A Variance–AM–GM Inequality Conjecture: A Call to the Mathematical Community

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Abstract

This note proposes a new inequality linking the classical arithmetic and geometric means with the variance of positive real variables. The conjectured relation, which we call the *Variance–AM–GM Inequality*, offers a quantitative bridge between additive and multiplicative dispersion. Its resolution may yield new insights in convex analysis, information theory, and statistical physics. The purpose of this paper is to formally state the conjecture, provide motivation for its structure, and invite the mathematical community to verify, refine, or refute it.

1 Introduction

The arithmetic–geometric mean inequality (AM–GM)

$$A \geq G$$

is among the foundational results of analysis, combinatorics, and probability. Numerous refinements of AM–GM have been developed (see, e.g., Mitrinović [1], Bullen [2]), yet an explicit closed-form lower bound on $A-G$ in terms of the variance of the underlying variables remains elusive.

In this note we introduce an inequality conjecture that attempts to fill this gap by directly coupling the *additive dispersion* measured by variance with the *multiplicative dispersion* captured by the AM–GM difference. The inequality is empirically motivated and numerically verified for many special cases, though a general proof (or counterexample) is presently unknown.

2 The Conjecture

Let $a_1, \ldots, a_n > 0$. Define

$$A = \frac{1}{n}\sum_{i=1}^{n} a_i, \quad G = \left(\prod_{i=1}^{n} a_i\right)^{1/n}, \quad \sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(a_i - A)^2.$$

The **Variance–AM–GM Inequality Conjecture** asserts that

$$A - G \geq \frac{\sigma^2}{A(n-1)}. \tag{1}$$

Equality is conjectured to be approached asymptotically when one entry tends to infinity while the remaining $n-1$ entries tend to zero in a balanced manner.

3 Motivation and Heuristic Evidence

The form of (1) is motivated by two classical limits:

1. **Small dispersion:** When $a_i \approx A$ for all i, a second-order Taylor expansion of $\log a_i$ around A yields $A - G \approx \frac{\sigma^2}{2A}$, suggesting the correct order of growth.

2. **Extreme dispersion:** When one variable dominates the rest, asymptotic calculations indicate $A - G \sim \frac{\sigma^2}{A(n-1)}$, giving rise to the constant $(n-1)^{-1}$.

This conjecture therefore interpolates between the two asymptotic regimes and proposes a simple, scale-homogeneous form.

4 Potential Impact

If proved, (1) would establish a direct, dimensionally consistent connection between the second central moment and multiplicative deviation. This could:

- provide improved error estimates in inequalities involving means,
- offer sharper stability bounds for optimization problems where AM–GM is applied,
- yield new variance-based characterizations of multiplicative inequalities in probability and information theory,
- and clarify the relationship between additive and logarithmic convexity.

5 Numerical Evidence and Next Steps

Preliminary computations suggest that (1) is tight for highly unbalanced vectors, but may fail for intermediate cases, indicating a need for refined constants or piecewise regimes. Future work should:

1. Determine whether a universal constant C_n exists such that
$$A - G \geq C_n \frac{\sigma^2}{A(n-1)}.$$

2. Identify extremal configurations (possibly two-valued distributions) that minimize or maximize this ratio.

3. Explore connections to log-Sobolev inequalities and entropy convexity.

6 Conclusion

The Variance–AM–GM Inequality represents an open challenge bridging variance and geometric mean analysis. Whether (1) holds in full generality, or only in limiting cases, its study may illuminate the geometry of inequalities and inspire new techniques across mathematical sciences.

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References

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