written

$$K_1 = C\nabla^2 w - C(1-\sigma)\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right),$$

$$K_2 = C\nabla^2 w - C(1-\sigma)\frac{\partial^2 w}{\partial r^2},$$

$$H = C(1-\sigma)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right);$$

and, as in the previous paper, these make the couples on the polar element at the corresponding point in the inverse solution

$$K_1' = C\left\{r^2\nabla^2 w + 4\left(w - r\frac{\partial w}{\partial r}\right)\right\}$$
$$- C(1-\sigma)\left\{r^2\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) + 2\left(w - r\frac{\partial w}{\partial r}\right)\right\},$$

$$K_2' = C\left\{r^2\nabla^2 w + 4\left(w - r\frac{\partial w}{\partial r}\right)\right\} - C(1-\sigma)\left\{r^2\frac{\partial^2 w}{\partial r^2} + 2\left(w - r\frac{\partial w}{\partial r}\right)\right\},$$

$$H' = -C(1-\sigma)r^2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right),$$

so that
$$K_1' = r^2 K_1 + 2C(1+\sigma)\left(w - r\frac{\partial w}{\partial r}\right),$$
$$K_2' = r^2 K_2 + 2C(1+\sigma)\left(w - r\frac{\partial w}{\partial r}\right),$$
$$H' = -r^2 H.$$

It follows that the resultant couple across a line-element $\delta s'$ of the inverse plate is compounded of (*a*) a couple equal to that across the corresponding line-element δs in the original plate and in the corresponding direction, (*b*) a couple of magnitude $2C(1+\sigma)\left(w - r\dfrac{\partial w}{\partial r}\right)\delta s'$ with its axis along the element. In particular, it follows again that the lines of principal flexure in the two plates correspond.

4. The deflection of a clamped circular plate of radius a due to a central load W is*

$$w = \kappa W\left(\frac{a^2 - r_1^2}{2} - r_1^2 \log\frac{a}{r_1}\right).$$

Invert with respect to the point O outside the plate as in § 2. If a is the radius of the new plate, C' its centre, and $O'C' = h$, then $a/c = h/a$; and therefore

$$w' = \kappa W c^2 \left\{ \tfrac{1}{2}\rho_1^2 \left(\frac{h^2\rho^2}{a^2\rho_1^2} - 1\right) - \rho_1^2 \log\frac{h\rho}{a\rho_1} \right\}.$$

Hence, changing the notation to a more convenient, the deflection w, due to a load W on a clamped circular plate of radius a at a point distant h from the centre, is given by

$$w = \kappa W r^2 \left\{ \tfrac{1}{2}\left(\frac{h^2 r'^2}{a^2 r^2} - 1\right) - \log\frac{h r'}{a r} \right\},$$

where r is the distance from the load and r' that from the inverse point.

If the radius of the circle is made infinite, we obtain the solution for the deflection of an infinite plate clamped along an infinite straight boundary, due to a concentrated load. The solution is

$$w = \kappa W r^2 \left\{ \tfrac{1}{2}\left(\frac{r'^2}{r^2} - 1\right) - \log\frac{r'}{r} \right\},$$

* See *e.g.*, St. Venant's *Clebsch*, p. 777.

where the inverse point is now the image of the load in the boundary. With my brother's assistance I have drawn the accompanying figures showing the contour lines (Fig. I.) for a circular plate loaded at a point bisecting a radius (indicated by a cross), (Fig. II.) for an infinite plate loaded at a point similarly indicated. Fig. I. (*a*) shows the deflection along the diameter of the circle on which the load is, Fig. I. (*b*) the deflection along the perpendicular diameter (inner curve), and the profile of the plate viewed from a distant point on the former diameter (outer curve).

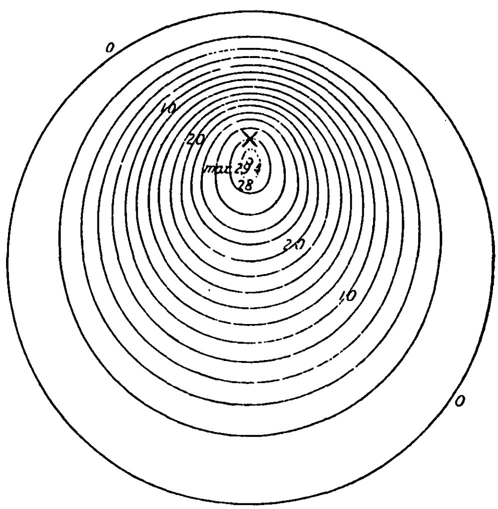

Fig. I.

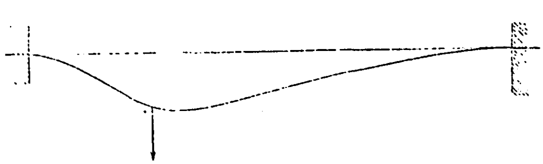

Fig. I. (*a*).

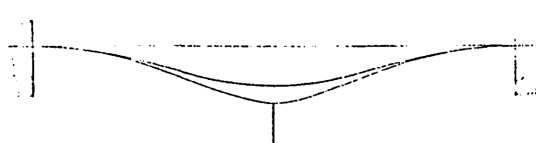

Fig. I. (*b*).

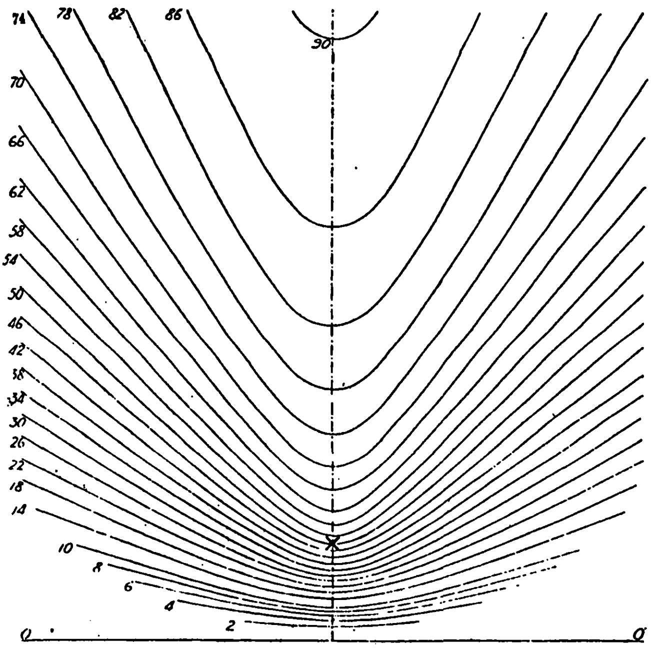

FIG. II.

It should perhaps be mentioned that I have verified the series of Clebsch as given in Todhunter and Pearson's *History*, Vol. II., § 1381, except for the terms $r_0^3/4b^2+r_0\log b+r_0$, which I make $r_0^3/4b^2-r_0\log b-r_0$.

5. As I have not been able to find any statement of the form of Green's function for $\nabla^4 w=0$, implied in the solution of the last section, it may be as well to put down the integrals which give the value of w within a circle, when the values of w and $\partial w/\partial n$ are given over the circumference.*

Taking $$w=\tfrac{1}{2}\left(\frac{h^2}{a^2}r'^2-r^2\right)-r^2\log\frac{hr'}{ar},$$

we readily deduce $$\nabla^2 w=2\frac{(a^2-h^2)^2}{a^2r^2},$$

$$\frac{\partial}{\partial n}\nabla^2 w=2\,(a^2-h^2)^2\frac{r^2+a^2-h^2}{a^3r^4},$$

as holding over the circumference.

* The method of Rayleigh, *loc. cit.*, solves the problem without the use of the Green's function.

Hence at the point O

$$w = -\frac{(a^2-h^2)^2}{4\pi a^2}\int\frac{\partial w}{\partial n}\,\frac{1}{r^3}\,ds+\frac{(a^2-h^2)^2}{4\pi a^3}\int w\,\frac{r^2+a^2-h^2}{r^4}\,ds$$

$$= -\frac{(a^2-h^2)^2}{4\pi a^2}\int\frac{\partial w}{\partial n}\,\frac{1}{r^3}\,ds+\frac{(a^2-h^2)^2}{2\pi a^2}\int w\,\frac{\cos\phi}{r^3}\,ds,$$

where ϕ is the inclination of δr to the outward normal δn.

It may be remarked that the Green's function for the equation $\nabla^4 w = 0$ within a sphere is

$$w = r - \frac{h}{a}\,r' + \frac{a^2-h^2}{2ah}\,\frac{a^2-\rho^2}{r'},$$

where ρ is the distance from the centre of the sphere, and the rest of the notation is the same as before.

There is a correspondingly simple formula for the stream function ψ for the slow motion of viscous liquid due to a source at the point O within a circular boundary together with an equal sink at the centre C. It is

$$\psi = \theta+\theta'-\vartheta+(a^2-\rho^2)\,\frac{\sin\theta'}{hr'},$$

where θ, θ', ϑ are the polar angles at O, O', C respectively. The stream function for the slow motion due to a line vortex at O in a circular cylinder is, with the same notation,

$$\psi = \log\frac{hr'}{ar}-(a^2-\rho^2)\,\frac{1}{h}\left(\frac{\cos\theta'}{r'}-\frac{1}{2}\,\frac{h}{a^2}\right).$$

Thursday, January 9th, 1902.

Dr. HOBSON, F.R.S., President, in the Chair.

Fourteen members and a visitor present.

The Rev. J. Cullen was admitted into the Society.

Major MacMahon, Vice-President, having taken the Chair, [the President communicated a paper "On Non-uniform Convergence, and the Integration of Series." Remarks upon the subject were made by Messrs. Larmor, Love, S. Roberts, Whittaker, and the Chairman.

The following papers were taken as read:—

On the Integrals of the Differential Equation

$$\frac{du}{\sqrt{f(u)}}+\frac{dv}{\sqrt{f(v)}}=0,$$

where $$f(x)\equiv ax^4+4bx^3+6cx^2+4dx+e,$$

considered geometrically: Prof. W. S. Burnside.

On the Fundamental Theorem of Differential Equations: Mr. W. H. Young.

The following presents were made to the Library:—

"Educational Times," January, 1902.

"Indian Engineering," Vol. xxx., Nos. 21-24, Nov. 23-Dec. 14, 1901.

Dickson, L. E.—"Theory of Linear Groups in an Arbitrary Field," 4to; 1901.

The following exchanges were received:—

"Proceedings of the Royal Society," Vol. LIX., No. 453; 1901.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. xxv., Heft 12; Leipzig, 1901.

"Rendiconti del Circolo Matematico di Palermo," Tomo xv., Fasc. 5, 6; 1901.

"Bulletin of the American Mathematical Society," Series 2, Vol. VIII., No. 3; New York, 1901.

"Bulletin des Sciences Mathématiques," Tome xxv., Nov., 1901; Paris.

"Rendiconto dell'Accademia delle Scienze Fisiche e Matematiche," Vol. VII., Fasc. 8-11, Aug.-Nov., 1901; Napoli.

"Archives Néerlandaises," Série 2, Tome IV., Liv. 4, 5; La Haye, 1901.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 2, Vol. x., Fasc. 11; Roma, 1901.

"Proceedings of the Physical Society," Vol. XVII., Pt. 7; Dec., 1901.

"Tōkyō Sūgaku-Butsurigaku Kwai Kiji," Maki 8, Dai 6; 1901.

"Periodico di Matematica," Anno XVI., Fasc. 3; Livorno, 1901.

"Supplemento al Periodico di Matematica," Anno v., Fasc. 2; Livorno, 1901.

On the Integrals of the Differential Equation

$$\frac{du}{\sqrt{f(u)}} + \frac{dv}{\sqrt{f(v)}} = 0,$$

where $f(x) \equiv ax^4 + 4bx^3 + 6cx^2 + 4dx + e$,

considered Geometrically. By W. Snow Burnside.
Received January 1st, 1902. Communicated January 9th, 1902.

If the coordinates of a point on a conic be expressed as quadric functions of a single variable ϕ, viz.,

$$\left.\begin{aligned} kx &= a_0\phi^2 + 2b_0\phi + c_0 \\ ky &= a_1\phi^2 + 2b_1\phi + c_1 \\ kz &= a_2\phi^2 + 2b_2\phi + c_2 \end{aligned}\right\}, \tag{1}$$

the equation of the chord joining two points u, v on the conic is

$$\begin{aligned} &x\left\{(a_1b_2)\,uv + (a_1c_2)\frac{u+v}{2} + (b_1c_2)\right\} \\ &+y\left\{(a_2b_0)\,uv + (a_2c_0)\frac{u+v}{2} + (b_2c_0)\right\} \\ &+z\left\{(a_0b_1)\,uv + (a_0c_1)\frac{u+v}{2} + (b_0c_1)\right\} = 0, \end{aligned}$$

and, if this chord touch any conic, we obtain an equation $\Sigma = 0$ quadric in the variables uv and $u+v$, whence we may write Σ in the following forms:—

$$\Sigma = L_0v^2 + 2L_1v + L_2 = M_0u^2 + 2M_1u + M_2,$$

where L_0, L_1, L_2 are quadric functions of u, and M_0, M_1, M_2 are quadric functions of v.

Now, differentiating the equation $\Sigma = 0$,

$$\frac{d\Sigma}{du}\,du + \frac{d\Sigma}{dv}\,dv = 0,$$

which, in virtue of the equation $\Sigma = 0$, may be written in the form

$$\sqrt{M_1^2 - M_0M_2}\;du + \sqrt{L_1^2 - L_0L_2}\;dv = 0.$$

Again, since Σ is symmetrical in u and v, we obtain an equation of the form

$$\frac{du}{\sqrt{F(u)}}+\frac{dv}{\sqrt{F(v)}}=0.$$

We have thus arrived at a differential equation of the same type as

$$\frac{du}{\sqrt{f(u)}}+\frac{dv}{\sqrt{f(v)}}=0,$$

the integral of the former being $\Sigma=0$. We now proceed to show that, if the two conics are properly selected, $F(u)$ becomes identical with $f(u)$, no reductions being required.

The conics which serve our purpose are

$$U\equiv ax^2+cy^2+ez^2+2dyz+2czx+2bxy,$$
$$V\equiv y^2-4zx,$$

and we proceed to find the condition that the chord u, v of V should touch $U-\rho V$.

Let the tangential equation of $U-\rho V$ be written in the form

$$\Sigma=\Sigma_0-2\rho\Phi+\rho^2\Sigma_1=0;$$

then $\Sigma_0=(ce-d^2)\lambda^2+(ae-c^2)\mu^2+(ac-b^2)\nu^2$

$$+2(bc-ad)\mu\nu+2(bd-c^2)\nu\lambda+2(cd-be)\lambda\mu,$$

$$2\Phi=e\lambda^2+4c\mu^2+a\nu^2-4b\mu\nu+2c\nu\lambda-4d\lambda\mu,$$

$$\Sigma_1=4(\gamma a-\beta^2),$$

where, if $\lambda x+\mu y+\nu z$ be the chord joining the points u and v on V, $\lambda=1$, $\mu=-\dfrac{u+v}{2}$, $\nu=uv$, the coordinates of a point on V, being given by the equations $\dfrac{x}{\phi^2}=\dfrac{y}{2\phi}=z$, the simplest form of equations (1).

Differentiating Σ and supposing ρ variable, we have now

$$\tfrac{1}{2}d\Sigma=\sqrt{M_1^2-M_0M_2}\,du+\sqrt{L_1^2-L_0L_2}\,dv+\sqrt{\Phi^2-\Sigma_0\Sigma_1}\,d\rho,$$

since Σ is a quadric function of each of the variables u, v, ρ considered separately.

We proceed to reduce the last equation to the form

$$d\Sigma=i\sqrt{\Delta(\rho)f(v)}\,du+i\sqrt{\Delta(\rho)f(u)}\,dv+\sqrt{f(u)f(v)}\,d\rho,$$

where
$$i^2 = -1,$$
$$\Delta(\rho) = 4\rho^3 - I\rho + J,$$
the discriminant of the conic $U-\rho V$.

In order to calculate $L_1^2 - L_0 L_2$, let the tangential form of $U-\rho V$ be
$$\Sigma \equiv A\lambda^2 + B\mu^2 + C\nu^2 + 2F\mu\nu + 2G\nu\lambda + 2H\lambda\mu$$
$$= \left(Cv^2 - Fv + \frac{B}{4}\right)u^2 - \left\{Fv^2 - \left(2G + \frac{B}{2}\right)v + H\right\}u + B\frac{v^2}{4} - Hv - A,$$
since
$$\lambda = 1, \quad \mu = -\frac{u+v}{2}, \quad \nu = uv,$$
whence
$$4(L_0L_2 - L_1^2) = (BC - F^2)v^4 + 4(FG - CH)v^3$$
$$+ 2\{(HF - BG) + 2(AC - G^2)\}v^2$$
$$+ 4(HG - AF)v + AB - H^2,$$
every term of which is, from the theory of determinants, divisible by
$$\Delta(\rho) \equiv 4\rho^3 - I\rho + J,$$
the discriminant of $U-\rho V$; whence, reducing, since
$$HF - BG = \Delta(\rho)(c + 2\rho),$$
$$AC - G^2 = \Delta(\rho)(c - \rho),$$
$$4(L_0L_2 - L_1^2) = \Delta(\rho)(av^4 + 4bv^3 + 6cv^2 + 4dv + e) = \Delta(\rho)f(v).$$
Similarly,
$$4(M_0M_2 - M_1^2) = \Delta(\rho)f(u).$$
It remains now only to reduce the coefficient of $d\rho$ in $d\Sigma$. And, since, from the theory of a system of two conics, it is, when equated to zero, the tangential equation of the points of intersection of the conics U and V, viz., the points $\alpha, \beta, \gamma, \delta$, when
$$f(x) = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta);$$
so we have
$$4(\Phi^2 - \Sigma_0\Sigma_1) \equiv \kappa \prod_1^4 (\lambda x_r + \mu y_r + \nu z_r).$$
By comparing the coefficients of ν^4 on both sides of this equation,
$$a^2 = \kappa . z_1 z_2 z_3 z_4.$$
Again, $\lambda \dfrac{x_r}{z_r} + \mu \dfrac{y_r}{z_r} + \nu = 1 . a^2 - 2\dfrac{u+v}{2}a + uv = (a-u)(a-v)$;

therefore, finally, $4(\Phi_2 - \Sigma_0\Sigma_1) = f(u)f(v)$.

Now, substituting these values for the coefficients of du, dv, $d\rho$ in $d\Sigma$ and changing the sign of $\rho = -\rho'$, we have

$$\Sigma \equiv \Sigma_0 + 2\rho'\Phi + \rho'^2\Sigma_1 = 0$$

which is a particular integral of the differential equation

$$\frac{du}{\sqrt{f(u)}} + \frac{dv}{\sqrt{f(v)}} - \frac{d\rho'}{\sqrt{4\rho'^2 - I\rho' - J}} = 0 \qquad \text{(A)}$$

and the general integral of

$$\frac{du}{\sqrt{f(u)}} + \frac{dv}{\sqrt{f(v)}} = 0,$$

when ρ' is an arbitrary constant.

When $\Delta(\rho_1) = 0$, the above investigation requires to be modified; for in this case

$$\Sigma \equiv \Sigma_0 - 2\rho_1\Phi + \rho_1^2\Sigma_1$$

is a perfect square of the form

$$\kappa(\lambda x_1 + \mu y_1 + \nu z_1)^2,$$

where x_1, y_1, z_1 are the coordinates of one of the vertices of the common self-conjugate triangle of the conics U and V, and, if this point be the intersection of the common chords (β, γ) and (α, δ),

$$\frac{x_1}{\beta\gamma(\alpha+\delta) - \alpha\delta(\beta+\gamma)} = \frac{y_1}{2(\beta\gamma - \alpha\delta)} = \frac{z_1}{\beta+\gamma-\alpha-\delta}.$$

$$\lambda x_1 + \mu y_1 + \nu z_1 = 0$$

then becomes

$$(\beta+\gamma-\alpha-\delta)\,uv - (\beta\gamma-\alpha\delta)(u+v) + \beta\gamma(\alpha+\delta) - \alpha\delta(\beta+\gamma) = 0.$$

In this way, corresponding to the three roots of $\Delta(\rho) = 0$, we obtain three particular integrals of the equation

$$\frac{du}{\sqrt{f(u)}} + \frac{dv}{\sqrt{f(v)}} = 0,$$

which can be easily verified directly, and when $u = v$ these integrals become the three quadratic factors of the sextic covariant of $f(u)$.

When the chord (uv) becomes a tangent to V it is interesting to ascertain how the previous results are modified. Therefore making $u = v$ in the equation

$$\Sigma \equiv \Sigma_0 + 2\rho'\Phi + \rho'^2\Sigma_1 = 0,$$

Σ_0 becomes the Hessian of $f(u)$, 2Φ becomes $f(u)$, and Σ_1 vanishes; whence

$$\rho' = -\frac{H(u)}{f(u)}.$$

Again, the differential equation (A) becomes

$$2\frac{du}{\sqrt{f(u)}} = \frac{d\rho'}{\sqrt{4\rho'^3 - I\rho' - J}},$$

which is the general elliptic differential reduced to Weierstrass's canonical form by Hermite's substitution.

A very complete discussion of this subject will be found in chap. v. of Greenhill's *Elliptic Functions*, from the analytic side, and I offer this geometrical view of the matter only in order to show how it depends on the contravariants of the conics U and V.

On the Fundamental Theorem of Differential Equations. By W. H. Young. Received and communicated January 9th, 1902.

The fundamental theorem of the modern theory of differential equations is Cauchy's existence theorem, dealing with the existence and uniqueness of a set of integrals satisfying given initial conditions, and the holomorphic character of the solution. This theorem has been stated in very precise language, and proved in various ways, by Picard and Painlevé, but some doubt has been expressed as to whether their proofs are rigorous. It has been suggested, in fact, that it has not been conclusively demonstrated that the holomorphic solution is unique even in the simplest case which can arise.

In the following note* it is proposed to give a brief account of the theorem in question, and to examine an example which has been put forward as typical of a large class of cases where the theorem fails.

* The note is substantially what I wrote in January, 1899, but did not publish, as I expected Painlevé or Picard to take the matter up. The former has now done so, but his treatment is too general to appeal to English readers. Indeed he does little more than repeat at length his previous definitions.