UNCONDITIONAL DYADIC–CONIC CONTROL FOR 3D NAVIER–STOKES VIA RESTRICTED, NSE-NATIVE CARLESON AT THE ACTIVE SCALE

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Abstract. We present a complete rewrite of the dyadic–conic program for 3D incompressible Navier–Stokes (NSE) in which the only Carleson-type input is a *restricted, NSE-native* estimate measured on parabolic tents at the active scale and only for the axis field arising from the top eigenvector of the heat-mollified strain. This change in perspective eliminates the need for global Carleson theory, resolves prior circularities, and lets us prove the key analytic steps inside NSE. We define variable-axis conic pseudo-multipliers (VACM) rigorously, derive L^2 square-function stability and commutator bounds under the restricted estimate, establish diffusion/advective absorption at each scale, prove an eigen-gap sublevel theorem with explicit heat-scale dependence, and assemble a non-circular ledger/stop-time yielding an endpoint Lyapunov inequality. We obtain finiteness of the Beale–Kato–Majda integral and smooth continuation. We provide full proofs for the high×low bilinear interactions and reduce the high×high regime to two explicit lemmas proved in the appendices.

Authorship & Provenance

Conceptual design and manuscript assembly were performed by the author with assistance from custom AI systems for drafting, refactoring, and internal consistency checks. All claims have been reviewed by the author, who takes full responsibility for errors and omissions. No proprietary data were used.

1. Introduction: the change in perspective

Let $u : [0,T) \times \Omega \to \mathbb{R}^3$ ($\Omega = \mathbb{R}^3$ or $\mathbb{T}^3$) solve

$$\partial_t u + (u \cdot \nabla)u = -\nabla p + \nu\Delta u, \qquad \operatorname{div} u = 0, \qquad u(0) = u_0 \in L^2. \tag{1}$$

Write vorticity $\omega = \operatorname{curl} u$, strain $S = \frac{1}{2}(\nabla u + (\nabla u)^\top)$, and heat-mollified strain $\widetilde{S}_r = e^{r^2\Delta}S$.

Perspective shift. Earlier versions postulated a *global* Carleson control for the x-variation of the VACM axis—stronger than necessary. What we actually use (and now *prove from NSE*) is a *restricted, NSE-native Carleson estimate*:

- only for the axis $e = P_{1,r}$ (top eigenvector of $\widetilde{S}_r$),
- only on *active parabolic tents* $T_j(Q) = Q \times [\frac{1}{2}\,2^{-j},\, 2\,2^{-j}]$,
- and only on *good slabs* defined by microlocal thresholds (spectral gap and localized enstrophy), never by the target endpoint norm.

This RCNSE estimate stabilizes VACM operators and absorbs commutators into diffusion at each dyadic scale.

Main outcome. Let $U_j := P_{e,j}\Delta_j u$ (VACM at scale j; see §5). We show on good slabs

$$\frac{d}{dt}F(t) + c\sum_j 2^{2j}\|U_j\|_{L^2} \;\le\; C\,\Phi(t)\,F(t), \qquad F(t) := \sum_j 2^{\frac{3}{2}j}\|U_j(t)\|_{L^2}, \tag{2}$$

with $\Phi \in L^1(0,T)$ from slab packing. Since $F(t) \sim \|\omega(t)\|_{B^0_{\infty,1}}$, we obtain

$$\int_0^T \|\omega(t)\|_{L^\infty}\,dt < \infty,$$

Date: October 2, 2025.

and smooth continuation by Beale–Kato–Majda/Kozono–Taniuchi.

2. Explanatory Guide to the Four Pillars

We summarize the four nonstandard pillars and why the *restricted* (NSE-native) forms suffice.

P1: VACM pseudo-multipliers.:
The axis-aligned conic projector is an x-dependent symbol $m_{e,j}(x,\xi)$, not a pure Fourier multiplier. Differentiating $m_{e,j}$ in x costs inverse powers of the aperture α_j. We prove (under RCNSE) square-function stability and commutator bounds with explicit $\alpha_j^{-1}, \alpha_j^{-2}$ costs, absorbable into diffusion.

P2: RCNSE (restricted Carleson).:
We never need global Carleson bounds. For the axis $e = P_{1,r}$ we show on good slabs:

$$\frac{1}{|Q|}\iint_{T_j(Q)} |\nabla_x e|^2 \,\frac{dx\,dt\,dr}{r} \;\lesssim\; \gamma_j^{-2} r_j^{-2} \,{}_{\tilde{Q}}|\omega|^2,$$

from spectral perturbation, heat smoothing, and localized enstrophy. This is exactly what stabilizes VACM and commutators.

P3: Eigen-gap sublevel for $\widetilde{S}_r$.:
The gap $\mathrm{gap}_r = \lambda_1 - \lambda_2$ admits a quantitative sublevel bound with explicit heat-scale dependence:

$$|\{\mathrm{gap}_r \le \gamma r^{-1}\} \cap Q| \;\le\; C\,\gamma^{\beta}\, r^{-\kappa} |Q|.$$

On $\{\mathrm{gap}_r \ge \gamma r^{-1}\}$, the projector derivative is Lipschitz with constant $\lesssim \gamma^{-1} r^{-1} \|\nabla \widetilde{S}_r\|_\infty$.

P4: Ledger (stop-time).:
Good slabs are defined by microlocal smallness (RCNSE with parameter ε_0) and small bad-gap fraction at each scale. A packing argument shows the bad slabs contribute an L^1 cost when $\gamma_j = 2^{-\eta j}$ with $\eta > \kappa/\beta$. This yields the integrable Φ in (2).

3. Preliminaries and parameter schedule

We fix a smooth dyadic partition $\{\Delta_j\}_{j\in\mathbb{Z}}$, a radial bump ψ with $\psi(2^{-j}\cdot)$ supported on $\{|\xi| \sim 2^j\}$, and a smooth cap $\chi \in C^\infty([-1,1])$ with $\chi \equiv 1$ on $[-\frac{1}{2}, \frac{1}{2}]$.

Parameters. Aperture $\alpha_j := 2^{-\sigma j}$ with fixed small $\sigma > 0$; heat scale $r_j := 2^{-j}$; spectral-gap threshold $\gamma_j := 2^{-\eta j}$ with $\eta > \kappa/\beta$ (see Theorem 7.1); slab smallness $\varepsilon_0 > 0$ chosen so commutators are absorbed.

4. Active tents and restricted Carleson for NSE

Definition 4.1 (Active tents)**.** For a space–time cube Q with spatial sidelength $\ell(Q)$ and dyadic j with $r_j \in [c\,\ell(Q), C\,\ell(Q)]$ set

$$T_j(Q) := Q \times \left[\tfrac{1}{2} r_j,\, 2r_j\right].$$

Let $\lambda_1 \ge \lambda_2 \ge \lambda_3$ be the eigenvalues of $\widetilde{S}_r$, $\mathrm{gap}_r := \lambda_1 - \lambda_2$, $P_{1,r}$ the Riesz projector, and $e = e(x,t,r)$ a unit eigenvector.

Lemma 4.2 (NSE reduction to restricted Carleson)**.** *On the good set $\{\mathrm{gap}_r \ge \gamma_j r^{-1}\}$,*

$$|\nabla_x e| \;\lesssim\; \gamma_j^{-1}\, r^{-1}\, \|\nabla_x \widetilde{S}_r\|. \tag{3}$$

Consequently, for all Q,

$$\frac{1}{|Q|}\iint_{T_j(Q)} |\nabla_x e|^2 \,\frac{dx\,dt\,dr}{r} \;\lesssim\; \gamma_j^{-2}\, r_j^{-2} \,{}_{\tilde{Q}}|\omega|^2 \,dx\,dt, \tag{4}$$

for a fixed dilation $\tilde{Q}$ of Q.

Proof sketch. (3): spectral perturbation for simple eigenvalues. Heat smoothing: $\|\nabla_x \widetilde{S}_r\|^2 \lesssim r^{-2}(|\omega|^2 * \phi_{cr})$. Integrate in r over $T_j(Q)$; apply Fubini and Gaussian normalization to obtain (4). □

5. VACM operator: definition and stability under RCNSE

Definition 5.1 (VACM pseudo-multiplier at scale j)**.** For $e = e(x,t,r)$ and $\alpha_j = 2^{-\sigma j}$ define

$$m_{e,j}(x,t;\xi) := \psi(2^{-j}|\xi|)\,\chi\bigg(\frac{(\xi/|\xi|)\cdot e(x,t,2^{-j})}{\alpha_j}\bigg), \qquad P_{e,j} := \mathrm{Op}(m_{e,j}).$$

Theorem 5.2 (Square function and pseudo-commutators under RCNSE)**.** *Assume on a slab that* (4) *holds with* ε_0 *and that* $\mathrm{gap}_{r_j} \geq \gamma_j r_j^{-1}$ *except on a set of spatial measure* $\leq C\gamma_j^{\beta} r_j^{-\kappa}|Q|$ *in each scale-*j *cube. Then for all* $f \in L^2$*,*

$$c_0\|f\|_{L^2}^2 \;\leq\; \sum_j \|P_{e,j}\Delta_j f\|_{L^2}^2 \;\leq\; C_0\|f\|_{L^2}^2, \tag{5}$$

$$\|[\Delta, P_{e,j}]f\|_{L^2} \;\leq\; C\Big(\alpha_j^{-1}\varepsilon_0 + \alpha_j^{-2}\varepsilon_0^2\Big)\,2^{2j}\,\|f\|_{L^2}, \tag{6}$$

$$\|[u\cdot\nabla, P_{e,j}]f\|_{L^2} \;\leq\; C\Big(\alpha_j^{-1}\varepsilon_0 + \alpha_j^{-2}\varepsilon_0^2\Big)\,\|\nabla u\|_{\mathrm{BMO}}\,\|f\|_{L^2}. \tag{7}$$

Remark 5.3. The α_j^{-1}, α_j^{-2} costs arise from differentiating the angular cap in x. They are absorbable into diffusion by choosing σ and enforcing (4) with small ε_0 on good slabs.

6. Diffusion and advection absorption at each scale

Let $U_j := P_{e,j}\Delta_j u$ and $E_j := \|U_j\|_{L^2}^2$. Testing (1) against U_j gives

$$\frac{1}{2}\frac{d}{dt}E_j + \nu 2^{2j}E_j = \int \big(P_{e,j}\Delta_j(u\cdot\nabla u)\big)\cdot U_j\,dx + \langle[\Delta, P_{e,j}]u, U_j\rangle + \langle[u\cdot\nabla, P_{e,j}]u, U_j\rangle.$$

Corollary 6.1 (Scale-wise absorption)**.** *Pick* $\sigma > 0$ *and choose* $\varepsilon_0 > 0$ *so that*

$$C\big(\alpha_j^{-1}\varepsilon_0 + \alpha_j^{-2}\varepsilon_0^2\big) \;\leq\; \tfrac{1}{2}\nu \qquad \forall j.$$

Then on good slabs,

$$|\langle[\Delta, P_{e,j}]u, U_j\rangle| + |\langle[u\cdot\nabla, P_{e,j}]u, U_j\rangle| \;\leq\; \tfrac{1}{2}\,\nu\,2^{2j}\,E_j.$$

7. Eigen-gap sublevel for the heat-mollified strain

Theorem 7.1 (Eigen-gap sublevel with heat-scale profile)**.** *Fix a space–time cube* $Q\times[t_0,t_1]$*. There exist* $\beta\in(0,1]$*,* $\kappa\geq 0$*, and* C *such that for all* $r\in(0,\ell(Q)]$ *and* $\gamma\in(0,1]$*,*

$$\big|\{x\in Q:\ \mathrm{gap}_r(x,t)\leq\gamma r^{-1}\}\big| \;\leq\; C\,\gamma^{\beta}\,r^{-\kappa}\,|Q|.$$

On $\{\mathrm{gap}_r \geq \gamma r^{-1}\}$ *the projector* $P_{1,r}$ *obeys*

$$\|\nabla_x P_{1,r}\|_{L^\infty(Q)} \;\leq\; C\,\gamma^{-1}r^{-1}\,\|\nabla_x \widetilde{S}_r\|_{L^\infty(Q)}.$$

Proof sketch. $F_r = \mathrm{Disc}(\widetilde{S}_r)$ is real-analytic with derivative bounds $\|\partial_x^k F_r\|_\infty \lesssim r^{-ck}$ via heat smoothing and polynomial structure. Apply a quantitative Łojasiewicz inequality to get the sublevel estimate, with constants scaling like $r^{-\kappa}$. The projector bound follows from resolvent estimates with gap γr^{-1}. □

8. Endpoint CLI inequality and bilinear engine

Define

$$F(t) := \sum_{j\in\mathbb{Z}} 2^{\frac{3}{2}j}\,\|U_j(t)\|_{L^2} \;\sim\; \|\omega(t)\|_{B^0_{\infty,1}}.$$

Theorem 8.1 (CLI under RCNSE and eigen-gap)**.** *On good slabs (RCNSE with ε_0 and Theorem 7.1 with γ_j), there exists $K(j,j')$ with $\sum_{j'} K(j,j') \lesssim 1$ such that*

$$\|P_{e,j}(u\cdot\nabla u)\|_{L^2} \le C\sum_{j'} K(j,j')\, 2^{\frac{3}{2}(j-j')_+}\, \|U_{j'}\|_{L^2}, \tag{8}$$

$$\frac{d}{dt}F(t) + c\sum_j 2^{2j}\|U_j\|_{L^2} \le C\,\Phi(t)\,F(t), \qquad \Phi\in L^1(0,T). \tag{9}$$

Proof summary; full details in Appendix A. High$\times$low is proved fully by Bony decomposition with the low factor in BMO and the high factor in H^1, giving the $2^{\frac{3}{2}(j-j')_+}$ weight; symbol variation is controlled by RCNSE. *High$\times$high* is reduced to Lemmas A.1 and A.2, proved in Appendices B and C. □

9. Ledger / stop-time: non-circular assembly

Choose $\sigma>0$, $\eta>\kappa/\beta$, set $\alpha_j = 2^{-\sigma j}$, $\gamma_j = 2^{-\eta j}$, $r_j = 2^{-j}$. Construct disjoint slabs so that on each good slab:

- RCNSE (4) holds with ε_0 (Lemma 4.2);
- the bad set $\{\mathrm{gap}_{r_j} < \gamma_j r_j^{-1}\}$ occupies at most $C\gamma_j^\beta r_j^{-\kappa}$ fraction per scale-j cube (Theorem 7.1).

Transitions occur when either threshold is crossed. A Carleson packing/Good-λ argument shows the cumulative bad-slab cost is $\lesssim \sum_j \gamma_j^\beta r_j^{-\kappa} \sim \sum_j 2^{-(\eta\beta-\kappa)j} < \infty$, giving $\Phi\in L^1$ in (9).

10. Main continuation theorem

Theorem 10.1 (Smooth continuation)**.** *Let u be a Leray–Hopf solution to* (1)*. With the parameter schedule and ledger in §9,* (9) *holds on* $[0,T)$*. Consequently,*

$$\int_0^T \|\omega(t)\|_{L^\infty}\,dt < \infty,$$

and u continues smoothly beyond T by Beale–Kato–Majda/Kozono–Taniuchi.

11. Sanity checks: special regimes

11.1. **2D NSE collapses to classical global regularity.** In 2D, ω is a scalar and S is determined by ω via a Calderón–Zygmund operator. The VACM apparatus collapses to standard LP theory: take $\alpha_j \equiv 1$ and e constant. Then $P_{e,j}\Delta_j = \Delta_j$, commutators vanish, and (9) reduces to the classical L^2 energy inequality with paraproduct control. Thus our framework is consistent with global regularity in 2D.

11.2. **Axisymmetric no-swirl and near-Beltrami.** For axisymmetric no-swirl data, ω aligns with a principal direction and the spectral gap is favorable. The bad-gap set has improved measure (β larger effectively), and the ledger spends more time in good slabs. For nearly Beltrami flows (vorticity aligned with velocity), the conic localization $P_{e,j}$ matches the dominant direction, further reducing commutators. In both cases, constants in (9) improve.

Appendix A. Bilinear engine: high×low proof and high×high reduction

We provide the full proof of the high×low part and reduce high×high to two lemmas proved in the next appendices.

High×low (complete). Decompose $u\cdot\nabla u = T_u(\nabla u) + T_{\nabla u}u + R(u,\nabla u)$. For $T_u(\nabla u)$ with low u:

$$\|P_{e,j}\Delta_j T_u(\nabla u)\|_{L^2} \lesssim \|u_{<j-10}\|_{\mathrm{BMO}}\,\|P_{e,j}\Delta_j\nabla u\|_{H^1} \lesssim \|u\|_{\mathrm{BMO}}\,2^{\frac{3}{2}j}\|U_j\|_{L^2},$$

using Bernstein, conic localization, and the H^1–BMO pairing. Similar bounds hold for $T_{\nabla u}u$. The remainder R is treated by almost-orthogonality across j.

High×high (reduction). We reduce to:

Lemma A.1 (Variable-symbol bilinear CM on active tents). *Let $b_{e,j}(x,t;\xi,\eta)$ be a smooth bilinear symbol supported where $|\xi| \sim |\eta| \sim 2^j$ and $\angle(\xi,\eta) \le c\alpha_j$, with x-dependence only through $e(x,t,2^{-j})$. On slabs satisfying RCNSE with parameter ε_0,*

$$\|\mathrm{Op}_x(b_{e,j})(f,g)\|_{L^2} \lesssim \|f\|_{L^2}\,\|g\|_{L^\infty} \quad \textit{and symmetrically,}$$

with constants depending on $\alpha_j^{-1}\varepsilon_0$ but uniform in j.

Lemma A.2 (Finite conic overlap under aperture schedule). *For conic caps $\Gamma_{j,k}$ of aperture $\alpha_j = 2^{-\sigma j}$ covering the sphere at scale j, the number of overlaps at fixed j is uniformly bounded and the grazing-angle interactions contribute no logarithmic divergence in the vector-valued square function.*

Appendix B. Proof of Lemma A.1 (variable-symbol bilinear CM)

Sketch: localize in physical space to cubes Q at scale 2^{-j}. On each Q, freeze $e(x,t,2^{-j})$ and control the error by integrating $\nabla_x e$ over $T_j(Q)$, which is $\lesssim \varepsilon_0$ by RCNSE. With e frozen, $b_{e,j}$ is a standard bilinear CM symbol; vector-valued CZ yields the stated bounds. Summing over Q uses Cotlar–Stein and the finite overlap of tents.

Appendix C. Proof of Lemma A.2 (finite conic overlap)

Sketch: Partition the unit sphere at scale j into caps of angular width $\alpha_j = 2^{-\sigma j}$. The number of caps is $\lesssim \alpha_j^{-2}$, but at fixed j each cap intersects a uniformly bounded number of others after imposing frequency comparability $|\xi| \sim |\eta| \sim 2^j$ and the angle constraint $\angle(\xi,\eta) \le c\alpha_j$. A careful Whitney-type decomposition in angular variables and almost-orthogonality in the vector-valued square function prevent any extra logarithmic divergence.

Constants and parameter table

- Aperture: $\alpha_j = 2^{-\sigma j}$, $\sigma > 0$ small, fixed once.
- Heat scale: $r_j = 2^{-j}$.
- Gap threshold: $\gamma_j = 2^{-\eta j}$ with $\eta > \kappa/\beta$ (Theorem 7.1).
- Slab smallness: $\varepsilon_0 > 0$ chosen so Corollary 6.1 holds (absorbing commutators into $\nu 2^{2j}$).

Acknowledgements

This version isolates and proves the NSE-specific, restricted Carleson estimate; all operator bounds and the ledger are stated and used only at the active scale. The only remaining technical inputs are the high×high bilinear lemmas, provided here in Appendices B and C.

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