Topological Unification of Lepton and Baryon Masses: The Color Imprint of the Non-Abelian Vacuum

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Abstract

We present a topological–geometric framework, formulated as effective field theories with solitonic solutions, that unifies mass scaling in leptons and baryons with very few global parameters. We show that a single power law $M = C\left[S_{\mathrm{top}} + \delta\right]^{\eta}$ emerges from soliton-based EFTs in leptons (vortices) and baryons (hopfions), with δ fixed by a coherence-window/RG mechanism (not fitted). The key ingredient is a universal offset—set by coherence-window and renormalization arguments, not by data fitting—that imprints the structure of the non-Abelian vacuum on both sectors. Calibrating with two anchors per sector, the scheme reproduces known masses and yields out-of-anchor predictions: the tau lepton at sub-percent level and light-baryon systematics at the ∼1% scale. The baryonic topological assignments are minimal and falsifiable. The approach is parsimonious and grounded in EFT and RG considerations. We conclude with proposed numerical/experimental tests and a set of relations designed to be refuted by additional data or variational simulations.

Keywords: baryons; leptons; topology; Hopf; QCD; non-Abelian vacuum; power law.

1. Introduction

1.1. Motivation: mass hierarchy and parsimony

The mass hierarchy of elementary particles—ranging from leptons (e, μ, τ) to the light baryons (p, n, Λ, Σ, Δ)—spans several orders of magnitude. In the Standard Model this hierarchy is encoded in a priori independent Yukawa couplings [15], effectively shifting the problem to a list of parameters without a simple organizing principle. In parallel, in condensed-matter theories, topological defects such as vortices and knots exhibit universal energetic laws governed by discrete invariants. This coexistence suggests a guiding question: can topology provide a minimal, unifying law for baryonic and leptonic masses? Our aim is to propose—and test—an affirmative answer with the smallest possible number of hypotheses and parameters.

1.2. Landscape: topological approaches in baryons and vortices

Solitonic models have captured qualitative—and at times quantitative—aspects of the baryon spectrum. The Skyrme model identifies baryons with solitons classified by $\pi_3(S^3) \neq 0$ (Skyrmions) [1,2]; within the Faddeev–Skyrme family one encounters Hopfions, configurations with Hopf charge Q that link preimages [7,14] and whose energy robustly follows a power law $E(Q) \sim Q^{3/4}$ (the Vakulenko–Kapitanskii law) across broad regimes

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[4–6]. In parallel, vortices in abelian $O(2)$-type effective gauge theories exhibit a winding number n_L as a phase invariant [10]. These two building blocks—vortices and knots—provide natural topological counters to organize energetic hierarchies. However, the literature typically treats them in isolation (baryons vs. vortex-like excitations) or requires a large parameter set to fit data. What is missing, therefore, is a compact, transversal law encompassing both leptons and baryons with a common geometric interpretation and a clear physical anchoring for the few parameters that remain.

1.3. Contribution of this work

Position relative to the state of the art.

(i) **Baryons (topology):** Skyrme and Faddeev–Skyrme models describe baryons as solitons with topological charge and capture spectral trends, but they do not address lepton masses.

(ii) **Leptons (vortex analogs):** Vortex-type configurations in Abelian theories (e.g., Abelian-Higgs/O(2) frameworks) are standard as EFTs and in condensed-matter analogs, yet a quantitative, unified mass law for leptons has not been proposed.

(iii) **Non-topological unifications:** GUT schemes (SU(5), SO(10), etc.) typically introduce new energy scales (e.g., the GUT scale) and symmetry structures; they do not leverage a universal geometric scaling law rooted in the vacuum. Likewise, flavor models relate masses via Yukawa textures and symmetries, rather than through topological invariants common to leptons and baryons.

Our contribution.

We introduce a topological–geometric framework, cast as solitonic EFTs, that unifies lepton and baryon mass scaling into a single power law across leptons (vortices) and baryons (hopfions). The key novelty is a **universal offset** δ, set *a priori* by coherence-window/RG arguments (not fitted to data), which acts as a common geometric counterterm. With two anchors per sector, the scheme reproduces known masses and delivers out-of-anchor predictions (τ and light baryons) with $O(1\%)$ errors. To our knowledge, this is the first approach that attributes lepton–baryon unification to a shared geometric rationale—soliton energies governed by topological invariants—with a universal offset grounded in non-Abelian-vacuum considerations.

1.4. Structure of the paper

In Section 2 we state the unified mass law and specialize it by sector, providing a geometric motivation for the 3/4 exponent in baryons and clarifying the operational role of the shift δ; in Section 3 we justify the origin and magnitude of δ from a "coherence window" argument and its connection to the non-Abelian vacuum, consistent with $N_c = 3$; Section 4 provides the theoretical rationale for the Q assignments (toroidal ansatz, $Q = pq$, baseline $T(3,2)$, decuplet $T(3,3)$; Section 5 details the calibration methodology, explaining how to fix (C,η) in each sector and how to extract g_s *and* g_{rad} while avoiding circularity; Section 6 presents the results—on the lepton side, the fit using (e,μ) and the prediction of m_τ with a sensitivity analysis; on the baryon side, the reconstruction of p, n, Λ, Σ, Δ and a cross-validation of the scheme; finally, Section 7 summarizes conclusions and sketches directions for future work.

2. EFT reading of the unified law (leptons and baryons)

Starting from two effective field theories—*O*(2) Abelian–Higgs vortices with a higher-derivative stabilizer in the leptonic sector, and CP^1/Faddeev–Skyrme in the baryonic sector—we obtain a single power law for particle masses:

$$M_X = C\left[S_{\mathrm{top}}(X) + \delta\right]^{\eta},$$

where S_{top} is a dimensionless topological functional, δ is an effective offset (coherence/renormalization), and C,η are sector-dependent constants (leptonic/baryonic).

In the EFT interpretation, η is fixed by the dominant stabilizer within the soliton's coherence window (*O*(2) vortices for leptons; CP^1/Faddeev–Skyrme hopfions for baryons), while δ is a universal counterterm that encodes coarse-graining between the object's internal scales. The construction and numerical value of δ are detailed in Sec. 3. We work in units $\hbar$=c=1.

Key symbols. M_X (mass), S_{top} (topological action), n_L (leptonic winding number), Q (Hopf charge), $f_{\mathrm{knot}}\, g_{\mathrm{flavor}}\, g_{\mathrm{rad}}$ (dimensionless factors), δ (shift), C and η (sector constants), β_0 (beta-function coefficient), N_c (number of colors), N_f (number of flavors).

2.1. Sector leptónico (vórtices O(2))

We now provide a field-theoretic basis for the leptonic side of the mass law. Unlike a mere ansatz, the power-law scaling arises naturally by minimizing the energy in a vortex–soliton model.

We consider an Abelian–Higgs effective theory with a higher-derivative stabilizer,

$$\mathcal{L}_\ell = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \left|D_\mu\phi\right|^2 - \frac{\lambda}{2}(|\phi|^2 - v^2)^2 + \frac{\Lambda^{2p-2}}{\xi^{2p}}|D^p\phi|^2$$

whose finite-core vortex solutions carry the winding

$$n_L = \frac{1}{2\pi}\oint_\Gamma \nabla\theta \cdot \mathrm{dl} \in Z$$

For a thin toroidal ring of length *L*, the line-reduced energy takes the form

$$E(L, n_L) \simeq \tau L + \frac{\kappa_{tw}}{L}(2\pi n_L)^2 + \frac{\kappa_{bend}}{L} + \frac{c_p}{L^{2p-1}}(2\pi n_L)^{2p}$$

In the regime where the string tension competes with the $2p$-order stabilizer, minimization over L yields

$$\mathcal{L}_* \propto n_L^p, \quad E_{min}(n_L) \propto n_L^p \equiv n_L^{\eta_\ell}.$$

Thus the power law is field-theoretic: the dominant stabilizer within the coherence window fixes $\eta_\ell = p$. The universal offset

$$\delta = -\frac{2\pi}{\beta_{\text{eff}}} \ln\left(\frac{r_c}{r_0}\right), \quad \beta_{\text{eff}} = \frac{4\pi}{\beta_0} c_{\text{geom}},$$

arises by coarse-graining modes between the core r_c and the outer coherence radius r_0. Consequently,

$$M_\ell = C_\ell (n_L + \delta)^{\eta_\ell}, \quad \eta_\ell = p,$$

placing the leptonic sector on the same footing as the hopfionic baryonic sector.

2.2. Baryonic sector: field model and energy law

Field Model (Faddeev–Skyrme / CP¹). We consider a unit vector $n(\boldsymbol{x})\colon R^3 \to S^2$ with $|\boldsymbol{n}| = 1$, or equivalently a normalized spinor $Z \in C^2$ con $Z^\dagger Z = 1$ such that $\boldsymbol{n} = Z^\dagger \boldsymbol{\sigma} Z$. In CP¹ notation we define the emergent abelian connection

$$a_\mu = \text{Im}\left(Z^\dagger \partial_\mu Z\right), \quad F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu,$$

And the Hopf charge

$$Q = \frac{1}{(2\pi)^2} \int a \wedge da = \frac{1}{(4\pi)^2} \int \boldsymbol{A} \cdot \boldsymbol{B} \, d^3x, \quad B_i = \epsilon_{ijk\boldsymbol{n}} \cdot \left(\partial_{j\boldsymbol{n}} \times \partial_{k\boldsymbol{n}}\right).$$

where $\nabla \times \boldsymbol{A} = \boldsymbol{B}$

A minimal effective Lagrangian that stabilizes hopfions is

$$\mathcal{L}_b = \frac{f^2}{2} \partial_\mu \mathbf{n} \cdot \partial^\mu \boldsymbol{n} - \frac{1}{4e^2} \left(\partial_\mu \mathrm{n} \times \partial_\nu \boldsymbol{n}\right)^2 + \sum_{p \geq 2} \frac{\zeta^{2p}}{\Lambda^{2p-4}} |\partial^p \mathbf{n}|^2 - U(\boldsymbol{n}),$$

with

$$|\partial^p \boldsymbol{n}|^2 \equiv \partial_{\mu_1} \cdots \partial_{\mu_p} \mathbf{n} \cdot \partial^{\mu_1} \cdots \partial^{\mu_p} \boldsymbol{n},$$

And where the "Skyrme" term quadratic in $F_{\mu\nu}$ can equivalently be written as $-\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu}$, The potential $U(\mathbf{n})$ selects the orientational vacuum(s), and the higher-derivative stabilizers of order *2p* (with p ≥ 2) capture relevant high-derivative corrections within the coherence window.

Energy law and the ¾ exponent.
For minimum–energy configurations at fixed Q, the Vakulenko–Kapitanskii bound and its near saturation give

$$E(Q) \sim \kappa(f, e, \dots)|Q|^{3/4}$$

as a result of the competition between the quadratic term $\int ||\nabla n|^2$ (which favors swelling the texture) and the Skyrme term $\int ||F|^2$ (which penalizes curvature), with anisotropic scale minimization. Higher-order stabilizers and core effects renormalize mainly the prefactor and can slightly shift the effective exponent to

$$E(Q) \propto |Q|^{\eta_b}, \quad \eta_b \in [3/4,1) \quad \text{(efectivo)}$$

consistent with the calibrated value $\eta_b \simeq 0.84$.

Toroidal topology and soft factors.
The minimal morphology is toroidal; classes T(p,q) (poloidal/toroidal windings) provide geometric guidance for Q and its energy "band." On top of the coarse topological count we introduce multiplicative soft factors (before raising to η_b) to capture substructure:

$$S_{\text{top}}^{(b)} = Q^{3/4} f_{\text{knot}}\, g_{\text{flavor}}\, g_{\text{rad}},$$

with f_{knot} (knot type; baseline =1), $g_{\text{flavor}} = g_s^{n_s}$ where n_s is the number of strange quarks (broken SU(3): multiplicative strangeness penalty), and $g_{\text{rad}} = r_{\text{rad}}^{n_s}$ (stiffening/radial factor for the decuplet, spin 3/2).

Universal shift δ.
Averaging modes between the core radius r_c and the outer coherence radius r_0, the effective action acquires an almost constant counterterm

$$\delta = -\frac{2\pi}{\beta_{\text{eff}}} \ln\left(\frac{r_c}{r_0}\right), \quad \beta_{\text{eff}} = \frac{4\pi}{\beta_0} c_{\text{geom}},$$

which encodes the imprint of the non-Abelian vacuum (via β_0) and geometric corrections (profile/curvature; $c_{\text{geom}} \sim 1$). This δ is the same as in the leptonic sector (universal), since it comes from coarse-graining across the coherence window rather than from hadron-specific flavor/color details.

Baryonic mass law (final form).
Starting from the Vakulenko–Kapitanskii scaling and numerical results for hopfions in Faddeev–Skyrme–type functionals, the static energy at fixed Hopf charge obeys $E_{min}(Q) \simeq \kappa|Q|^{3/4}$. Identifying the baryon mass with the soliton's static energy, $M_b \equiv E_{min}(Q)$, and

grouping geometric/flavor effects into a dimensionless "topological action" $S_{\text{top}} \equiv Q^{3/4} f_{\text{knot}}\, g_{\text{flavor}}\, g_{\text{rad}}$, we incorporate the coherence-window renormalization $[r_c, r_0]$ through the universal shift δ. This yields the operative mass law

$$M_b = C_b \left[S_{\text{top}} + \delta\right]^{\eta_b}.$$

In the reference model one expects η_b close to 3/4; small deviations encode finite-core curvature, higher-order stabilizers, and collective-coordinate dressing, and are therefore bounded by the two non-strange anchors.

Additive shift from renormalization. The additive structure of the shift, $\left[S_{\text{top}} + \delta\right]^{\eta}$, is not an ansatz but the direct outcome of Wilsonian renormalization. In the unified EFT (Appendix A), the soliton energy is a homogeneous functional of a dimensionless *topological phase–length* S_{top}. Coarse-graining over the internal coherence window $[r_c, r_0]$ induces a finite **additive** counterterm for this operator,

$$S_{\text{top}} \to \mathrm{S_R} = \mathrm{Z_S} S_{\text{top}} + \delta,$$

where the finite part δ is universal (see Sec. 3 for $\delta = \frac{\beta_{\text{eff}}}{2\pi} \ln \frac{r_c}{r_0}$. The (smooth, nearly constant) field-strength renormalization Z_S can be absorbed into the prefactor *C*, while the additive δ\ remains. Since the scaling exponent η is fixed by the derivative balance of the stabilizer terms, the renormalized energy law becomes $E_{min} = C[S + \delta]^{\eta}$ which justifies the operative mass law used in both sectors.

3. Origin and magnitude of δ: coherence/renormalization counterterm

At one loop in the RG we have:

$$\frac{d\alpha_s}{d \ln \mu} = -\frac{\beta_0}{2\pi}\,\alpha_s^2, \qquad \alpha_s \equiv \frac{g^2}{4\pi}$$

Any one–loop correction carries a factor $(g^2/(16\pi^2) = \alpha_s/(4\pi))$. Factoring the color weight β_0, a typical logarithmic contribution reads,

$$(\text{one-loop weight}) \sim \frac{\beta_0\, \alpha_s}{4\pi}$$

Averaging α_s over the coherence window $\mu \in [\mu_c, \mu_0]$ and absorbing geometric/scheme details (shell profile, torus curvature CP^1 normalization) into a dimensionless $(O(1))$ factor, we define:

$$\beta_{\text{eff}} \equiv \frac{\beta_0}{4\pi}\, c_{\text{geom}}$$

Converting this weight into a phase/action shift with the natural 2π normalization gives:

$$\delta \;=\; \frac{\beta_{\text{eff}}}{2\pi}\,\ln\frac{r_c}{r_0} \;=\; -\,\frac{\beta_0}{8\pi^2}\,c_{\text{geom}}\,\ln\frac{r_0}{r_c}$$

We may take a moderate window $\ln(r_0/r_c) \approx \ln 10 \simeq 2.3026$, for three reasons:

i. **Geometric/conventional:** It is natural to choose units so that the vortex's coherent region spans roughly one decade between the outer scale r_0 and the core r_c; this prevents the result from depending on an arbitrary unit choice and captures that the object is compact but non-singular.

ii. **Phenomenological–topological:** In compact vortices and hopfions, the balance between gradient and curvature energies favors moderate scale separations (effective ratios $\frac{r_0}{r_c} \sim 8{-}12$), which place $\ln\left(\frac{r_0}{r_c}\right)$ in [ln 8, ln 12] centered around ln10.

iii. **RG reading:** a moderate window corresponds to a typical coupling consistent with a small negative shift.

With $\mathrm{N_c} = 3,\ \mathrm{N_f} = 3 \Rightarrow \beta_0 = 9$ and $\ln(r_0/r_c) \simeq 2.3026$

$$\delta \;\approx\; -\,\frac{9}{8\pi^2}\,c_{\text{geom}}\,\ln 10 \;\approx\; -\,0.2625\,c_{\text{geom}}$$

Taking $c_{\text{geom}} = 1.00$ by natural normalization yields δ ≈ −0.262. Allowing $\mathrm{c_{geom}} = 1.00 \pm 0.05$ (shell+curvature) gives $\delta \in [-0.249, -0.275]$. Varying only the window $L = \ln\left(\frac{r_0}{r_c}\right)$ in [ln 8, ln 12] gives $\delta \in [-0.24, -0.29]$.

To quantify the uncertainty, we combine the contributions from c_{geom} and the window *L:*

$$\sigma_\delta(\mathrm{c}) = \frac{\beta_0}{8\pi^2}\ln 10\,(0.05) \approx 0.0131,$$

$$\sigma_L = \frac{\ln 12 - \ln 8}{\sqrt{12}} \;\approx\; 0.117 \quad\Rightarrow\quad \sigma_\delta^{(L)} = \frac{\beta_0}{8\pi^2}\,\sigma_L \;\approx\; 0.0133$$

Adding in quadrature, we obtain:

$$\sigma_\delta = \sqrt{\left(\sigma_\delta^{(c)}\right)^2 + \left(\sigma_\delta^{(L)}\right)^2} \approx 0.019$$

Therefore,

$$\boxed{\delta = -0.27 \pm 0.02}$$

This establishes a direct link between the non-Abelian vacuum structure, characterized by β_0, and the mass spectrum of both colored and colorless particles.

3.1 UV motivation and matching to SMEFT

We work in an effective (SMEFT) framework where new physics at scale Λ is integrated out, leaving dimension-six operators with Wilson coefficients c_i. We highlight two minimal UV realizations—a heavy scalar singlet and a vector-like fermion—that generate $(H^\dagger H)^3$ and $\overline{Q_3}\widetilde{H}t_R$, respectively, with natural signs and magnitudes.

(i) Operator $(H^\dagger H)^3$ from a heavy scalar singlet
Consider a real singlet S coupled to the Higgs:

$$\mathcal{V}(H,S) \supset \frac{1}{2}M_S^2\,S^2 + \frac{1}{3}\mu_S\,S^3 + a\,S(H^\dagger H) + \,\cdots$$

Integrating S at tree level (regime E ≪ M_S) yields in SMEFT:

$$\mathcal{L}_{\text{SMEFT}} \supset \frac{c_6}{\Lambda^2}(H^\dagger H)^3, \quad \frac{c_6}{\Lambda^2} = c_*\frac{\mu_S a^3}{M_S^6} \;=\; \frac{c_*}{M_S^2}\,\hat{\mu}\,\hat{a}^3$$

where $\hat{a} \equiv \frac{a}{M_S}$, $\hat{\mu} \equiv \frac{\mu_S}{M_S}$ y $c_* \sim O(1)$ collects normalization choices and mild corrections. The sign is fixed by sign(c_6) = sign($\mu_S a^3$).

Natural range (illustrative). With $|\hat{a}| \lesssim 2,\ |\hat{\mu}| \lesssim 4,\ \Lambda = M_S \in (2{,}4)$ TeV. (This corridor suffices to shift the Higgs sector at an observable level while preserving EFT validity.)

(ii) Operator $\overline{Q_3}\widetilde{H}t_R$ (top-aligned) with universal phase δ
A vector-like fermion T~(3,1,2/3) mixing with the top induces at tree level:

$$\mathcal{L}_{\text{SMEFT}} \supset \frac{C_{tH}}{\Lambda^2}\,H^\dagger H\,\overline{Q_3}\,\widetilde{H}\,t_R + \text{h.c.}$$

In the framework considered here, the imaginary part of the coefficient is set by the same universal phase introduced in this work:

$$\operatorname{Im} C_{tH} = \kappa_t\,\delta$$

It is convenient to define the adimensional combination that captures the CP-odd content at the electroweak scale:

$$\widetilde{\kappa_t} \equiv \frac{v^2}{\Lambda^2}\operatorname{Im} C_{tH} = \kappa_t\,\delta\,\frac{v^2}{\Lambda^2}$$

Comment. This parametrization keeps the low-energy analysis independent of UV details; it suffices to work in terms of (($\frac{c_6}{\Lambda^2}$, $\widetilde{\kappa_t}$). Concrete applications to CP-odd observables will be presented elsewhere.

Validity of the EFT: $\frac{v^2}{\Lambda^2} \ll 1$in the quoted range, and dimension-8 effects remain subdominant.

Flavor alignment (MFV): choosing κ_t aligned with the top sector avoids flavor tensions and concentrates the effects in the operator shown.

4. A minimal theoretical rationale for the *Q* assignments

Before calibration, we state the working hypothesis for the integer Hopf charge Q. The assignment is grounded in standard toroidal/CP^1 constructions and in the morphology of minimal-energy hopfions within the Faddeev–Skyrme family; the detailed rationale and its mapping to torus-knot sectors are developed in items (i)–(iv) below.

(i) Toroidal ansatz and $Q = pq$. In toroidal coordinates, both CP^1 knotted fields and Faddeev–Skyrme hopfions admit representations in which the map n: $S^3 \rightarrow S^2$ carries two integers (*p,q*) counting poloidal and toroidal windings. In such constructions the Hopf invariant is

$$Q = pq \ ,$$

i.e., the linking number of preimages equals the product of the two winding numbers (see the explicit CP^1/electromagnetic knotted fields of Rañada and the Faddeev–Skyrme minimizers reported in Refs. [4–7,14]). This matches the empirical observation that low-Q energy minimizers are toroidal/knotted and organize naturally along torus-knot families *T*(*p,q*).

(ii) Baseline choice and $B = 1$ **Skyrmion projection.** The SU(2) Skyrme model identifies baryons with maps U: $S^3 \rightarrow S^3$ of degree B (Skyrme baryon number). Projecting to $\mathrm{n} = \mathrm{U}\sigma_3\mathrm{U}^\dagger$ (Hopf fibration) yields n: : $S^3 \rightarrow S^2$ with linked preimages. While B and Q are distinct invariants, the projection provides a geometric bridge: the $B = 1$ Skyrmion supports a toroidal core whose CP^1 image naturally sits in the lowest toroidal band; adopting the torus-knot baseline *T*(3,2) (three poloidal strands braided twice) gives $Q = 3\times2 = 6$ as the minimal nontrivial class consistent with a three-strand (color/flavor) toroidal geometry. We therefore set

$$\mathrm{Q(p) = Q(n) = 6} \quad \left(\text{baseline } \mathrm{T}(3{,}2)\right),$$

using $p = 3$ to reflect the threefold structure and $q = 2$ as the minimal closed braid consistent with a compact torus.

(iii) Octet vs. decuplet and the ladder $6 \rightarrow 7 \rightarrow 8 \rightarrow 9$. Numerical studies of the Faddeev–Skyrme model find stable hopfions for all integers Q, with minimal-energy shapes organized along toroidal/knotted families [4,5,7]. Within this toroidal band, increasing the toroidal winding by one unit from the baseline $T(3,2)$ naturally selects the next torus knot $T(3,3)$ with $Q = 9$. We therefore place the spin-3/2 decuplet at $Q = 9$ (baseline $T(3,3)$), while the octet remains at $Q = 6$ (baseline $T(3,2)$); the intermediate classes $Q = 7$ and $Q = 8$ accommodate the minimal additional twist/writhe associated with SU(3) flavor structure

$$\mathrm{Q}(\Delta, \Sigma^*, \Xi^*, \Omega) = 9 \quad \big(\text{baseline T(3,3)}\big),$$

and use the intermediate classes

$$\mathrm{Q}(\Lambda) = 7, \quad \mathrm{Q}(\Sigma) = 8,$$

to represent the minimal additional twist/writhe required by flavor structure (SU(3) breaking) before the full $T(3,3)$ excitation is reached. This stepwise assignment $6 \rightarrow 7 \rightarrow 8 \rightarrow 9$ is consistent with known stable hopfion charges and preserves falsifiability: a different spectroscopy pattern would force a different Q ladder.

(iv) Separation of roles: topology vs. smooth sector modifiers. The Hopf charge Q labels the global topological class (linking/twist) of the configuration. In contrast, smooth, dimensionless multiplicative factors g_{flavor} and g_{rad}—introduced later—collect the effects of SU(3) flavor breaking and of radial/excitation systematics within a given class. This separation prevents double counting: integer changes in Q track genuine topological steps along the toroidal band, whereas in-band variations due to symmetry breaking or excitation are absorbed by those modifiers without altering Q.

(v) Anchors and calibration robustness. We calibrate (C_b,η_b) with the non-strange pair (p,Δ) by design: both states have $n_s = 0$, baseline $f_{\text{knot}} = 1$, and in our scheme $g_{\text{flavor}} = g_{\text{rad}} = 1$, so their splitting probes predominantly the topological step along the toroidal ladder, Q: 6⟶9. This makes the calibration insensitive to strangeness penalties or radial stiffening. At the two-point level one has

$$\eta_b = \frac{\ln\left(\dfrac{M_\Delta}{M_p}\right)}{\ln\left(\dfrac{Q_\Delta^{3/4} + \delta}{Q_p^{3/4} + \delta}\right)}, \qquad C_b = \frac{M_p}{\left(Q_p^{3/4} + \delta\right)^{\eta_b}}$$

so η_b depends only on the ratio of the renormalized topological actions $S \equiv Q^{3/4} + \delta$. We verified stability by re-anchoring with alternative non-strange or "octet vs. decuplet" pairs (e.g. N with Σ^*): the extracted (C_b,η_b) and the ensuing predictions for Σ, Ξ, Ω remain within our quoted uncertainties. This supports the claim that the octet–decuplet gap at $n_s = 0$ is dominantly captured by the topological increment along the toroidal/knotted family, consistent with Faddeev–Skyrme numerics where the minimal-energy morphology evolves monotonically with Q.

Remark(testability). A more microscopic confirmation (variational/lattice) should evaluate Q from minimizing configurations constrained by baryon quantum numbers; if the resulting (p,q) sector differs from (3,2) and (3,3), the present ladder can be revised or ruled out. This makes the assignment directly testable through lattice QCD or variational calculations that compute the Hopf charge of static configurations with the quantum numbers of the nucleon or delta.

5. Calibration of the unified law

Goal of the calibration methodology. Fix a minimal set of global parameters and then evaluate out-of-anchor predictions with well-defined error bands, avoiding circularity.

i. **Universal shift δ (no fit to data):** δ is fixed *a priori* from the physical–geometric consistency of the "coherence window" between scales r_0 and r_c and its non-Abelian vacuum reading. We use δ ≃−0.27 (set by the coherence window and vacuum considerations; see Sec. 3.2). No experimental masses are used to determine δ.
ii. **Sector constants** *(C,η)* **with two anchors:**

   −Leptons: (C_ℓ,η_ℓ) are set with *(e, μ)* using $S_{\text{top}} = n_L$ and n_L *(e,μ,τ)* = (1,2,3).
   −Baryons: (C_b,η_b) are set with *(p,n)* using $S_{\text{top}} = Q^{3/4} f_{\text{knot}}\, g_{\text{flavor}}\, g_{\text{rad}}$ and the baseline $f_{\text{knot}} = 1$, $g_{\text{flavor}} = 1$, $g_{\text{rad}} = 1$ for anchors without strangeness.

iii. **Flavor factor g_s (strangeness):** We define $g_{flavor} = F(g_s, n_s)$, where n_s is the number of strange quarks. We extract g_s using baryons with $n_s = 1$ (e.g. Λ and Σ), via direct fit or least squares, keeping δ and $(C_b,\ \eta_b)$ fixed. We then validate on the remaining species without introducing new parameters.
iv. **Off-anchor validation:** With δ and *(C,η)* fixed, and g_s determined, we report predicted masses for the τ lepton and for light baryons outside the anchor set, comparing them with the experimental values.

With this protocol, the masses reported in the results sections are out-of-anchor predictions (except for the species explicitly used to set parameters), and the sensitivity to δ and to g_s is quantified transparently.

5.1. Leptonic sector (two–point fit and prediction)

Using two anchors (X_1,X_2) = *(e, μ)* with $\mathrm{S_i} = n_L(X_i)$ and masses M_i, we solve in closed form:

$$\eta_\ell = \frac{\ln \mathrm{M_2} - \ln \mathrm{M_1}}{\ln(\mathrm{S_2}+\delta) - \ln(\mathrm{S_1}+\delta)}\ ;\ \mathrm{C}_\ell = \frac{\mathrm{M_1}}{(\mathrm{S_1}+\delta)^{\eta_\ell}}$$

The parameter-free prediction is $\widehat{M_\tau} = \mathrm{C}_\ell\, [\mathrm{S}_\tau + \delta]^{\eta_\ell}$

With $\mathrm{S_{top}} = n_L$ and $\big(n_L(\mathrm{e}), n_L(\mu), n_L(\tau)\big) = (1,2,3)$ we obtain:

$$\eta_\ell \approx 6.179, \quad C_\ell \approx 3.573\ \mathrm{MeV}$$

Phenomenology. Calibrating η_ℓ and C_ℓ on (e,μ) while keeping δ fixed by RG geometry, the τ\ mass emerges as an out-of-anchor prediction. The large fitted value $\eta_\ell \simeq 6$ indicates that, within the leptonic coherence window, a higher-order stabilizer (or its RG-dressed combination) dominates—an explicit, falsifiable target for variational simulations of the effective action.

5.2. Baryonic sector (anchors without strangeness and without excitation)

Topological hypothesis. We use the unified law:

$$M = C_b\left[S_{\mathrm{top}} + \delta\right]^{\eta_b}, \quad S_{\mathrm{top}} = \mathrm{Q}^{3/4} f_{\mathrm{knot}}\, \mathrm{g}_{\mathrm{flavor}}\, \mathrm{g}_{\mathrm{rad}}$$

Flavor ansatz. We take $SU(3)$ breaking by strangeness as a multiplicative factor applied before the exponent, i.e. encoded in:

$$\mathrm{g}_{\mathrm{flavor}} = \mathrm{g}_{\mathrm{s}}^{\mathrm{n}_{\mathrm{s}}}$$

where n_s is the number of *s* quarks. This form is the most parsimonious: for small deviations,

$$(S+\epsilon)^{\eta_b} = S^{\eta_b}\left(1+\frac{\epsilon}{S}\right)^{\eta_b} \approx S^{\eta_b}\left(1+\frac{\eta_b\epsilon}{S}\right), \quad \text{for } |\epsilon| \ll S.$$

so it reproduces the "increment per *s*-quark" of a linear approach while still respecting the power law.

We take two anchors with no strangeness and in the ground state—the proton *p* and the Δ—defining:

$$\mathrm{S_p} \equiv \mathrm{Q_p^{3/4} f_{knot}}, \quad \mathrm{Q_p} = 6, \quad ; \quad S_\Delta \equiv Q_\Delta^{3/4} f_{\mathrm{knot}}, \quad Q_\Delta = 9,$$

with $f_{\mathrm{knot}} = 1$ in the baseline $\mathrm{g}_{\mathrm{flavor}} = \mathrm{g}_{\mathrm{rad}} = 1$ for these anchors.

The theoretical rationale for the *Q* ladder 6→7→8→9 is discussed in Sec. 4.:

$$Q(p) = 6,\ Q(n) = 6,\ Q(\Lambda) = 7,\ Q(\Sigma) = 8,\ Q\ (\Delta, \Sigma^*, \Xi^*, \Omega) = 9$$

Calibration of (C_b, η_b) with (*p*, Δ) with $\delta = -0.27$ (fixed):

$$\eta_b = \frac{\ln M_\Delta - \ln M_p}{\ln(S_\Delta + \delta) - \ln\left(S_p + \delta\right)}; \ C_b = \frac{M_p}{\left(S_p + \delta\right)^{\eta_b}}$$

Fixing g_s with Λ(uds) where $n_s(\Lambda) = 1$, $Q(\Lambda) = 7$. In the octet $g_{\mathrm{rad}} = 1$, hence $\mathrm{S}_\Lambda = \mathrm{Q}_\Lambda^{3/4} \mathrm{f_{knot}}\, \mathrm{g_s}$ and

$$M_\Lambda = C_b\left[Q_\Lambda^{3/4} g_s + \delta\right]^{\eta_b} \;\Rightarrow\; \mathrm{g_s} = \frac{1}{\mathrm{Q_\Lambda^{3/4} f_{knot}}}\left[\left(\frac{\mathrm{M_\Lambda}}{\mathrm{C}_b}\right)^{1/\eta_b} - \delta\right]$$

With M_Λ= 1115.68 MeV and $\mathrm{Q}_\Lambda^{3/4} = 7^{3/4}$, this gives:

$$\eta_b \approx 0.8412,\; C_b \approx 322.16\ \text{MeV},\;\; g_s \approx\; 1.080$$

Strangeness ansatz. For the *J* = 3/2 states (decuplet) we take

$$g_{\text{rad}} = r_{\text{rad}}^{n_s}$$

with n_s the number of *s*-quarks (0,1,2,3). We adopt this geometric scaling for a physical–geometric reason: in the topological ansatz, each strange quark locally stiffens the "tube" of the vortex/knot (increasing its tension/rigidity and shortening the braid pitch), and these contributions compose almost independently; in the effective energy this appears as a multiplicative insertion factor. Moreover, after undoing the exponent η_b, the steps $\Delta \;\to \Sigma^* \to \Xi^* \to \Omega$ display nearly constant ratios, supporting $r_{\text{rad}}^{n_s}$ over linear schemes.

Determination of *r* (common degree/excitation factor in the decuplet) using the exact relation with Δ and Ω,

$$r_{\text{rad}} = \frac{1}{\mathrm{g_s}}\left(\frac{\left(\frac{\mathrm{M}_\Delta}{\mathrm{C}_b}\right)^{1/\eta_b} - \delta}{\left(\frac{\mathrm{M}_\Omega}{\mathrm{C}_b}\right)^{1/\eta_b} - \delta}\right)^{1/3} \quad \approx 1.040$$

and refining r_{rad} via least squares with Σ^*, Ξ^*, Ω we obtain

$$r_{\text{rad}} \approx 1.0423 \quad \text{(dispersion } \pm 0.009 \text{ approx.)}$$

The "± 0.009 dispersion" corresponds to the sample standard deviation of the individual *r* estimates for the decuplet, i.e., of $\{\,\mathrm{r}_{\Sigma^*},\, \mathrm{r}_{\Xi^*},\, \mathrm{r}_\Omega\,\}$.

Summary of calibrated parameters (baryonic sector):

$$\boxed{\delta = -0.27,\; \eta_b = 0.8412019,\; C_b = 322.1601\ \text{MeV},\; g_s = 1.08010,\;\; r_{\text{rad}} = 1.0423}$$

Bias control and robustness. We treat δ as universal (not fitted to masses) and fix (*C, η*) with two anchors per sector (leptons: e, μ; baryons: p, Δ), keeping f_{knot}=1. The flavor factor is determined once with Λ(uds), yielding g_s= $O(1)$; in the decuplet we use $g_{\text{rad}} = r^{n_s}$ (calibrated by least squares on $\{\Sigma^*, \Xi^*, \Omega\,\}$ and in the octet ground states we set $g_{\text{rad}} = 1$. As a sensitivity guide, the relative mass variation under a shift Δδ is $\frac{\Delta M}{M} \simeq \frac{\eta}{S+\delta}\,\Delta\delta$; it typically stays $\lesssim$1% for light baryons with Δδ = ±0.02. A systematic study (sweeping δ, permuting anchors, and exploring alternative *Q*-assignments) is left to supplementary material/future work.

6. Results

We report the mass results by sector. Experimental masses are taken from the PDG [13].

6.1. Leptonic sector (calibrated on e, μ; prediction for τ)

Table 1. Mass predictions for leptons

Particle	n_L	$Mass_{exp}$ (MeV)	$Mass_{pred}$ (MeV)	Rel. error
e (anchor)	1	0.5110	0.5110	0.00%
μ (anchor)	2	105.6584	105.6584	0.00%
τ (pred.)	3	1776.86	**1770.53**	**−0.36%**

Remarks: With δ = −0.27 the τ prediction is ∼1770.5 MeV, about 0.36% below the experimental value. The sensitivity to δ is $\Delta M_\tau/M_\tau \approx \eta_\ell\, \Delta\delta/(n_L(\tau)+\delta)$; for Δδ = ±0.02, the shift would be ∼ ± 2.6% (cf. ≈ ±4.5 if C, η are held fixed). This is why we treat δ as a universal, pre–fixed parameter.

6.2. Baryonic sector (baseline scenario)

The mapping Q ← (concrete knot/soliton topology) must ultimately be confirmed by a more detailed geometric model; here it is used as a minimal working hypothesis.

Table 2. Mass predictions for baryons

Baryon	Q	n_L	$Mass_{exp}$ (MeV)	$Mass_{pred}$ (MeV)	Rel. error
Octet (spin 1/2)					
p (uud)	6	0	938.27	**938.27**	**+0.00%** *(anchor)*
n (udd)	6	0	939.57	938.27	−0.14%
Λ (uds)	7	1	1115.68	**1115.68**	**+0.00%** *(fixed g_s)*
Σ^+ (uus)	8	1	1189.37	1219.74	+2.55%
Σ^0 (uds)	8	1	1192.64	1219.74	+2.27%
Σ^- (dds)	8	1	1197.45	1219.74	+1.86%
Ξ^0 (uss)	8	2	1314.86	1306.49	−0.64%

Baryon	Q	n_L	$Mass_{exp}$ (MeV)	$Mass_{pred}$ (MeV)	Rel. error
Ξ^- (dss)	8	2	1321.71	1306.49	−1.15%
Decuplet (spin 3/2)					
Δ (uuu, uud, udd, ddd)	9	0	1232.00	**1232.00**	**+0.00%** *(anchor)*
Σ^* (uus, uds, dds)	9	1	1385.00	1368.13	−1.22%
Ξ^* (uss, dss)	9	2	1530.00	1518.93	−0.72%
Ω^- (sss)	9	3	1672.45	1685.80	+0.80%

Remarks

Octet: In the model without electromagnetism (EM) and isospin breaking, p and n are identified, hence the neutron comes out at −0.14% (as expected: EM corrections and $(\mathrm{m_d} - \mathrm{m_u})$ split the physical masses). $SU(2)$ isospin symmetry makes p (uud) and n (udd) degenerate. Λ fixes g_s, and Σ (with one s) lies at +2–3%: reasonable at first order (hyperfine and EM not included). The Ξ (two s) falls within ∼1%.

Decuplet: With a single r for the whole multiplet, Σ^*, Ξ^*, Ω are within ±1.2%, with the worst case Ω at +0.8%. This is very good performance for a parsimonious power law.

Exponent and shift: $\eta_b = 0.841$ is close to 3/4 (Vakulenko–Kapitanskii law), and the universal δ maintains continuity with the leptonic sector. This supports a topological—geometric reading of the mass scaling.

7. Conclusions and Future Work

We have presented a unified framework that addresses the mass hierarchy of elementary particles through topology and the geometry of the non-Abelian vacuum. Unlike Standard Model descriptions that depend on many a priori independent Yukawa couplings, our single power law

$$M = C\left[S_{\mathrm{top}} + \delta\right]^{\eta}$$

captures lepton and baryon masses parsimoniously.

The model's key elements have direct physical meaning:

- **Topological action.** Leptons and baryons are characterized by topological invariants—the winding number for leptons (vortices) and the Hopf charge for baryons (knots)—in agreement with the Vakulenko–Kapitanskii law.
- **Renormalization shift.** The universal shift δ ≈ −0.27 is not a fitted parameter but a natural counterterm associated with the coherence window of the vacuum. Its value

is consistent with the beta-function coefficient of the non-Abelian vacuum, linking colored and colorless sectors.

This framework highlights a broad universality principle: QCD-vacuum properties influence the particle mass spectrum independently of color charge.

Future work will include: (a) variational construction of vortices and hopfions with controlled boundary conditions to constrain δ and Q; (b) systematic study of $f_{\text{knot}} \neq 1$ and g_{rad} in excitations; (c) extension to mesons; (d) uncertainty analysis of sectoral constants; (e) lattice and variational computations to extract δ and test the scaling $E \sim Q^{3/4}$.

A speculative outlook is to interpret the framework in terms of an angular field Θ(x,t), where the angle acts as an effective degree of freedom. Such a field could admit stable localized configurations (solitons) when the potential supports prolonged angular coherence. In this picture, effective mass may correspond to stable angular curvature or to a dynamic reorientation rate, in line with the topological functional S_{top}. Unitary maps n: $R^3 \to S^2$ (or $SU(2) \simeq S^3 \to S^2$) would naturally generate three-dimensional defects—vortices and hopfions—characterized by n_L and the Hopf charge Q. This interpretation, while still exploratory, suggests that the observed universality of the mass law may reflect an underlying microgeometric structure of the vacuum, with the shift δ reflecting a scale imprint of non-Abelian dynamics.

References

1. Skyrme, T. H. R. (1961). A non-linear field theory. *Proceedings of the Royal Society A, 260*(1300), 127–138.
2. Adkins, G. S., Nappi, C. R., & Witten, E. (1983). Static properties of nucleons in the Skyrme model. *Nuclear Physics B, 228*(3), 552–566.
3. Faddeev, L., & Niemi, A. J. (1997). Stable knot-like structures in classical field theory. *Nature, 387*(6628), 58–61.
4. Battye, R. A., & Sutcliffe, P. M. (1998). Knots as stable soliton solutions in a three-dimensional classical field theory. *Physical Review Letters, 81*(22), 4798–4801.
5. Hietarinta, J., & Salo, P. (2000). Ground state in the Faddeev–Skyrme model. *Physical Review D, 62*(8), 081701.
6. Vakulenko, A. F., & Kapitanskii, L. V. (1979). Stability of solitons in the class of maps from $S^3 \to S^2$. *Soviet Physics Doklady, 24*, 432–434.
7. Manton, N. S., & Sutcliffe, P. (2004). *Topological Solitons*. Cambridge University Press.
8. Gross, D. J., & Wilczek, F. (1973). Ultraviolet behavior of non-Abelian gauge theories. *Physical Review Letters, 30*(26), 1343–1346.
9. Politzer, H. D. (1973). Reliable perturbative results for strong interactions? *Physical Review Letters, 30*(26), 1346–1349.

10. Nielsen, H. B., & Olesen, P. (1973). Vortex-line models for dual strings. *Nuclear Physics B, 61*, 45–61.

11. White, J. H. (1969). Self-linking and the Gauss integral in higher dimensions. *American Journal of Mathematics, 91*(3), 693–728.

12. Fuller, F. B. (1971). The writhing number of a space curve. *Proceedings of the National Academy of Sciences, 68*(4), 815–819.

13. Particle Data Group. (2024). Review of Particle Physics. *Progress of Theoretical and Experimental Physics, 2024*, 083C01.

14. Rañada, A. F. (1989). A topological theory of the electromagnetic field. *Journal of Physics A: Mathematical and General, 23*(16), L815–L820.

15. ATLAS Collaboration. (2022). A detailed map of Higgs boson interactions using 139 fb^{-1} of *pp* collisions at $\sqrt{s} = 13$ TeV. *Nature, 607*, 52–59

Appendix A: Unified Angular EFT (one theory, two reductions)

Master field. We work with a field $U(\boldsymbol{x}) \in \mathrm{SU}(2)$ and a fiber phase $(\boldsymbol{x}) \in S^1$, with gauge symmetry

$$\theta \to \theta + \alpha(\boldsymbol{x}), \quad U \to U e^{i\alpha(\boldsymbol{x})\sigma_z}$$

Define

$$L_\mu \equiv U^\dagger \partial_\mu U \in \mathfrak{su}(2), \quad a_\mu \equiv -\tfrac{i}{2}\mathrm{Tr}(\sigma_z L_\mu), \quad \boldsymbol{n} \equiv U\sigma_z U^\dagger \in S^2,$$

and the covariant derivative of the fiber phase

$$D_\mu \theta \equiv \partial_\mu \theta - a_\mu$$

Master Lagrangian (compact):

$$\mathcal{L}_{\mathrm{ui}} = \frac{f^2}{2}\mathrm{Tr}(L_\mu L^\mu) - \frac{1}{4e^2}\mathrm{Tr}\left(\left[L_\mu, L_\nu\right]^2\right) + \frac{\kappa}{2}\left(D_\mu\theta\right)^2 + \sum_{p\geq 2}\frac{\xi_{2p}}{\Lambda^{2p-2}}(D^p\theta)^2 - U(\boldsymbol{n},\theta)$$

- The sigma term (f) and the Skyrme term (e) stabilize textures of **n** (hopfions).
- The $2p$-derivative stabilizers in the fiber (ξ_{2p}) control vortex rigidity (leptonic sector).
- The potential $U(\boldsymbol{n},\theta)$ selects the regime (see below) and softly lifts degeneracies.

Topology and charges.
– **Base (baryons):** $\boldsymbol{n} = U^\dagger \boldsymbol{\sigma} U$ with Hopf charge

$$Q = \tfrac{1}{(2\pi)^2}\int a \wedge da = \tfrac{1}{(4\pi)^2}\int \boldsymbol{A}\cdot\boldsymbol{B}\, d^3x, \quad B_i = \epsilon_{ijk\boldsymbol{n}}\cdot\left(\partial_{j\boldsymbol{n}} \times \partial_{k\boldsymbol{n}}\right),$$

with $\nabla\times\mathbf{A}=\mathbf{B}$.
– **Fiber (leptons):** θ-vortices with winding $n_L = \frac{1}{2\pi}\oint \nabla\theta \cdot dl$

Two limits (two sectors) from the same theory.

Leptonic limit (O(2) vortices): Freeze **n** (e.g. $U \to e^{i\theta\sigma_z}$)) and leave θ dynamical:

$$\mathcal{L}_{\mathrm{ui}} \to \frac{\kappa}{2}(\partial\theta)^2 + \sum_{p\geq 2}\xi_{2p}\,\Lambda^{-(2p-2)}(\partial^p\theta)^2$$

Minimization on thin rings (length L) with tension vs. $2p$-order stabilizer yields

$$E_{min}(n_L) \propto n_L^p \equiv n_L^{\eta_1}, \quad \eta_1 = p$$

Baryonic limit (CP¹ hopfions): Fix θ\ (fiber frozen, $D\theta \simeq 0$) and excite **n**

$$\mathcal{L}_{\mathrm{ui}} \to \frac{2}{f^2}(\partial \boldsymbol{n})^2 - \frac{1}{4e^2}\big(\boldsymbol{n}\cdot(\partial\boldsymbol{n}\times\partial\boldsymbol{n})\big)^2$$

Minimization at fixed Q reproduces the robust law

$$E(Q) \sim \kappa(f,e)|Q|^{3/4} \Rightarrow \eta_b \simeq \frac{3}{4} \quad (\text{efectivo } \simeq 0.8 - 0.9)$$

Universal offset δ (coherence/RG). In both limits, integrating modes between the core radius r_c and the outer coherence radius r_0 generates an additive counterterm on the topological functional (fiber/base):

$$\delta = -\frac{2\pi}{\beta_{\mathrm{eff}}}\ln\left(\frac{r_c}{r_0}\right), \quad \beta_{\mathrm{eff}} = \frac{4\pi}{\beta_0}c_{\mathrm{geom}}.$$

It is universal (species–independent) because it arises from the same coarse-graining window and the same vacuum weight (the β imprint).

Unified mass law (EFT output).

Leptons: $M_\ell = C_l(n_L+\delta)^{\eta_\ell}, \quad \eta_\ell = p$ (determined by the dominant stabilizer in the coherence window)
Baryons: $M_b = C_b\big(Q^{3/4} f_{\mathrm{knot}} g_{\mathrm{flavor}} r_{\mathrm{grad}} + \delta\big)^{\eta_b}, \ \eta_b \simeq 0.84$ (effective).

Thus a single theory (with S^1 fiber and S^2 base of the Hopf fibration) yields both sectors as different regimes, explains why δ is common, and why the exponents η differ.