On the general Theory of Stationary Motion in an Infinite System of Molecules. By S. H. BURBURY, M.A., F.R.S. Received and read January 13th, 1898.

I begin this paper with a confession of error made in a former paper of mine "On Boltzmann's Law of the Equality of Mean Kinetic Energy." I obtained a correct result, and then drew a wrong inference. I considered, namely, the case of two sets of elastic spheres having masses M and m. The velocities in x of the M spheres are U_1, U_2, &c., and of the m spheres u_1, u_2, &c. Then I assumed the law of distribution of the velocities to be

$$\epsilon^{-hQ}\, dU_1 \ldots du_1 \ldots dw_n,$$

and
$$Q = A\,(U_1^2 + U_2^2 + \&\text{c.}) + a\,(u_1^2 + u_2^2 + \&\text{c.}) + B\Sigma\Sigma UU' + b\Sigma\Sigma uu' + \beta\Sigma\Sigma Uu,$$

with a similar expression for the velocities in directions y and z. And I proved that in stationary motion the relation between the coefficients A, a, β must be

$$2Am - 2aM + \beta\,(m - M) = 0;$$

and therefore
$$\frac{A}{a} \neq \frac{M}{m}.$$

That is correct. I inferred that

$$A\overline{U^2} = a\overline{u^2};$$

and therefore
$$M\overline{U^2} \neq m\overline{u^2},$$

which was incorrect.

Plan of this Paper.

Boltzmann, Watson, and other writers take for the foundation of their theory of gases a certain assumption, which may be expressed as follows. The chance of any molecule having velocities $u_1, \ldots u_1 + du_1$, &c., is independent of the velocities which any other molecule may have, unless the two molecules are, or recently have been, within one another's sphere of action. I call this assumed state of things "Condition A."

If the chance of a group of n molecules having velocities respectively $u_1, \ldots u_1+du_1, \ldots w_n \ldots w_n+dw_n$ be denoted by

$$\epsilon^{-hQ} du_1 \ldots dw_n,$$

then on this assumption Q must contain squares only, in the form

$$Q = \Sigma m\,(u^2+v^2+w^2).$$

Condition A implies that, and that implies Condition A.

It is proved by what is known as the H theorem that this is the only distribution which can exist consistently with the continued existence of Condition A.

It is not proved, but it is always assumed, that the state of things described by Condition A can and does continue to exist.

I proved in my last paper that, if the molecules be equal elastic spheres having finite diameter, the motion cannot be stationary unless Q contains products as well as squares of the velocities; in other words, Condition A cannot continue to exist. I admit, of course, that, if it exists in fact—if, for instance, it could be maintained artificially—the H theorem and all its consequences follow.

I propose to show in the present paper (see Arts. 1–10) that the assumption of Condition A is unnecessary, because all the theoretical results obtained by assuming it can be obtained by the more general assumption of Art. 2, namely,

$$Q = \Sigma a\,(u^2+v^2+w^2) + \Sigma\Sigma b\,(uu'+vv'+ww').$$

It is unnecessary (see Art. 11), because all the practical results obtained by assuming it, *e.g.*, for diffusion, viscosity, &c., depend on the calculation of mean free path, which for gases under ordinary condition would have very approximately the same value whether the coefficients b exist or not.

Arts. 12–25.—I propose to justify the above general assumption by extending the method of my former paper, which deals only with the case where the molecules are equal elastic spheres, to other forms of molecule.

The Assumed Form of Molecule.

1. Let the molecules be material particles, between any two of which acts a finite force. This force is assumed to be a continuous function of the distance r between the two molecules, acting in the line r, becoming repulsive and infinite, either at some finite distance, or ultimately as r is indefinitely diminished, and becoming

insensible when r exceeds a certain very small distance, which may be called the *radius of action* of a molecule. This includes the case of elastic spheres as a limiting case. If it be true that a molecule has finite size, or that there is a distance within which two molecules cannot approach one another, that appears to be inconsistent with the ultimate force when the distance is indefinitely diminished being attractive.

The Law of Distribution assumed.

2. I assume the law of distribution of the coordinates and the velocities to be as follows, viz., the chance that the n molecules in any infinite space shall at a given instant have coordinates

$$x_1, \dots x_1 + dx_1, \dots z_n \dots z_n + dz_n,$$

and component velocities $u_1, \dots u_1 + du_1$, &c., is

$$ACe^{-hQ}\, dx_1 \dots dz_n\, du_1 \dots dw_n,$$

in which

$$Q = a_1 u_1^2 + a_2 u_2^2 + \&c. + a_1 v_1^2 + \&c.$$
$$+ b_{12}(u_1 u_2 + v_1 v_2 + w_1 w_2) + b_{13}(u_1 u_3 + v_1 v_3 + w_1 w_3) + \&c.$$

Here the a coefficients are supposed to be independent of the position of the molecules and of the time, but every b, as b_{pq}, is a function of r_{pq}, the distance at the given instant between the molecules whose velocities are u_p, &c., and u_q, &c., and, moreover, is a continuous function of r_{pq}, which diminishes as r_{pq} increases, and becomes evanescent when r_{pq} exceeds a certain very small finite value.

Further, C is that function of h, and of the coefficients a, b which makes

$$\iiint_{-\infty}^{+\infty} \dots Ce^{-hQ}\, du_1 \dots dw_n = 1.$$

That is,
$$C = \sqrt{D}\left(\frac{h}{\pi}\right)^{\frac{3}{2}n},$$

$3n$ being the number of the variables u, v, w, and D the determinant of the coefficients a, b set out in Art. 3. It is a function of the coordinates only as contained in the bs.

Further, A is a constant or an explicit function of the coordinates. With these meanings of A, C, Q, let

$$f = ACe^{-hQ}.$$

In stationary motion

$$\left(\frac{dx_1}{dt}\frac{d}{dx_1} + \dots + \frac{dz_n}{dt}\frac{d}{dz_n} + \frac{du_1}{dt}\frac{d}{du_1} + \dots + \frac{dw_n}{dt}\frac{d}{dw_n}\right) f = 0.$$

But also in stationary motion Q must be constant, or

$$\Sigma \frac{dQ}{du} \frac{du}{dt} = 0$$

and

$$\Sigma \frac{dQ}{db} \frac{db}{dt} = 0.$$

3. We now prove that on average, given b_{12}, &c.,

$$\overline{u_1 \frac{dQ}{du_1}} = \overline{u_2 \frac{dQ}{du_2}} = \text{\&c.} = \frac{1}{h}.$$

For

$$u_1 \frac{dQ}{du_1} = u_1 (2a_1 u_1 + b_{12} u_2 + b_{13} u_3 + \text{\&c.}).$$

Let us now find the mean value, given u_1, of

$$u_1 (2a_1 u_1 + b_{12} u_2 + b_{13} u_3 + \text{\&c.}).$$

To that end we perform the integration

$$A \iiint \dots du_2 \dots dw_n \, \epsilon^{-hQ} u_1 (2a_1 u_1 + b_{12} u_2 + b_{13} u_3 + \text{\&c.}), \qquad (a)$$

and divide the result by

$$A \iiint \dots du_2 \dots dw_n \, \epsilon^{-hQ}. \qquad (b)$$

As the result of the integration, the index becomes in both (a) and (b), $h \frac{D}{2D_{11}} u_1^2$, where D is the determinant

$$D = \begin{vmatrix} 2a_1 & b_{12} & b_{13} & \dots \\ b_{12} & 2a_2 & b_{23} & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix},$$

and D_{11} its first coaxial minor.

It follows that the mean value required is what

$$u_1 (2a_1 u_1 + b_{12} u_2 + b_{13} u_3 + \text{\&c.})$$

becomes when all the integrations are effected.

4. The integration can be effected thus :—Let

$$2a_1 u_1 + b_{12} u_2 + \text{\&c.} = P.$$

Then, before the integration begins, we see that the coefficient of u_1 in P is twice the coefficient of u_1^2 in Q, but the coefficient of any other u, as u_2 in P, is equal to the coefficient of $u_1 u_2$ in Q. It will be found that this relation holds after the integration according to u_n, and therefore after all the integrations. And so, finally, when Q is reduced to $\frac{D}{2D_{11}}u_1^2$, P must be $\frac{D}{D_{11}}u_1$, and

$$\overline{u_1\frac{dQ}{du_1}} = \frac{D}{D_{11}}\overline{u_1^2},$$

which is the mean value sought.

For, to integrate according to u_n, we proceed as follows. The terms in Q which contain u_n are

$$a_n u_n^2 + (b_{1n}u_1 + b_{2n}u_2 + \&c.)\,u_n,$$

which may be put in the form

$$\left(u_n\sqrt{a_n} + \frac{b_{1n}u_1 + b_{2n}u_2 + \&c.}{2\sqrt{a_n}}\right)^2 - \frac{(b_{1n}u_1 + b_{2n}u_2 + \&c.)^2}{4a_n},$$

or, as we may write for brevity,

$$\left(u_n\sqrt{a_n} + \frac{\beta}{2\sqrt{a_n}}\right)^2 - \frac{\beta^2}{4a_n},$$

and $\quad P = 2a_1 + b_{12}u_2 + \&c. + b_{1n}u_n$

$$= 2a_1 + b_{12}u_2 + \&c. + \frac{b_{1n}}{\sqrt{a_n}}\left(u_n\sqrt{a_n} + \frac{\beta}{2\sqrt{a_n}} - \frac{\beta}{2\sqrt{a_n}}\right).$$

Then we take $\left(u_n\sqrt{a_n} + \frac{\beta}{2\sqrt{a_n}}\right)$ for variable, and integrate for it between $+\infty$ and $-\infty$. The result is, omitting the numerical factors, that, in Q, all the terms involving u_n disappear, and are replaced by

$$-\frac{(b_{1n}u_1 + b_{2n}u_2 + \&c.)^2}{4a_n},$$

and, in P, $b_{1n}u_n$ is replaced by

$$-b_{1n}\frac{b_{1n}u_1 + b_{2n}u_2 + \&c.}{2a_n}.$$

Expanding the new term in Q, we find that the new coefficient of u_1u_q is

$$b_{1q} - \frac{b_{1n}b_{qn}}{2a_n},$$

which is the coefficient of u_q in P. Also the new coefficient of u_1^2 in Q is $a_1 - \frac{b_{1n}^2}{4a_n}$, and the new coefficient of u_1 in P is $2a_1 - \frac{b_{1n}^2}{2a_n}$. The same relation then subsists after integration as before between the respective coefficients in Q and in P; and this holds for every subsequent integration. And ultimately, when all the integrations are effected to u_1 inclusive,

$$Q = \frac{D}{2D_{11}} u_1^2, \quad u_1 P = \frac{D}{D_{11}} u_1^2,$$

or

$$u_1 \left(\overline{2a_1 u_1 + b_{12} u_2 + \&c.}\right) = \frac{D}{D_{11}} u_1^2.$$

We have then on average

$$\overline{u_1 \frac{dQ}{du_1}} = \frac{D}{D_{11}} \overline{u_1^2}, \quad \overline{u_2 \frac{dQ}{du_2}} = \frac{D}{D_{22}} \overline{u_2^2}, \ \&c.$$

5. This is true on average of all possible values of $u_1 \dots u_n$, treating all the b's as constants. That is, it is true for every separate set of values of $b_{12} \dots b_{1n}$, &c. But it is also true that for each set of values of $b_{12} \dots b_{1n}$, &c.,

$$\overline{u_1^2} = \frac{D_{11}}{hD}, \quad \overline{u_2^2} = \frac{D_{22}}{hD}, \ \&c.;$$

and therefore

$$\overline{u_1 \frac{dQ}{du_1}} = \frac{1}{h} = \overline{u_2 \frac{dQ}{du_2}} = \overline{u_3 \frac{dQ}{du_3}} = \&c.$$

This is the general proposition of which

$$m_1 \overline{u_1^2} = m_2 \overline{u_2^2} = \&c.,$$

with

$$Q = \Sigma m\,(u^2 + v^2 + w^2),$$

is a particular case.

6. But, again, in the system of an infinite number of molecules, not only is Q constant, but the kinetic energy, $T = \frac{1}{2}\Sigma m u^2$, is also separately constant; and therefore we may write

$$Q = \mu T,$$

where μ is constant, and

$$\overline{Q} = \frac{3n}{2h}, \quad T = \frac{3n}{2\mu h}.$$

We have then

$$\Sigma \frac{dQ}{du} \frac{du}{dt} = 0,$$

and

$$\Sigma m u \frac{du}{dt} = 0,$$

and, if we write $\dfrac{dQ}{du_1} = \mu m_1 u_1, \quad \dfrac{dQ}{du_2} = \mu m_2 u_2$, &c.,

and $$m_1 \overline{u_1^2} = m_2 \overline{u_2^2} = \text{\&c.},$$

we obtain consistent results.

On Stationary Motion as affected by the b-Coefficients.

7. Let us next consider the second condition of Art. 2 for stationary motion, namely,

$$\Sigma \frac{dQ}{db} \frac{db}{dt} = 0.$$

That asserts that, if we regard the system at time t, and again at time $t+dt$, there will in the infinite number of molecules be as many pairs of molecules at the second instant as at the first whose distances from each other are between r and $r+dr$, and as many molecules at the second instant as at the first whose neighbours are distant from them respectively $r_1 \dots r_1+dr_1$, $r_2 \dots r_2+dr_2$, &c., and that the distribution of the velocities has generally the same relation to the distances at the second as at the first instant.

In my last paper I found a solution for the case of equal elastic spheres as follows. Let f be a function of r, the distance from any point P, which satisfies the conditions

$$f = 1, \quad \text{if} \quad r < r_0,$$

$$f = \frac{r_0}{r}, \quad \text{if} \quad r > r_0,$$

or, instead of $f = \dfrac{r_0}{r}$, we may use any continuous function of r for which $\dfrac{df}{dr}$ is negative. Then, at any point P, let

$$\xi = \frac{\Sigma fu}{\Sigma f}, \quad \eta = \frac{\Sigma fv}{\Sigma f}, \quad \zeta = \frac{\Sigma fw}{\Sigma f},$$

the summation including all the molecules of the system. When the molecules are material points between which no collisions occur, let $\overline{\xi^2}$ be the mean value of ξ^2. Then, when the molecules become spheres of finite diameter c, stationary motion requires that $\overline{\xi^2}$ shall become

$$\overline{\xi^2}+\overline{\xi'^2}, \quad \text{and} \quad \overline{\xi'^2} = \frac{\kappa}{1-\kappa}\overline{\xi^2}.$$

Here ρ is the number of molecules per unit volume, and

$$\kappa = \tfrac{2}{3}\pi c^3 \rho.$$

Further, $\overline{\xi'^2}$ necessarily introduces into Q the term

$$\Sigma\Sigma b\,(uu'+vv'+ww').$$

If the b's are on the average of all molecules invariable with the time, then $\overline{\xi'^2}$ is invariable with the time, and *vice versa*. The constancy of $\overline{\xi'^2}$ involves the condition

$$\Sigma \frac{dQ}{db}\frac{db}{dt} = 0.$$

Further, I found that, if

$$b = -\tfrac{2}{3}\frac{c^5}{r^5},$$

then the condition $\qquad \overline{\xi'^2} = \kappa\overline{\xi^2},$

or, so long as κ is very small,

$$\overline{\xi'^2} = \frac{\kappa}{1-\kappa}\overline{\xi^2}$$

is satisfied.

Of the System under Conservative External Forces and of the Coefficient A.

8. Let us now suppose our system to be placed in a field of conservative external forces, whose components at any point are X, Y, Z, and let us consider what modifications are required in the above results. We have

$$\frac{d}{dt}(AC\epsilon^{-hQ}) = 0,$$

in which A is now to be regarded as an explicit function of the coordinates, C as a function of the coordinates as they are involved in the bs. We have then

$$\frac{d}{dt}(AC\epsilon^{-hQ}) = \Sigma\left(u\frac{d}{dx}+v\frac{d}{dy}+w\frac{d}{dz}+X\frac{d}{du}+Y\frac{d}{dv}+Z\frac{d}{dw}\right)AC\epsilon^{-hQ}.$$

That gives $$\left(\frac{dA}{dx_1}u_1-hAX\frac{dQ}{du_1}\right)Ce^{-hQ}$$
$$+\left(\frac{dA}{dx_2}u_2-hAX\frac{dQ}{du_2}\right)Ce^{-hQ}$$
$$+\&c.$$
$$+u_1\left\{\frac{dC}{db_{12}}-hC(u_1u_2+v_1v_2+w_1w_2)\right\}\frac{db_{12}}{dx_1}Ae^{-hQ}$$
$$+u_2\left\{\frac{dC}{db_{12}}-hC(u_1u_2+v_1v_2+w_1w_2)\right\}\frac{db_{12}}{dx_2}Ae^{-hQ}$$
$$+\&c.=0.$$

The terms derived from differentiation of the b coefficients vanish in pairs. For any b, as b_{pq}, appears twice only, namely, in $u_p\dfrac{db_{pq}}{dx_p}$, and in $u_q\dfrac{db_{pq}}{dx_q}$, and, since every b is a function of r only, we have generally

$$\frac{db}{dx_1}=\frac{db}{dr}\frac{dr}{dx_1}=\frac{x_2-x_1}{r}\frac{db}{dr},$$
$$\frac{db}{dx_2}=\frac{x_1-x_2}{r}\frac{db}{dr};$$

therefore the terms in $\dfrac{df}{dt}$ derived from differentiation of the bs according to the coordinates resolve themselves into pairs of the form

$$Ch\,(u_1-u_2)(u_1u_2+v_1v_2+w_1w_2)\frac{db}{dr}\frac{x_1-x_2}{r}$$

and $$(u_1-u_2)\frac{db}{dr}\frac{x_1-x_2}{r},$$

all which pairs vanish, because there will be on average just as many cases in which any given velocity u is at x_1, y_1, z_1 as at x_2, y_2, z_2.

We have, then, dividing by AC,

$$\frac{1}{AC}\frac{d}{dt}(ACe^{-hQ})$$
$$=\left\{\frac{d\log A}{dx_1}u_1-hX(2a_1u_1+b_{12}u_2+\&c.)\right\}e^{-hQ}$$
$$+\left\{\frac{d\log A}{dx_2}u_2-hX(2a_2u_2+b_{12}u_1+\&c.)\right\}e^{-hQ}$$
$$+\&c.$$
$$=0$$

$$= \left(\frac{d\log A}{dx_1} - hm_1 X \frac{D}{D_{11}m_1}\right) u_1 e^{-hQ}$$

$$+ \left(\frac{d\log A}{dx_2} - hm_2 X \frac{D}{D_{22}m_2}\right) u_2 e^{-hQ} + \&c.$$

$$= \left\{ \left(\frac{d\log A}{dx_1} + h\mu\frac{d\chi}{dx_1}\right) u_1 + \left(\frac{d\log A}{dx_2} + h\mu\frac{d\chi}{dx_2}\right) u_2 + \&c. \right\} \epsilon^{-hQ},$$

if χ be the potential of the external forces, and

$$\frac{1}{A}\frac{d}{dt}(AC\epsilon^{-hQ}) = 0,$$

if $$A = \epsilon^{-h\mu\chi}, \quad f = \epsilon^{-h\mu\chi} C e^{-hQ}.$$

Comparing the state of the system at two different levels of potential, we may take the average of Q, or $Q = \mu T$. Then the law becomes

$$C e^{-h\mu(\chi+T)},$$

agreeing with Boltzmann's result, with $h\mu$ for his h.

Application to Intermolecular Forces.

9. The method of Art. 8 would apply equally to the case where χ is the potential of intermolecular forces instead of external forces only. The following difficulty has, however, been raised:—

Unless in the case of very small density, where the encounters are binary, χ can be completely expressed only as a single function involving the coordinates of all the molecules. It cannot be treated as the sum of the potentials of distinct pairs or groups. It seems to me that the following method (which is substantially that used by Dr. Watson in his *Kinetic Theory of Gases*, 2nd edition, p. 70) enables us to remove the difficulty, if any.

Consider any molecule m_1, whose coordinates are x_1, y_1, z_1. Let χ_{12} be the potential at x_1, y_1, z_1 of the forces due to the molecule m_2 at x_2, y_2, z_2; and so on. Then $\chi_{12} = 0$ unless the two molecules are very near each other.

Let $$\chi_{12} + \chi_{13} + \&c. = \chi_1.$$

Then we may write $$\chi = \chi_1 + \chi',$$

where χ' is the potential of mutual action of all the molecules except m_1. Given χ_1, χ' may have many different values according to the relative positions *inter se* of the other molecules. But on average it will be constant.

Then we obtain consistent results if we assume that the chance of m_1 being in a position relative to the other molecules in which its potential is χ_1 is, *cæteris paribus*, proportional to

$$e^{-h\mu\chi_1}.$$

That assumption enables us to prove that our gas obeys the Second Law of Thermodynamics, which, in case of intermolecular forces, has never, I believe, been proved on any other assumption.

10. The proof that I have given of the proposition of Art. 8 is, with slight modifications, the same which Boltzmann gives in his *Vorlesungen* for the case in which Q is a sum of squares only; that is, in which Condition A is assumed. It is necessary to extend the proof to my case, where I have begun by denying that assumption, because Boltzmann has said (*Vorlesungen*, p. 20) that without making that assumption it is impossible to prove any theorem in the kinetic theory of gases. I submit that I have proved two, at least, of these theorems without making Boltzmann's assumption, and without making any assumption except that Q is always positive.

11. Boltzmann has also calculated the rate of diffusion, &c., on his own assumption, such calculations being based on the determination of the "mean free path." Now, for gases under the ordinary conditions, to which Boltzmann's calculation of mean free path apply, my coefficients b would be very small, and the mean free path determined from my formula would differ so little from the mean free path determined from his formula that the results arrived at for diffusion, &c., would agree with his, at least, within the limits of experimental error.

Elastic Spheres of Unequal Masses.

12. I proved in my former paper the necessary existence of the b coefficients in a system of equal elastic sphere molecules. Before extending the theory to any more general form of molecule, I worked out the corresponding problem for two sets of elastic spheres of unequal masses and diameters. As the process is laborious, I here give the results only. Let m_1, m_2 be the masses, c_{11}, c_{22} the diameters, of the spheres of the two classes respectively. Also let

$$c_{12} = \frac{c_{11} + c_{22}}{2}.$$

Let ρ_1 be the number per unit of volume of the m_1s, ρ_2 of the m_2s. Let ξ_1, η_1, ζ_1 be the functions ξ, η, ζ for the m_1s, ξ_2, η_2, ζ_2 the same functions for m_2. If the spheres were all material points, so that no collisions should occur, the motion of the m_1 system would be independent of that of the m_2 system, so that

$$\overline{\xi_1\xi_2} = \overline{\eta_1\eta_2} = \overline{\zeta_1\zeta_2} = 0.$$

I then found that, as the condition for stationary motion, ξ_1 must become $\xi_1+\xi_1'$, &c., ξ_2 must become $\xi_2+\xi_2'$, &c., and, as in my former paper, ξ_1', ξ_2', &c., must satisfy the equations

$$\overline{\xi_1'^2} = \tfrac{2}{3}\pi c_{11}^3\rho_1\overline{\xi_1^2},$$

$$\overline{\xi_2'^2} = \tfrac{2}{3}\pi c_{22}^3\rho_2\overline{\xi_2^2}.$$

These result from collisions between m_1 and m_1, or m_2 and m_2; and, for collisions between m_1 and m_2, we have

$$\overline{\xi_1'\xi_2'} = \tfrac{2}{3}\pi c_{12}^3\frac{2m_1m_2}{m_1+m_2}\,\frac{\rho_1\rho_2}{m_1\rho_1+m_2\rho_2}\,(\overline{\xi_1^2}+\overline{\xi_2^2}).$$

The two systems cease to be independent when collisions take place between m_1 and m_2, so that

$$\overline{\xi_1'\xi_2'} \neq 0.$$

The above expressions are accurate only on the assumption that squares and products of $\pi c_{11}^3\rho_1$, $\pi c_{22}^3\rho_2$, and $\pi c_{12}^3\overline{\rho_1+\rho_2}$ are to be neglected.

If B denote the coefficient of u_1u_1' in Q, b that of u_2u_2', and β that of u_1u_2, then the values of B, b, β, which lead to the above results, are

$$B = -\tfrac{2}{3}\frac{c_{11}^5}{r^5},$$

$$b = -\tfrac{2}{3}\frac{c_{22}^5}{r^5}.$$

$$\beta = -\frac{4m_1m_2}{(m_1+m_2)^2}\,\tfrac{2}{3}\,\frac{c_{12}^5}{r^5}.$$

Consideration of the Case in which the Molecules are Centres of Force.

13. Next let us assume the molecules to be centres of force under the conditions described in Art. 1; and, first, let the forces be repulsive only.

Further, we will begin by assuming the density to be so small that the encounters are binary.

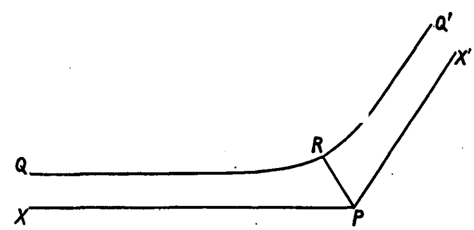
As in my former paper, the functions ξ, η, ζ, and M having the same meanings as before, we find, for a system of material points between which no mutual action takes place, the equation for stationary motion

$$\frac{dM}{dt} = 0 = uw\frac{d}{dt}\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right) - \frac{1}{2h}\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right)^2.$$

We shall find that, when the molecules exert mutual action on each other, the effect is to introduce into $\frac{dM}{dt}$ new terms, which are necessarily positive, and so to disturb the stationary character of the motion; and stationary motion can be restored only by increasing ξ^2, &c., in a certain ratio, which increase involves the introduction into Q of the terms $b(uu'+vv'+ww')$. But we shall find that in this case the b's are functions of density and temperature.

Centres of Force with Binary Encounters.

14. First, suppose a molecule P fixed in space.



Let PX be any line through P. Let Q be a point whose distance from PX is β, and is less than the radius of action, but whose distance from P is greater than the radius of action. A molecule is projected from Q with velocity q in direction parallel to XP towards P. An encounter then occurs, and, P being fixed, the molecule Q describes a symmetrical curved path, QRQ', as in the figure, R being the apse, and the asymptotes are parallel to PX and PX'. Let the angle

$$RPX = \psi = \tfrac{1}{2}XPX',$$

and ψ must be less than $\frac{\pi}{2}$.

The integral effect of the repulsive central force at P, or, as we may call it, the effect of the encounter, is equivalent, so far as the change of direction of the moving particle is concerned, to an impulse on the moving particle in direction PR, whose value is

$$mq\sqrt{2}\sqrt{1+\cos 2\psi} = 2mq\cos\psi,$$

m being the mass of the particle. Therefore, also, the resultant force on P is $2mq\cos\psi$ in direction RP; and the resultant force on P in direction perpendicular to XP is $2mq\cos\psi\sin\psi$.

Let, now, $\rho f(q)\,dq$ be the number per unit of volume of molecules whose velocities parallel to XP are between q and $q+dq$. Then the number of such molecules which in unit of time cross an element of unit area at right angles to XP at Q is $\rho q f(q)\,dq$.

And so the mean force on P in direction perpendicular to XP is, for all the molecules which pass through the element of area with velocity $q \dots q+dq$, in XP, equal to

$$F = \rho\, 2mq^2 f(q)\,dq\cos\psi\sin\psi,$$

which is necessarily positive.

15. As q^2 increases, β remaining unchanged, ψ increases, and therefore $\cos\psi$ diminishes. Also, as ψ increases, if $\psi < \frac{\pi}{4}$, $\cos\psi\sin\psi$ increases; if $\psi > \frac{\pi}{4} < \frac{\pi}{2}$, $\cos\psi\sin\psi$ diminishes. Now, in order that ψ may be less than $\frac{\pi}{4}$, β must be less than a certain value β_0, known if the law of force is known, and β_0 diminishes as q^2 increases. It follows that, for sufficiently great values of $\overline{q^2}$, $\cos\psi\sin\psi$ diminishes on average as $\overline{q^2}$ increases; and therefore F increases less rapidly than $\overline{q^2}$. If we write

$$F = \kappa_1\overline{q^2},$$

κ_1 is a quantity which diminishes as $\overline{q^2}$, *i.e.*, the temperature, increases.

16. If the molecules are in stationary motion with $\frac{3}{2h}$ for mean square velocity, we must write $\frac{1}{2h}$ for $\frac{m\overline{q^2}}{2}$ in the above expressions (I here use h instead of μh above).

If the molecule P, instead of being fixed, is itself moving relatively to the system of molecules with velocity u in direction PX, we must write $\frac{1}{2h}+\frac{1}{2}mu^2$ for $\frac{m\overline{q^2}}{2}$ in the above expressions.

17. Now let us suppose that at P taken for origin $\xi = \xi_0$, and at any point whose z coordinate is ∂z

$$\xi = \xi_0 + \frac{d\xi}{dz}\,\partial z.$$

That means that the molecules at positive ∂z, say at P', have a mean velocity $\frac{d\xi}{dz}\,\partial z$ in x relative to the molecules for which ∂z is negative, say at P'' (not shown in figure).

Therefore the mean force on P, moving in direction PX with velocity u, due to the molecules at P' is

$$\frac{\kappa_1}{2h} + \kappa_1 \frac{m}{2}\left(u + \frac{d\xi}{dz}\,\partial z\right)^2,$$

and the mean force on P due to the molecules at P'' is

$$\frac{\kappa_1}{2h} + \kappa_1 \frac{m}{2}\left(u - \frac{d\xi}{dz}\,\partial z\right)^2$$

in the opposite direction. The resultant force on P in direction z in plane of the motion is proportional to

$$2\kappa_1 mu \frac{d\xi}{dz}\,\partial z.$$

If the plane of the motion make angle ϕ with that of xz, we multiply the last expression by $\cos\phi$. Also, if $XQ = \beta$, we have

$$\partial z = \beta \cos\phi,$$

and the resultant force due to all encounters, given β, is

$$2\kappa_1 mu\beta \frac{d\xi}{dz}\,\overline{\cos^2\phi} = \kappa_1 mu\beta \frac{d\xi}{dz}.$$

Since κ_1 is an unknown function of β, we will now write κ for $\kappa_1\beta$.

18. We have, then, if v, w be the velocities of P in y and z respectively,

$$\frac{dw}{dt} = \kappa mu \frac{d\xi}{dz},$$

whence

$$u\frac{dw}{dt} = \kappa mu^2 \frac{d\xi}{dz}.$$

Similarly,

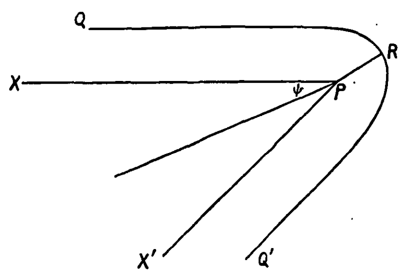
$$w\frac{du}{dt} = \kappa mw^2 \frac{d\zeta}{dx};$$

and therefore $\left(\frac{d\xi}{dz} + \frac{d\zeta}{dx}\right)\frac{d}{dt}(uw) = \kappa m \frac{1}{2h}\left(\frac{d\xi}{dz} + \frac{d\zeta}{dx}\right)^2.$

This expression is of the same form as I obtained in the case of elastic spheres, except that κ is now an unknown function depending on the law of force, and is a function of temperature as well as density, and generally diminishes as temperature increases.

Case of Attractive Forces considered.

19. If, for some values of r, the distance between two molecules, the force be attractive instead of repulsive, these conclusions require modification as follows: Let us use the notation of Art. 14. The force being attractive, it may be that, for some values of q, the molecules P and Q, if alone in space, would form a permanent union, revolving round one another in a circular or other orbit. But, if q be great enough, the Q molecule, P being fixed, will describe a path as shown in the figure.



All that has been said concerning the path described in Art. 14 applies equally to this, except that the force on P will now be in direction PR, and its resolved part, at right angles to PX, will be $-2mq\cos\psi\sin\psi$, instead of $2mq\cos\psi\sin\psi$. Consequently we shall have in this case, so far as regards the result of these encounters, $-\xi'^2$ instead of ξ'^2, and the b coefficients would, were encounters of this kind alone concerned, be positive instead of negative.

We may assume, however, that, for sufficiently small values of r, the mutual force is repulsive, and, ultimately, as r diminishes indefinitely, infinite. It follows that, as the density increases and the temperature increases, the repulsive forces must become predominant. And the general effect will be of the same form as if only repulsive forces existed; the attractive forces, if they exist, only serving to make the increase in absolute value of the b coefficients with increasing density less than it otherwise would be.

Another Method of Proving the Terms which appear in $\frac{dM}{dt}$ as the Effect of Mutual Action, to be Positive.

20. I will now treat the case in the same way as in my former paper I treated the case of elastic spheres in collision, in order to show that the new terms in $\frac{dM}{dt}$ are positive.

Let V be the half relative velocity of two molecules immediately before their encounter, λ, μ, ν its direction cosines before the encounter commences, λ', μ', ν' after it has ceased. Let

$$\lambda V = V_x, \quad \mu V = V_y, \quad \nu V = V_z.$$

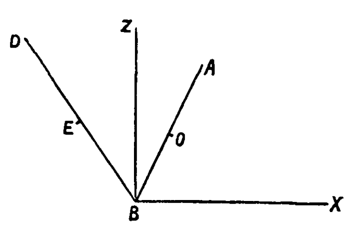
Let c be the distance of their centres apart when the encounter commences. Then the number of encounters per unit of volume and time is $2\pi c^2 \rho V$. Hence, for the change with the time of $V_x V_z$, due to encounters with given V, we have

$$2\pi c^2 \rho V^3 (\overline{\lambda'\nu'} - \overline{\lambda\nu}),$$

the bar denoting mean values. We have now to find the mean values of $\overline{\lambda'\nu'}$ and $\overline{\lambda\nu}$, given V. At the beginning of the encounter, let the centres of the two encountering molecules be A, B. Let BD be the relative velocity. Or, if $DE = BE$, one molecule has velocity DE, and the other BE, in addition to the velocity of their common centre of inertia, whatever that velocity may be. Let the angle

$$DBA = \theta;$$

let O be the middle point of AB. Let BX, BZ be the directions of



the axes of x and z, and let the angle between the planes DBA and DBZ be ϕ. When the plane DBA, turning round DB, contains the axis of z, $\phi = 0$ for that position of BA which makes the least angle with the axis of positive z.

Similarly, let the angle between the planes DBA and DBX be ϕ'. Then, if x_0, z_0 be the x and z coordinates of A referred to O as origin, and if the direction DE be that of λ, μ, ν,

$$z_0 = -\nu \frac{c}{2} \cos\theta + \frac{c}{2} \sqrt{1-\nu^2} \sin\theta \cos\phi,$$

$$x_0 = -\lambda \frac{c}{2} \cos\theta + \frac{c}{2} \sqrt{1-\lambda^2} \sin\theta \cos\phi'.$$

Now let ξ_0, ζ_0 be the values of ξ, ζ at O. Then at A

$$\xi - \xi_0 = \frac{c}{2} \left(-\nu \cos\theta + \sqrt{1-\nu^2} \sin\theta \cos\phi\right) \frac{d\xi}{dz},$$

$$\zeta - \zeta_0 = \frac{c}{2} \left(-\lambda \cos\theta + \sqrt{1-\lambda^2} \sin\theta \cos\phi'\right) \frac{d\zeta}{dx}.$$

Then we find, as in my former paper, that the mean value of $V_x V_z$, for all encounters with given V, is

$$\overline{V_x V_z} = \overline{\lambda\nu}\, V^2 = -\frac{2hc}{3} V^3 \frac{1}{3.5} \left(\frac{d\xi}{dz} + \frac{d\zeta}{dx}\right).$$

21. We have next to calculate $\overline{\lambda'\mu'}$. When the plane DBA, turning round DB, contains the axis of z, let A_z, A'_z be the two positions in which that plane is cut by the circle which A describes. Let A_z be nearer to the positive axis of z than A'_z. Then, for A_z, $\phi = 0$; for A'_z, $\phi = \pi$. Similarly, when the plane DBA, turning round DB, contains the axis of x, A_x and A'_x are the two positions in which that plane is cut by the circle which A describes. And, if A_x is nearer to the positive axis of x than A'_x, for A_x $\phi' = 0$; for A'_x $\phi' = \pi$.

Let ϵ be the angle between the plane of DB and z and the plane of DB and x. Then

$$\cos\epsilon = \frac{-\lambda\nu}{\sqrt{1-\lambda^2}\sqrt{1-\nu^2}}, \quad \sin\epsilon = \frac{\mu}{\sqrt{1-\lambda^2}\sqrt{1-\nu^2}}.$$

In order to find λ' and ν' we must now use 2ψ of Art. 14 instead of 2θ, which we used in the case of elastic spheres. We have the two symmetrical systems

$$\left.\begin{aligned} \nu' &= -\nu \cos 2\psi + \sqrt{1-\nu^2} \sin 2\psi \cos\phi \\ \lambda' &= -\lambda \cos 2\psi - \frac{\lambda\nu}{\sqrt{1-\nu^2}} \sin 2\psi \cos\phi \end{aligned}\right\},$$

$$\left.\begin{aligned} \nu' &= -\nu \cos 2\psi - \frac{\lambda\nu}{\sqrt{1-\lambda^2}} \sin 2\psi \cos\phi' \\ \lambda' &= -\lambda \cos 2\psi + \sqrt{1-\lambda^2} \sin 2\psi \cos\phi' \end{aligned}\right\}.$$

The term involving $\sin\epsilon$ will disappear because $\sin\epsilon$ contains the factor μ. We find then that, for given V,

$$V^2\overline{\lambda'\nu'} = V^2\int_0^{\frac{1}{2}\pi} d\theta\int_0^{\pi} d\phi\, 2\sin\theta\cos\theta\,(1+2h\lambda V\overline{\xi-\xi_0}+2h\nu V\overline{\zeta-\zeta_0})\,\lambda'\nu'\,;$$

but, with the above values of $\lambda'\nu'$,

$$\int_0^{\frac{1}{2}\pi} d\theta\int_0^{\pi} d\phi\, 2\sin\theta\cos\theta\,\lambda'\nu' = \lambda\nu\int_0^{\frac{1}{2}\pi} d\theta\int_0^{\pi} d\phi\sin\theta\cos\theta(\cos^2 2\psi-\tfrac{1}{2}\sin^2 2\psi),$$

ψ being a function of θ and V known if the law of force is known. This is zero on average, because

$$\overline{\lambda\nu} = 0.$$

Also
$$\int_0^{\frac{1}{2}\pi} d\theta\int_0^{\pi} d\phi\, 2\sin\theta\cos\theta\,\lambda\overline{\xi-\xi_0}\,\lambda'\nu'$$

$$\begin{aligned}
= \frac{d\xi}{dz}\int_0^{\frac{1}{2}\pi} d\theta\int_0^{\pi} d\phi\, 2\sin\theta\cos\theta\,\frac{c}{2}\lambda\,(-\nu\cos\theta+\sqrt{1-\nu^2}\sin\theta\cos\phi)&\\
\times(-\nu\cos 2\psi+\sqrt{1-\nu^2}\sin 2\psi\cos\phi)&\\
\times\left(-\lambda\cos 2\psi-\frac{\lambda\nu}{\sqrt{1-\nu^2}}\sin 2\psi\cos\phi\right)&
\end{aligned}$$

$$\begin{aligned}
= \frac{d\xi}{dz}\int_0^{\frac{1}{2}\pi} d\theta\, 2\sin\theta\cos\theta\,\lambda^2\nu^2\,(-\cos\theta\cos^2 2\psi+\tfrac{1}{2}\sin\theta\sin 2\psi\cos 2\psi&\\
+\tfrac{1}{2}\cos\theta\sin^2 2\psi)&
\end{aligned}$$

$$-\frac{d\xi}{dz}\lambda^2(1-\nu^2)\int_0^{\frac{1}{2}\pi} d\theta\, 2\sin\theta\cos\theta\sin\theta\,(\tfrac{1}{2}\sin 2\psi\cos 2\psi)$$

$$= \frac{d\xi}{dz}\lambda^2\nu^2\int_0^{\frac{1}{2}\pi} d\theta\, 2\sin\theta\cos\theta\,(\tfrac{1}{2}\cos\theta-\tfrac{3}{2}\cos\theta\cos^2 2\psi+\sin\theta\sin 2\psi\cos 2\psi)$$

$$-\frac{d\xi}{dz}\lambda^2\int_0^{\frac{1}{2}\pi} d\theta\, 2\sin\theta\cos\theta\sin\theta\,\tfrac{1}{2}\sin 2\psi\cos 2\psi,$$

and, since on average
$$\lambda^2 = 5\lambda^2\nu^2,$$

this becomes

$$\frac{d\xi}{dz}\lambda^2\nu^2\int_0^{\frac{1}{2}\pi} d\theta\, 2\sin\theta\cos\theta\,(\tfrac{1}{2}\cos\theta-\tfrac{3}{2}\cos\theta\cos^2 2\psi-\tfrac{3}{2}\sin\theta\sin 2\psi\cos 2\psi).$$

Hence, by symmetry between x and z,

$$V^2(\overline{\lambda'\nu'}-\overline{\lambda\nu}) = 2hcV^3\lambda^2\nu^2\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right)$$

$$\times\left\{\tfrac{2}{3}+\int_0^{\frac{1}{2}\pi} d\theta\, 2\sin\theta\cos\theta\,(\tfrac{1}{2}\cos\theta-\tfrac{3}{2}\cos\theta\cos^2 2\psi-\tfrac{3}{2}\sin\theta\sin 2\psi\cos 2\psi)\right\}.$$

The last factor is equal to

$$1-\tfrac{3}{2}\int_0^{\frac{1}{2}\pi} 2\sin\theta\cos^2\theta\cos^2 2\psi\, d\theta-\tfrac{3}{2}\int_0^{\frac{1}{2}\pi} 2\sin^2\theta\cos\theta\sin 2\psi\cos 2\psi\, d\theta.$$

22. Now ψ cannot be less than θ, and cannot be greater than $\frac{\pi}{2}$. Therefore the greatest possible negative value which the two last terms can have is when

$$\psi=\frac{\pi}{2},$$

and that greatest possible negative value is -1. And this is a limiting case that can never occur, because

$$\psi=\frac{\pi}{2},\quad 2\psi=\pi$$

expresses the fact that neither molecule suffers from the encounter any deflexion. It follows that on the average of all encounters which do produce deflexion $V^2(\overline{\lambda'\nu'}-\overline{\lambda\nu})$ is positive, and proportional to $\frac{d\xi}{dz}+\frac{d\zeta}{dx}$; and therefore $\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right)\frac{\partial}{\partial t}\overline{V_xV_z}$ is $\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right)^2$ multiplied by a necessarily positive coefficient.

23. Next we must consider in the same way the term

$$uw\frac{d}{dt}\left(\frac{d\xi}{dz}+\frac{d\zeta}{dx}\right).$$

We have now to take into account not only the variation of λ, μ, ν due to the encounter, but also the variation of position of the molecules, that is, of f. It is sufficient to deal with $\frac{d}{dt}\frac{d\xi}{dz}$. Suppose, then, an encounter to take place between two molecules, their centre of inertia being at the origin. About the origin describe a sphere of radius a, and let PP' be an ordinate parallel to z, P and P' being on the surface of the sphere. Let ξ, ξ' be the values of ξ at P and P'. If Ω be the volume of that sphere,

$$\Omega=\iint PP'\,dx\,dy,$$

and, if $\frac{d\xi}{dz}$ be the mean value of that function throughout the space enclosed by the sphere,

$$\frac{d\xi}{dz} = \iint dx\,dy\,(\xi - \xi') \div \Omega;$$

and therefore $$\frac{d}{dt}\frac{d\xi}{dz} = \iint dx\,dy\left(\frac{d\xi}{dt} - \frac{d\xi'}{dt}\right) \div \Omega.$$

24. Let λ, μ, ν be the direction cosines of the relative velocity before the encounter; λ', μ', ν' the same after the encounter has ceased. Let f be the value at the origin of that function referred to P as centre; f_1, f_2 its values at the centres of the two molecules respectively. At the end of the encounter let f_1, f_2 become f_1', f_2'. Then, in the notation above employed, we have for the change in the value of ξ due to the encounter, omitting the factor $\frac{1}{\Sigma f}$,

$$\partial\xi = (f_1' - f_2')\,\lambda' V - (f_1 - f_2)\,\lambda V,$$

if f be referred to P. Also

$$\lambda' V = \lambda V + \left(-2\lambda\cos^2\psi - \frac{\lambda\nu}{\sqrt{1-\nu^2}}\sin 2\psi\cos\phi\right) V.$$

Therefore $$\partial\xi = +(f_1' - f_2')\,\lambda V - (f_1 - f_2)\,\lambda V$$

$$+ (f_1' - f_2')\left(-2\lambda\cos^2\psi - \frac{\lambda\nu}{\sqrt{1-\nu^2}}\sin 2\psi\cos\phi\right).$$

Also $$f_1' - f_2' = -c\,(-\nu\cos 2\psi + \sqrt{1-\nu^2}\sin 2\psi\cos\phi)\,\frac{df}{dz},$$

$$f_1 - f_2 = -c\,(-\nu\cos\theta + \sqrt{1-\nu^2}\sin\theta\cos\phi)\,\frac{df}{dz};$$

therefore, since, on integrating with reference to ϕ, $\cos\phi$ disappears,

$$\partial\xi = -c\lambda\nu\,(\cos\theta - \cos 2\psi)\,V\frac{df}{dz}$$

$$-c\int_0^\pi d\phi\,(-\nu\cos 2\psi + \sqrt{1-\nu^2}\sin 2\psi\cos\phi)$$

$$\times\left(-2\lambda\cos^2\psi - \frac{\lambda\nu}{\sqrt{1-\nu^2}}\sin 2\psi\cos\phi\right) V\frac{df}{dz};$$

that is, $$\partial\xi = -c\lambda\nu\,(\cos\theta - \cos 2\psi)\,\frac{z}{r}\frac{df}{dr}\,V$$

$$-c\lambda\nu\,(2\cos^2\psi\cos 2\psi - \tfrac{1}{2}\sin^2 2\psi)\,\frac{z}{r}\frac{df}{dr}\,V.$$

The number of encounters per unit of volume and time is $\pi c^2 \rho V$. So, given V,

$$\frac{\partial}{\partial t}\frac{d\xi}{dz} = -2\pi c^3 \rho V^2 \lambda \nu \frac{z}{r}\frac{df}{dr}(\cos\theta - \cos 2\psi + 4\cos^4\psi - 4\cos^2\psi + 2\cos^4\psi)$$

$$= -2\pi c^3 \rho V^2 \lambda \nu \frac{z}{r}\frac{df}{dr}(\cos\theta - 6\cos^2\psi + 1 + 6\cos^4\psi),$$

and $\frac{\partial}{\partial t}\frac{d\xi'}{dz}$ is the same expression with $-z$ for z; therefore

$$\frac{\partial}{\partial t}\frac{d\xi}{dz} = -4\pi c^3 \rho V^2 \lambda \nu \frac{z}{r}\frac{df}{dr}(\cos\theta - 6\cos^2\psi + 6\cos^4\psi + 1),$$

and the expression $\cos\theta - 6\cos^2\psi + 6\cos^4\psi + 1$,

where $\psi > \theta$ and $\psi < \frac{\pi}{2}$, is necessarily positive. For it is equal to

$$(\cos\theta - \cos^2\psi) + (6\cos^4\psi - 5\cos^2\psi + 1).$$

The first term is always positive. The second is negative between certain limits. But it will be found that the greatest negative value of the second term between those limits is less in absolute magnitude than the least value of the positive term between the same limits. Therefore also $\left(\psi \text{ being a known function of } \theta \text{ always } > 0 < \frac{\pi}{2}\right)$

$$\int_0^{\frac{1}{2}\pi} 2\sin\theta\cos\theta\, d\theta\, (1 + \cos\theta + 6\cos^4\psi - 6\cos^2\psi)$$

is positive. Therefore, also $\frac{\partial}{\partial t}\frac{\partial\xi}{\partial z}$ is of the opposite sign to $\lambda\nu\frac{df}{dr}$, *i.e.*, to $uw\frac{df}{dr}$. Therefore also $uw\frac{\partial}{\partial t}\frac{d\xi}{dz}$ is of the opposite sign to $\frac{df}{dr}$, and is therefore positive because $\frac{df}{dr}$ is negative. The same is true of $uw\frac{\partial}{\partial t}\frac{d\zeta}{dx}$.

25. We see, then, that, whereas for a system of material points without mutual action

$$\frac{dM}{dt} = 0 = \overline{uw}\frac{d}{dt}\left(\frac{d\xi}{dz} + \frac{d\zeta}{dx}\right) - \frac{1}{2h}\left(\frac{d\xi}{dz} + \frac{d\zeta}{dx}\right)^2,$$

when they exert finite forces on each other $\frac{dM}{dt}$ is increased by necessarily positive terms. In order to make $\frac{dM}{dt}$ again zero, and so restore the stationary character of the motion, we must, as in the case of elastic spheres, use $\overline{\xi^2}+\overline{\xi'^2}$ instead of $\overline{\xi^2}$, and this requires that Q shall contain the terms

$$\Sigma\Sigma b\,(uu'+vv'+ww').$$

26. In my former paper I drew the conclusion, from the form assumed in stationary motion by the function Q, that molecules which at any instant happen to be near to one another are, for that reason, more likely to be moving in the same direction; hence that any mass of contiguous molecules have on the average greater energy of the motion of their common centre of inertia, and correspondingly less energy of relative motion, than they would have as material particles without mutual action in Maxwell's system. But the following conclusion may also be drawn, namely, that local inequalities of density are more probable, or greater and more frequent, in the system of finite molecules than they are in Maxwell's system. For, in Maxwell's system, whatever at any instant be the velocities, the molecules are as likely to have any assigned positions in space as any other assigned positions. Suppose two spheres, each of radius a, not intersecting each other, and the average number of molecules which each contains is n. The chance that, at a given instant, one shall contain $n+q$, and the other $n-q$, molecules can be calculated; and, if q be considerable, is much smaller than that each shall contain n. But, whatever it be, the effect of introducing the terms $\Sigma\Sigma b(uu'+vv'+ww')$ into Q is to increase it in a finite ratio. For, if b vary as $\frac{x^5}{r^5}$ and is negative, the mean value of b, for all pairs of points within one of the a spheres, varies as $\frac{x^3}{a^3}$.

We may assume the two spheres to be so far apart that, for any pair of points, one in one a sphere and one in the other, b is negligible. Then, of all the pairs which can be formed out of the $2n$ molecules in the two spheres, $\frac{\overline{n+q}\,\overline{n+q-1}}{2}$ pairs in one sphere, and $\frac{\overline{n-q}\,\overline{n-q-1}}{2}$ in the other sphere, have on average $-\frac{c^3}{a^3}$ for b, and all the other pairs have zero b. Now

$$\frac{\overline{n+q}\,\overline{n+q-1}}{2}+\frac{\overline{n-q}\,\overline{n-q-1}}{2}=n^2+q^2-n,$$

and increases as q increases, and has its greatest value when $q = n$. The effect, therefore, of the inequality of density is to increase the absolute value of $\Sigma\Sigma b(uu'+vv'+ww')$, and, therefore, to diminish Q. And, to that extent, the inequalities become more probable. So that, as the general density increases, the system tends more and more to assume the form of a number of denser masses moving through a comparatively rare medium.

Thursday, February 10th, 1898.

Professor E. B. ELLIOTT, F.R.S., President, in the Chair.

Thirteen members and a visitor present.

Mr. B. Hopkinson was admitted into the Society.

Lt.-Col. Cunningham read a paper "On Aurifeuillians," which Mr. Bickmore supplemented at some length by a treatment of the subject from another point of view. A discussion took place on the two communications.

Lt.-Col. Cunningham having (*pro tem.*) taken the Chair, the President communicated a paper by Mr. J. E. Campbell, entitled "On the Transformations which leave the Lengths of Arcs on Surfaces unaltered."

Mr. Hargreaves made a brief impromptu communication.

The following presents were made to the Library:—

"Proceedings of the Royal Society," Vol. LXII., Nos. 382, 383.

"Proceedings of the Royal Irish Academy," Vol. IV., No. 4; Dublin, 1897.

"Bulletin de la Société Mathématique de France," Tome XXV., Nos. 8, 9, et dernier; Paris, 1897.

"Journal of the Institute of Actuaries," Vol. XXXIII., Pt. 6; January, 1898.

"Monatshefte für Mathematik und Physik," 1898, Pt. 1; Wien, 1898.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. XIII., No. 3; Coimbra, 1897.

"Bulletin des Sciences Mathématiques," Tome XXI., December, 1897; Tome XXII., January, 1898.

Laisant, C. A.—"La Mathématique, Philosophie-Enseignement," 8vo; Paris, 1898.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XXI., St. 11, 12, 1897; Bd. XXII., St. 1, 1898; Leipzig.

"Proceedings of the Physical Society," Vol. xv., Pt. 12, No. 87; December, 1897.

"Wiskundige Opgaven met de Oplossingen door de Leden van het Wiskundige Genootschap Amsterdam," Deel VII., St. 4; Amsterdam, 1898.

"Nieuw Archief voor Wiskunde," Deel III., St. 3; Amsterdam, 1897.

"Revue Semestrielle des Publications Mathématiques," Tome VI., 1897, Av.–Oct.; Amsterdam, 1898.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Vol. III., Fasc. 12; Napoli, 1897.

"Bulletin of the American Mathematical Society," Series 2, Vol. IV., No. 4; January, 1898.

"Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen," Geschäftliche Mittheilungen, Heft 2, Mathematisch-Physikalische Klasse, 1897, Heft 3.

"Annali di Matematica," Tomo XXVI., Fasc. 4; Milano, December, 1897.

"Atti della reale Accademia dei Lincei—Rendiconti," Sem. 2, Vol. VI., Fasc. 12; Sem. 1, Vol. VII., Fasc. 1; Roma.

Wundt, W.—"Die Geometrisch Optischen Täuschungen," 8vo; Leipzig, 1898.

"Transactions of the Cambridge Philosophical Society," Vol. XVI., Pt. 3; 1898.

"Proceedings of the Cambridge Philosophical Society," Vol. IX., Pt. 7; 1898.

"Journal für die reine und angewandte Mathematik," Bd. CXIX., Heft 1; Berlin, 1898.

"Educational Times," February, 1898.

"Indian Engineering," Vol. XXII., No. 26; Vol. XXIII., Nos. 1–3; Dec. 25, 1897–Jan. 15, 1898.

Transformations which leave the Lengths of Arcs on Surfaces unaltered. By J. E. Campbell. Received and read February 10th, 1898.

In this paper it is shown, by an application of Lie's method of contact transformations, that the infinitesimal transformations which have the property of leaving unaltered the lengths of arcs on any given surface, $z = f(x_1, \dots x_n)$, in space of $n+1$ $(n>2)$ dimensions are in general mere motions in $n+1$ dimensional space (§ 3). For a special class of surfaces there is a more general transformation than a mere motion; such special surfaces must be the envelopes of planes

$$z = ax_1 + bx_2 + {}_3f(a, b)\, x_3 + \dots + {}_nf(a, b)\, x_n + {}_{n+1}f(a, b),$$