COLS—Listening to Integers: A Geometric-to-Acoustic Theory of Modular Resonances

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Abstract

Units. *All angles are in radians*; we write $A_{\mathrm{b}}(n) = A(n)/(\pi/2)$.

Starting from the COLS framework, we estimate per-factor *arithmetic periods* T_r on signals $S_r = 1 - h_r$ (not on geometric aggregators), map $T_r \mapsto f_r = K/T_r$ to obtain audible voices, and state an acoustic density law derived from the mixed-torus equidistribution: the expected proportion of note hits in fixed pitch-class windows factorises as *area* $\times$ *arithmetical mass* (with a precise correction when $\gcd(M, 2\lambda) > 1$). The estimator fixes $(T_{\min}, T_{\max})$, uses a sinusoidal scan with *Hann* apodisation and an ACF-based *harmonic-to-fundamental promotion* to avoid small-T bias. Practical recipes cover listening to a given M, labelling its chord, and searching integers near a target chord.

1 Minimal background from COLS

Let $M \geq 3$ be odd, $a \in (\mathbb{Z}/M\mathbb{Z})^{\times}$, and n even with $t = n/2$. On the unit circle, the GOLS triangle (S_0, S_N, S_n) with $\theta = 2\pi n/M$ has a right angle at S_n; define

$$A(n) = \arcsin\big(\,|\sin(\pi n/M)|\,\big) \in [0, \pi/2], \qquad A_{\mathrm{b}}(n) = \frac{A(n)}{\pi/2} \in [0, 1]. \tag{1}$$

For each prime power $r = p^e \mid M$, define the modular proximity

$$|x|_r = \min(x \bmod r,\ r - (x \bmod r)), \qquad h_r(n) = \frac{1}{r} \min\Big(\,\Big|a^t - 1\Big|_r,\ \Big|a^t + 1\Big|_r\,\Big), \tag{2}$$

and the arithmetic signal $S_r(n) = 1 - h_r(n)$. Aggregators like $= A_{\mathrm{b}}\big(1 - \min_{r|M} h_r\big)$ or $E_{\mathrm{bi}} = A_{\mathrm{b}}\big(1 - \max_{r|M} h_r\big)$ quantify unilateral/bilateral coincidences but are *not* used for T_r estimation here.

2 Estimating arithmetic periods on S_r

2.1 Sinusoidal scan with bounds, Hann, and ACF-promotion

Fix an analysis window $\mathcal{W} = \{n_0, \ldots, n_0 + N - 1\}$ and bounds

$$T_{\min} = 2, \qquad T_{\max} = \min\{\text{window length in } n,\ 400\}. \tag{3}$$

For each integer $T \in [T_{\min}, T_{\max}]$ form the Hann-windowed periodogram

$$\mathcal{P}(T) = \Big|\sum_{n \in \mathcal{W}} w(n)\, S_r(n)\, e^{-2\pi i n/T}\Big|^2, \quad w(n) = \tfrac{1}{2}\Big(1 - \cos\frac{2\pi(n - n_0)}{N - 1}\Big). \tag{4}$$

Let $T^\star = \arg\max_T \mathcal{P}(T)$ and compute the (biased) autocorrelation $\rho(\tau) = \sum_n w(n)S_r(n) \cdot w(n+\tau)S_r(n+\tau)$. If a strong ACF peak appears at $\tau = mT^\star$ for $m \in \{2,3,4\}$, *promote* the candidate to the fundamental: $T^\star \leftarrow T^\star/m$. The *period of* r is the constant $T_r := T^\star$.

2.2 From periods to audible frequencies

Fix $K > 0$, a gain $\alpha > 0$ and an octave shift $\mathrm{oct} \in \mathbb{Z}$ and define

$$f_r = \frac{K}{T_r} \cdot \alpha \cdot 2^{\mathrm{oct}}. \tag{5}$$

Optionally quantise to 12-TET by

$$\widehat{f_r} = 440 \cdot 2^{\mathrm{round}(12 \log_2(f_r/440))/12}, \tag{6}$$

or operate in an *interval-only* mode that evaluates pairwise differences of $\{\log_2(f_r)\}$ within a tolerance (in cents).

3 Acoustic equidistribution and density of note hits

Let $\mathrm{PC} = \{0, 1, \ldots, 11\}$ be the 12-TET pitch classes and $\pi(f) \in \mathrm{PC}$ the nearest class to f. For tolerance $\delta \in [0, \frac{1}{2})$ (in semitones), let $W_c(\delta)$ be the set of $f > 0$ within δ semitone of class c. Let $\lambda_r = \mathrm{ord}_{p^e}(a)$, $\lambda = \mathrm{lcm}_r\, \lambda_r$, and $d = \gcd(M, 2\lambda)$. Index even exponents by $n = 2s$ and consider a full cycle of length $\mathrm{lcm}(M, 2\lambda)$.

Definition 3.1 (Arithmetical mass)**.** For a pitch-class set $S \subseteq \mathrm{PC}$ and tolerance δ, define

$$\mu_{\mathrm{arith}}(S;\delta) = \frac{1}{2\lambda} \sum_{s \in \mathbb{Z}/(2\lambda)\mathbb{Z}} \mathbf{1}\left\{ \exists\, r = p^e \mid M,\ \exists\, c \in S:\ \frac{K}{T_r} \in W_c(\delta) \right\}. \tag{7}$$

Theorem 3.2 (Acoustic density factorisation)**.** *Fix $\tau \in (0,1)$, a base a, a pitch-class set $S \subseteq \mathrm{PC}$ and tolerance δ. Under the mixed-torus equidistribution of COLS (see COLS, Thm. 3.3), the expected proportion of even indices n such that*

$$A_{\mathrm{b}}(n) \geq \tau \quad \textit{and} \quad \pi\left(\frac{K}{T_r}\right) \in S \textit{ for at least one } r = p^e \mid M$$

factorises as

$$\mathbb{P}\Big(A_{\mathrm{b}} \geq \tau \ \&\ \pi(K/T_r) \in S \textit{ for some } r\Big) = (1-\tau) \cdot \mu_{\mathrm{arith}}(S;\delta) + \varepsilon(M,a), \tag{8}$$

where $\varepsilon(M,a) = 0$ for $d = \gcd(M, 2\lambda) = 1$, and otherwise equals the class-wise average on the alignment classes of the mixed torus.

Remark 3.3. Tolerance δ amounts to a measurable coarsening after the pushforward $T \mapsto f = K/T \mapsto \pi(f)$. The interval-only variant replaces pitch classes by difference sets; the factorisation still applies after passing to differences.

4 Algorithms and practical recipes

A. Listening to a given M. Inputs: M, base a (e.g. 2), window factor W (so $n \in [0, WM)$), $(K, \alpha, \mathrm{oct})$, 12-TET ON/OFF. Factorise M to prime powers $r = p^e$ for voicing; build S_r; estimate T_r by Section 2; map by Section 2.2; synthesise a chord or arpeggio (amplitudes from an aggregator such as ; micro-fades and a short crossfade allow seamless looping).

B. Labelling the chord of M. Select k voices (top-k by power); compute $P = \{\pi(f_{ri})\}$; identify the chord by scoring against triad/7th templates with penalties for extras/misses.

C. Searching integers for a target chord. Given a target $S \subseteq \mathrm{PC}$, scan M; run the pipeline; keep the best by class hits, interval quality, and prior power.

5 Empirical notes

Windows $W \in \{1, 2\}$ stabilise T_r in typical bases; larger windows refine long fundamentals. 12-TET ON helps labelling; OFF preserves raw colour (report cent offsets). Allowing $T_{\min} = 2$ and using ACF-promotion prevents small-T bias on spiky S_r. Applying a Hann window reduces spectral leakage and homogenises results with the COLS note.

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