On Closed Sets of Points defined as the Limit of a Sequence of Closed Sets of Points. By W. H. Young.

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§ 1. Let $G_1, G_2, \ldots, G_n, \ldots$ be a countably infinite sequence of sets of points, each of which is closed and nowhere dense and is contained in all the succeeding sets.

This sequence of sets defines a limiting set G, defined by the two properties: (1) Every point of G_n is a point of G for all values of n; (2) no point of G exists which does not belong to some definite G_n of the sequence. This limiting set G may or may not be itself closed. Moreover, if we express it as the sum of sets of points thus,

$$G = G_1+(G_2-G_1)+(G_3-G_2)+ \ldots +(G_n-G_{n-1})+ \ldots, \text{ ad inf.},$$

the terms of the series may or may not be closed sets.

As regards the latter property, we make no assumption. We assume, however, in what follows that G is closed and nowhere dense. Notice the advantage of regarding G as defined by a sequence instead of by a series. We can in this way confine our attention to closed sets.

§ 2. The theorem we propose to prove is the following:—

Given any small positive quantity σ, we can determine an integer m and a small positive quantity ϵ so that, for all values of $n \geqslant m$, all the intervals $\geqslant \epsilon$ of G_n are identical with all the intervals $\geqslant \epsilon$ of G; and the sum of the remaining intervals of $G_n < \sigma$.

This result throws light on the mode in which G_n approaches its limit G. Moreover, it gives us as an immediate corollary Osgood's content theorem *that the content of G is the limit when n is infinite of the content of G_n.*

§ 3. Since each set is closed and nowhere dense, the complementary points fill up a set of intervals, which, in the enunciation of the theorem, we have called the intervals of the set. We shall in what follows refer to them as the "black intervals" of the set, the points of the set being conceived as white on a black ground. These intervals are "open," or, in other words, the end points of the intervals are points of the set.

§ 4. It is, perhaps, here the place to point out a confusion which is easily made in the case when G is open.

Suppose Γ to be the set got by closing G. Then *

$$\Gamma = \mathfrak{M}(G, G'),$$

where G' is the first derived of G. If Γ is perfect,

$$\Gamma = G'.$$

We have by definition

$$\mathrm{Lt}\, G_n = G;$$

but we cannot deduce that

$$\mathrm{Lt}\, G'_n = G' = \Gamma$$

when Γ is perfect. In fact, as G_n is closed, G'_n will be contained in G_n, and cannot, therefore, generate more than, at most, G. Also, since by hypothesis G is open, G' contains points which are not contained in G. Thus $\mathrm{Lt}\, G'_n$ cannot be G' when G is open; while G' and Γ are not identical unless Γ is perfect.

It is probably this confusion which has led to the incorrect presentation of the subject of this paper in Schoenflies's *Bericht über die Mengenlehre*, p. 91. The following is a literal translation of the passage referred to:—

"If Γ be a closed set, and G_n a component set of Γ, and if G_n, as n increases, becomes dense everywhere † in Γ, then the limit of the content of G_n is the content of Γ. For the content of G_n is equal to that of G'_n, (the first derived of G_n). And, if G_n, as n increases, becomes dense everywhere in Γ, *then* G'_n *converges towards* Γ itself, whence the theorem follows."

The fallacy in the proof is in the words in italics. We may add, it will appear from the sequel that the *statement* of the theorem given by Schoenflies is incorrect.

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* That is, the set consisting of all the elements of G and G', common elements being counted once only, *Math. Ann.*, p. 355, Schoenflies, p. 6.

† It is not quite clear in what sense the expression "If G_n is dense everywhere in Γ" is used. On p. 80, *loc. cit.*, Schoenflies gives the definition: A component set U of a closed set Q is said to be dense everywhere in Q, if the first derived set U' of U coincides with Q. This definition is clearly untenable, however, for, in the passage referred to, Schoenflies especially includes the consideration of closed sets which are not perfect; now for such a set Q the definition could have no meaning, for no component set of it could possibly have Q for its derived set. In what follows we shall ourselves always say that the component set U of a closed set Q is dense everywhere in Q, if U coincides with Q when the remaining limiting points of U are added to U. In other words, when $Q = \mathfrak{M}(U, U')$.

§ 5. A simple example has been given by Osgood, in which G is unclosed. Let G_n stand for all the proper fractions with n as denominator. Then G consists of all the rational numbers between 0 and 1. Evidently the content of G_n is 0, however great n may be. Here Γ is the continuum from 0 to 1, and its content is therefore unity.

§ 6. In the above Γ is dense everywhere. In the following example Γ is nowhere dense. We give it at some length, as it illustrates the essential features of the matter in hand and throws light on the mode of proof adopted in the sequel.

It will be noted that (1) G' may contain a more than countable set of points which are not points of any G'_n; (2) the content of G' may be different from the limit of the content of G'_n when n is made infinite.

§ 7. Consider the following sequence of sets :—

G_1 is H. J. S. Smith's ternary closed set* of the first kind in the segment (0, 1).

By means of repetitions of the processes by which G_1 was constructed in the segment (0, 1), we propose to construct a series of closed sets whose limit G, when closed by the addition of those limiting points not already included in it, is identical with H. J. S. Smith's ternary closed set of the second kind. This we denote by Γ. For this purpose the following should be noticed :—

(1) If we divide the segment in which an H. J. S. Smith's set of the first kind is given into 3^n equal parts, certain of them will be entirely black for the set, and in each of the others there is an H. J. S. Smith's set of the first kind.

(2) If we divide the segment in which an H. J. S. Smith's set of the second kind is given into $3^{\frac{1}{2}n(n+1)}$ parts, certain of them will be entirely black for the set, and in each of the others the given set has precisely the same form, though this form is not an H. J. S. Smith's set of the second kind, because the largest black interval in each part is not $\frac{1}{3}$ of that part.

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* *Proc. Lond. Math. Soc.*, Vol. VI., p. 948; *cf. Proc. Lond. Math. Soc.*, Vol. XXXIV., p. 286, footnote, for the reason of the insertion of the term "closed," *i.e.*, the set in question is that got by adding to H. J. S. Smith's ternary set of the first kind its limiting points. See, for a full discussion of these sets, the preceding paper by the present author on "Sets of Intervals."

Having premised this, we proceed to the construction. (See Fig. 1.) The largest black interval of G_1 is the same as the largest

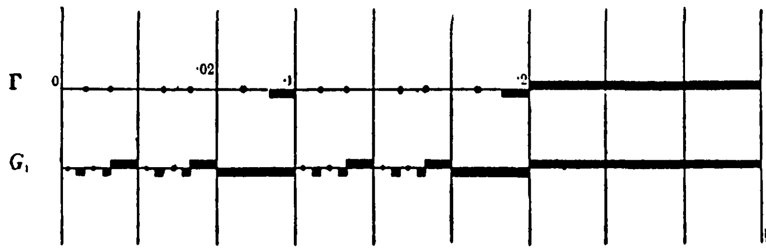

FIG. 1.

black interval of Γ, and in each of the two remaining segments (0, ·1) and (·1 to ·2) in the ternary notation, Γ has the same form, and G_1 consists of an H. J. S. Smith's set of the first kind. We need, therefore only consider what modification is necessary in the segment (0, ·1), the same modification being supposed made in the segment (·1, ·2), and the segment (·2, 1) being left unaltered. In the segment (0, ·1) Γ has its largest black interval of length $\frac{1}{3^{1+2}}$ on the extreme right; in all the other segments of the same length,

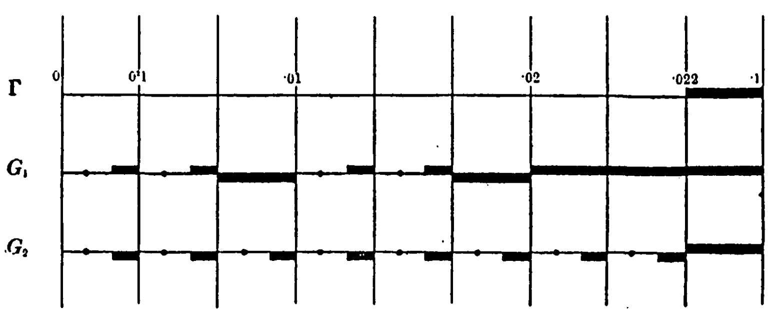

FIG. 2.

(0, ·$0^2$1), &c., it has the same form, whereas G_1 has the form of an H. J. S. Smith's set of the first kind in only some of these segments, (·$0^2$2, ·01), (·012, ·02), and (·02, ·1) being entirely black. If, however, in each of these three largest black intervals of G_1 we insert an H. J. S. Smith's set of the first kind, then the extreme right-hand segment (·022, ·1) will be entirely black for the new set G_2, and in each of the other eight segments G_2 will consist of an H. J. S. Smith's set of the first kind.

We can, therefore, as before, consider what modifications are necessary in the segment $(0, \cdot 0^{2}1)$ only, the same modification being supposed made in the other seven segments, and the segment $(\cdot 022, \cdot 01)$ being left unaltered.

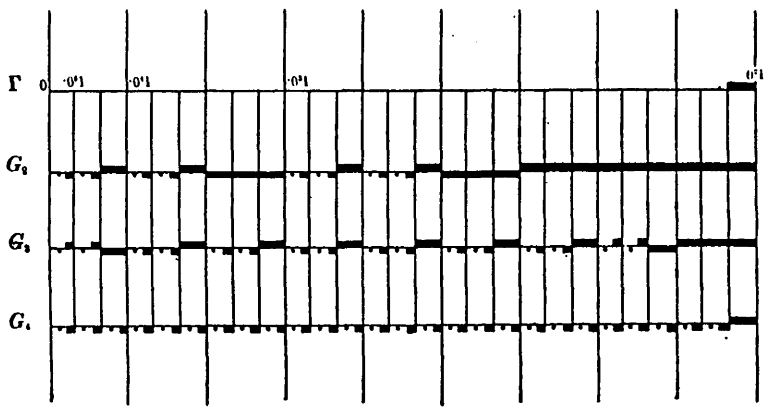

FIG. 3.

By a precisely similar argument as before, it is easily seen that, if we form G_3 by inserting in each of the three largest intervals of G_2 an H. J. S. Smith's set of the first kind, and G_4 by inserting in each of the 3^2 largest intervals of G_3 an H. J. S. Smith's set of the first kind, the extreme right-hand segment of length $\frac{1}{3^{1+2+3}}$, $(\cdot 0^{3}222, \cdot 0^{2}1)$, will be entirely black, as it is for Γ, and, in all the other segments of length $\frac{1}{3^{1+2+3}}$, G_4 will consist of an H. J. S. Smith's set of the first kind.

The general law is now obvious. We shall only have to consider the modifications necessary in $G_{1+1+2+3+\dots+n}$ in the segment $(0, \cdot 0^{1+2+\dots+n}1)$, in which it consists of an H. J. S. Smith's set of the first kind, the same modification being made in all the other segments of the same length except those which are by our construction already black for Γ.

The modification will consist in inserting H. J. S. Smith's sets of the first kind in the $3, 3^2, 3^3, \dots 3^{n+1}$ largest intervals in turn to form $G_{\frac{1}{2}[n(n+1)]+2}$, $G_{\frac{1}{2}[n(n+1)]+3}$, up to $G_{\frac{1}{2}[(n+1)(n+2)]+1}$.

8. Comparing this series of sets with H. J. S. Smith's ternary closed set of the second kind, we see that, given any small quantity ϵ,

we can assign a stage m in the series such that for all values of $n \geqslant m$ all the black intervals of G_n which are $\geqslant \epsilon$ are identical with those $\geqslant \epsilon$ of H. J. S. Smith's ternary closed set of the second kind. But the content of G_n *is always zero*; while that of H. J. S. Smith's ternary closed set of the second kind lies between $\frac{2}{3}$ and $\frac{1}{2}$.

The set G obviously consists of all the isolated points and limiting points on one side only of H. J. S. Smith's ternary set of the second kind, together with some of its limiting points on both sides. The set got by closing G *is* therefore H. J. S. Smith's ternary closed set of the second kind; that is, Γ.

We will now prove that G does not contain all the points of Γ, and is therefore unclosed. To do this it is sufficient to prove that the point

$$P = \cdot 1212212221\ldots$$

(where the right-hand side of the symbolic equation represents a ternary fraction, the number of 2's between consecutive 1's increasing each time by one), which is a limiting point on both sides of Γ (and a limiting point on one side only of Γ'), is an *internal* point of a definite black interval of G_n for *every* value n, and is therefore *not* a point of G. This interval is $(\cdot 12, \cdot 2)$ of G_1, $(\cdot 1212, \cdot 122)$ of G_2, $(\cdot 12122, \cdot 122)$ of G_3, $(\cdot 1212212, \cdot 121222)$ of G_4, $(\cdot 12122122, \cdot 12122)$ of G_5, $(\cdot 121221222, \cdot 121222)$ of G_6, $(\cdot 12122122212, \cdot 1212212222)$ of G_7, and so on. The general law is now evident, and hence the assertion is proved. Similarly it is evident that any ternary fraction of Γ which involves an infinite number of 2's * cannot belong to any G_n, since, apart from a finite number of figures at the beginning, the numbers of G_n involve only the figures 0 and 1. Each such point is a limiting point on both sides of Γ and is interior to a black interval of G_n whose length diminishes indefinitely as n increases indefinitely.

Hence the set G which we have constructed as the limiting set of a sequence of closed sets, nowhere dense, is open, and the content of the set obtained by closing G is different from that of the limit of G_n when n is made infinite.

9. In the following sections we proceed to prove the theorem enunciated in § 2. To do this we shall first state and prove a series of simple facts about the sets of the sequence.

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* Other than 2, of course.

10. *Any black interval (P, Q) of G or of G_n "leads up" to a definite black interval (P', Q') of G_m $(m \leqslant n)$, which is either identical with (P, Q) or of which (P, Q) forms a part.*

This follows at once from the fact that no point of G_m can lie between P and Q, since every point of G_m is a point of G_n and of G.

11. Hence *the sum of any number of black intervals $\leqslant$ the sum of the black intervals to which they lead up.*

12. *To each positive quantity ϵ there corresponds a perfectly definite finite integer m such that for all values of $n \geqslant m$ all the black intervals of G_n which are $\geqslant \epsilon$ are identical with all those $\geqslant \epsilon$ of G.*

For let (P, Q) be any black interval of G. Then, by the definition of G, there is a definite integer m_1 such that P and Q are points of G_{m_1}, but not both of G_{m_1-1}. The black interval (P, Q) will then belong to the set of black intervals of G_n for all values of $n \geqslant m_1$.

Now the number of black intervals of $G \geqslant \epsilon$ is finite, and hence it follows, by an immediate extension, that we can assign a definite finite integer m_2 such that all the black intervals $(P, Q) \geqslant \epsilon$ of G belong to the set of black intervals of G_n for all values of $n \geqslant m_2$. But G_{m_2} may have other black intervals $\geqslant \epsilon$.

Let (P', Q') be such an interval. Then P' and Q' are both points of G; but (P', Q') is not a black interval of G, nor does there lie between P' and Q' any interval $\geqslant \epsilon$ of G. Between P' and Q' we can mark a finite number of points of G, say $P_1, P_2, \ldots, P_k$, such that no one of the distances between successive points $P'P_1, P_1P_2, \ldots, P_kQ'$ is $\geqslant \epsilon$. Treating each such interval (P', Q') in the same way, we shall get a finite number of such points $P_1, P_2, \ldots$, since the number of *all—a fortiori, of some*—of the black intervals G_{m_2} which are $\geqslant \epsilon$ is finite. We can therefore determine a definite integer $m_3 \geqslant m_2$ such that G_{m_3} contains all these points, and G_{m_3} will therefore have no black interval $\geqslant \epsilon$ other than those belonging to G. G_{m_2} will be the first of our sequence of sets having this property, or else there will be a definite first set between it and G_{m_3} (this inclusive) having this property. Call this set G_m. Then m is evidently the integer sought.

13. To each positive quantity ϵ we have shown to correspond a definite integer m such that the black intervals of G_n $(n \geqslant m)$ and G differ only

in those intervals which are $< \epsilon$. It remains to show that we can choose ϵ so that the sum of the black intervals which are $< \epsilon$ is less than σ. Denote these sums by $R_n(\epsilon)$ and $R(\epsilon)$ respectively. We have only to refer to the example constructed in § 7 to convince ourselves that, without restrictions of some kind on the nature of G, this will not be the case, even when, without G being closed, the theorem of § 12 holds. Examination of that example suggests that the validity of this second part of the whole theorem enunciated in § 2 depends essentially on those points, which, without being end points of black intervals, are limiting points of G, and therefore limiting points *on both sides*. It is, in fact, evident that, provided G contains all its limiting points on *one side only*, (end points of black intervals), the first part of the theorem (§ 12) will hold, as it does in the example of § 7, but not necessarily the second part.

14. From the theorem of § 12 it follows immediately that, conversely, to each integer n there corresponds uniquely a positive quantity ϵ_n such that all the black intervals of G_n which are $\geqslant \epsilon_n$ are identical with all the black intervals of G which are $\geqslant \epsilon_n$, and ϵ_n is the *least* quantity of which this is true.

15. This being so, $\epsilon_m \geqslant \epsilon_n$, $(n > m)$, *and the quantities* ϵ_n *decrease without limit as* n *is increased indefinitely*. For, n being $> m$, any black interval of G_n which is $\geqslant \epsilon_m$ must lead up to a black interval of G_m which is $\geqslant \epsilon_m$; that is, to an interval which, being itself a black interval of G, contains no point of G except its end points; hence any black interval of G_n which is $\geqslant \epsilon_m$ must be identical with the black interval of G_m to which it leads up, and therefore is a black interval of G. On the other hand, any black interval of G which is $\geqslant \epsilon_m$ must lead up to a black interval $\geqslant \epsilon_m$ of G_n (§ 10), which is, therefore, identical with itself. Hence all the black intervals which are $\geqslant \epsilon_m$ of G and of G_n are identical, and therefore, by the definition of ϵ_n, $\epsilon_m \geqslant \epsilon_n$.

That the quantities ϵ_n form a sequence with zero as limit follows from the fact that, however small ϵ be taken, we can find a corresponding integer m, as in § 12, so that, by the definition of ϵ_m and by what has just been proved, $\epsilon_n \leqslant \epsilon_m \leqslant \epsilon$ $(n \geqslant m)$.

16. Now, since every black interval of $R_n(\epsilon_n)$ leads up to a black interval of $R_m(\epsilon_m)$, we have $R_n(\epsilon_n) \leqslant R_m(\epsilon_m)$; so *that the quantities*

$R_n(\epsilon_n)$ *form a sequence* whose limit is not greater than any one of them. If the limit of this sequence is zero, then it is evident that the theorem enunciated in § 2 is true, and we have a method of determining the integer m, while the quantity there denoted by ϵ might be taken as no other than ϵ_m, where we have determined m so that $R_m(\epsilon_m) < \sigma$.

17. Let us then, assuming that all that we have proved so far remains true, leave *out the assumption that G is closed*,* and assume that the limit of the sequence of quantities $R_n(\epsilon_n)$ is other than zero. We know, by the example of § 7, that there is no contradiction in these assumptions.

We will now show that it is a consequence of the last assumption that G is unclosed. By § 16 this will complete the proof of the theorem of § 2.

18. Let the limit of the sequence $R_n(\epsilon_n)$ be the positive quantity λ. Then, assuming any small positive quantity $\tau < \frac{\lambda}{2}$ and any integer r, we can determine an integer m_r, such that, for all values of $n \geqslant m_r$,

$$R_n(\epsilon_n) - \lambda < \frac{\tau}{2^r}.$$

Now, taking, instead of the whole sequence $R_n(\epsilon_n)$, the partial sequence $R_{m_1}(\epsilon_{m_1})$, $R_{m_2}(\epsilon_{m_2})$, ..., and calling them for convenience $R_1(\epsilon_1)$, $R_2(\epsilon_2)$, ..., $R_n(\epsilon_n)$, ..., we have, for all values of n,

$$R_n(\epsilon_n) - \lambda < \frac{\tau}{2^n}. \tag{1}$$

Now take any small positive quantity $\sigma < \frac{\lambda}{2}$. We can determine a *finite* number, k_n, of the intervals $R_n(\epsilon_n)$, say, $d_{n,r}$, $(r = 1, 2, ..., k_n)$, such that the sum of the remaining intervals of $R_n(\epsilon_n)$ is less than $\frac{\sigma}{2^n}$. It follows that the sum of the intervals $d_{n,r}$, being greater

* The assumptions are, stated at length, as follows:—G_1, G_2, ... are such that (1) each set is contained in all the preceding sets, (2) in any given segment, however small, it is possible to specify a part of it which is entirely black for every set G_n (so that G is nowhere dense), and (3), given any positive quantity ϵ, there corresponds a perfectly definite finite integer m, such that, for all values of $n \geqslant m$, all the black intervals of G_n which are $\geqslant \epsilon$, are identical with all those $\geqslant \epsilon$ of G.

than $R_n(\epsilon_n)-\frac{\sigma}{2^n}$, is greater than

$$\lambda-\frac{\sigma}{2^n}. \tag{2}$$

Now the intervals $d_{2,s}$, being intervals of $R_2(\epsilon_2)$, lead up to intervals of $R_1(\epsilon_1)$; and the sum of those intervals of $R_1(\epsilon_1)$ is, by (2), greater than $\lambda-\frac{\sigma}{2^2}$, and is therefore certainly greater than $\frac{\sigma}{2}$. Hence some of these intervals of $R_1(\epsilon_1)$ must be intervals $d_{1,r}$, because the sum of the remaining intervals of $R_1(\epsilon_1)$ was less than $\frac{\sigma}{2}$. For the same reason the sum of those intervals, (if any), $d_{2,s}$ which do not lead up to intervals $d_{1,r}$ is also less than $\frac{\sigma}{2}$. Hence, by (2), *the sum of those intervals $d_{2,s}$ which lead up to intervals $d_{1,r}$ is greater than $\lambda-\frac{\sigma}{2^2}-\frac{\sigma}{2}$; and the same is true of those intervals $d_{1,r}$ led up to by intervals $d_{2,s}$.* (3)

Now let us proceed a step further. The sum of the intervals $d_{3,t}$ is, as before, greater than $\lambda-\frac{\sigma}{2^3}$. The sum of those intervals of $R_2(\epsilon_2)$ which are not intervals $d_{2,s}$, or which, being intervals $d_{2,s}$, do not lead up to intervals $d_{1,r}$, is, by (3), less than

$$R_2(\epsilon_2)-\lambda+\frac{\sigma}{2}+\frac{\sigma}{2^2};$$

and therefore, by (1), is less than

$$\frac{\tau}{2^2}+\frac{\sigma}{2}+\frac{\sigma}{2^2}.$$

Hence the sum of those intervals $d_{3,t}$ which lead up to such intervals of $R_2(\epsilon_2)$ is also less than

$$\frac{\tau}{2^2}+\frac{\sigma}{2}+\frac{\sigma}{2^2}.$$

Therefore the sum of those intervals $d_{3,t}$ which lead up to intervals $d_{2,s}$ contained in intervals $d_{1,r}$ is greater than

$$\lambda-\frac{\sigma}{2^3}-\frac{\tau}{2^2}-\frac{\sigma}{2}-\frac{\sigma}{2^2},$$

which is $$\lambda-\left\{\frac{\sigma}{2}+\frac{\sigma}{2^2}+\frac{\sigma}{2^3}\right\}-\frac{\tau}{2^2},$$

while the same is true of the intervals $d_{2,s}$, or the intervals $d_{1,r}$ to which they lead up.

Similarly, going a step further, the sum of the remaining intervals of $R_3(\epsilon_3)$, being less than

$$R_3(\epsilon_3)-\lambda+\left(\frac{\sigma}{2}+\frac{\sigma}{2^2}+\frac{\sigma}{2^3}\right)+\frac{\tau}{2^2},$$

is, by (1), less than

$$\frac{\tau}{2^2}+\frac{\tau}{2^3}+\left(\frac{\sigma}{2}+\frac{\sigma}{2^2}+\frac{\sigma}{2^3}\right).$$

It follows that *the sum of those intervals $d_{4,u}$ which lead up to intervals $d_{3,t}$, which lead up to intervals $d_{2,s}$, contained in intervals $d_{1,r}$, is greater than*

$$\lambda-\frac{\sigma}{2^4}-\frac{\tau}{2^2}-\frac{\tau}{2^3}-\left\{\frac{\sigma}{2}+\frac{\sigma}{2^2}+\frac{\sigma}{2^3}\right\}$$

or

$$\lambda-\left\{\frac{\sigma}{2}+\frac{\sigma}{2^2}+\frac{\sigma}{2^3}+\frac{\sigma}{2^4}\right\}-\left\{\frac{\tau}{2^2}+\frac{\tau}{2^3}\right\},$$

while the same is true of the intervals $d_{3,t}$, or $d_{2,s}$, or $d_{1,r}$, to which they lead up.

In precisely the same manner it now follows by induction that there are a finite number of the intervals $d_{n,s}$ which lead up in turn to intervals $d_{m,r}$ for all values of $m < n$, and that the sum of these intervals $d_{n,s}$ is greater than

$$\lambda-\left\{\frac{\sigma}{2}+\frac{\sigma}{2^2}+\ldots+\frac{\sigma}{2^n}\right\}-\left\{\frac{\tau}{2^2}+\frac{\tau}{2^3}+\ldots+\frac{\tau}{2^{n-1}}\right\},$$

while the same is true of the intervals $d_{m,r}$ (for any fixed $m < n$) to which the intervals $d_{n,r}$ lead up. *A fortiori*, any one of these sums is greater than $\lambda-\sigma-\tau$, where σ and τ are as small as we please.

19. During this process we have been weeding out in turn (1) those intervals $d_{1,r}$ which do not contain intervals $d_{2,s}$; (2) those which, containing intervals $d_{2,s}$, do not contain intervals $d_{3,t}$ inside those intervals $d_{2,s}$; (3) those which, containing intervals $d_{2,s}$ with intervals $d_{3,t}$ inside them, do not contain intervals $d_{4,u}$ inside those intervals $d_{3,t}$; and so on. The sum of the intervals $d_{1,r}$ which are not weeded out, however, is, and remains always, greater than $\lambda-\sigma-\tau$. Hence the weeding process cannot, as far as it affects the intervals $d_{1,r}$, go on *ad infinitum;* but, after a finite number of stages, it comes to an

end. Let those intervals $d_{1,r}$ which are then left be denoted by

$$D_{1,r}, \quad (r = 1, 2, \ldots, \kappa_1;\ \kappa_1 \leqslant k_1).$$

Simultaneously we have been weeding out the intervals $d_{2,s}$. We have, firstly, weeded out intervals $d_{2,s}$ which are not contained in intervals $D_{1,r}$. But it may be that some of the intervals $d_{2,s}$ which lead up to intervals $D_{1,r}$ do not lead down to intervals $d_{n,t}$ for *all* values of $n > 2$. We have, however, weeded out from the intervals $d_{2,s}$ those which have not this property, just as we did the intervals $d_{1,r}$, and, since the sum of the intervals left is always $> \lambda-\sigma-\tau$, it follows, as before, that the weeding process, as far as it affects the intervals $d_{2,s}$, must also come to an end after a finite number of stages. Let us denote the intervals $d_{2,r}$ left at the end of the weeding process by

$$D_{2,r}, \quad (r = 1, 2, \ldots, \kappa_2;\ \kappa_2 \leqslant k_2).$$

Similarly we determine sets

$$D_{n,r}, \quad (r = 1, 2, \ldots, \kappa_n;\ \kappa_n \leqslant k_n)$$

for each successive value of n. The characteristics of these sets $D_{n,r}$ are (1) each interval $D_{n,r}$ leads up to intervals $D_{m,r}$ for all values of $m < n$; (2) each interval $D_{m,r}$ is led up to by intervals $D_{n,r}$ for all values of $n > m$; (3) the sum of all the intervals $D_{m,r}$ for given m is always $> \lambda-\sigma-\tau$; (4) the upper limit of the length of the intervals $D_{n,r}$, for given n, being $\leqslant \epsilon_n$, decreases without limit as n is indefinitely increased. Hence, if we choose one of these intervals for each successive value of n so that each of the chosen intervals lies within the preceding, they will define a limiting *point* P which will *not be exterior* to any one of the defining intervals. The end points of these defining intervals being points of G, P will certainly be a limiting point of G, (that is, a point of Γ), so that, unless P is an *end point* of each of the defining intervals after a definite stage, P, being *interior* to a black interval of every G_n, will belong to no G_n, and therefore not to G, so that G will be unclosed.

Since, however, the sum of the intervals $D_{n,r}$ for any assigned n is greater than $\lambda-\sigma-\tau$, it is easy to show that there must be limiting points which are *not* end points of the defining intervals. This is a consequence of a theorem which, being of general application, is stated and proved in the following section.

20. Theorem.—*If we have a finite number of non-overlapping intervals $D_{n,r}$, $(r = 1, 2, \ldots, k_n)$, for every integral value of n such that all the*

intervals $D_{n,r}$, for given n, lie inside the intervals $D_{m,r}$, for every value of $m < n$, and contain intervals $D_{p,r}$, for all values of $p > n$, then, if the sum S_n of all the intervals $D_{n,r}$, for every n, is greater than some assignable positive quantity λ, there are points such that each is interior to an interval $D_{n,r}$ for every value of n.

Let μ be any assigned small positive quantity. From each end of each interval $D_{n,r}$ let us cut off a fraction $\frac{1}{2^{n+r+1}}\frac{\mu}{S_n}$ of its length. The sum of the pieces cut off is less than $\frac{\mu}{2^n}$, and that of the curtailed intervals is greater than $S_n - \frac{\mu}{2^n}$.

Now inside the curtailed intervals $D_{m,r}$, $\left(\text{whose sum is} < S_m - \frac{\mu}{2^m}\right)$, there will fall parts of some, but not necessarily all, of the uncurtailed intervals $D_{m+1,r}$. But, since the sum of the parts cut off from the $D_{m,r}$ was less than $\frac{\mu}{2^m}$, it follows that the sum of the uncurtailed intervals and parts of intervals $D_{m+1,s}$ inside the curtailed intervals $D_{m,r}$ is greater than $S_{m+1} - \frac{\mu}{2^m}$. Hence the sum of the curtailed intervals and parts of intervals $D_{m+1,s}$ in the same is greater than

$$S_{m+1} - \frac{\mu}{2^m} - \frac{\mu}{2^{m+1}}.$$

The sum of the remaining intervals and part of intervals $D_{m+1,r}$ (uncurtailed) is less than $\frac{\mu}{2^m} + \frac{\mu}{2^{m+1}}$. Hence the sum of the uncurtailed intervals and parts of intervals $D_{m+2,t}$ which fall inside those curtailed intervals $D_{m+1,s}$ which lie in curtailed intervals $D_{m,r}$ is greater than $S_{m+2} - \frac{\mu}{2^m} - \frac{\mu}{2^{m+1}}$; and the sum of the *curtailed* intervals and parts of intervals $D_{m+2,t}$ in the same is greater than $S_{m+2} - \frac{\mu}{2^m} - \frac{\mu}{2^{m+1}} - \frac{\mu}{2^{m+2}}$.

In this way it appears that the sum of the curtailed intervals $D_{n,r}$ which fall inside curtailed intervals $D_{n-1,r}$, which themselves fall inside curtailed intervals $D_{n-2,r}$, and so on back to $D_{1,r}$, is greater than $S_n - \frac{\mu}{2} - \frac{\mu}{2^2} - \dots - \frac{\mu}{2^n}$ and is therefore certainly greater than $S_n - \mu$; *a fortiori*, than $\lambda - \mu$.

Considering then the new sets of curtailed intervals, and leaving out all such as do not lie in the preceding and contain all the following sets, we get a new series of sets, which we may denote by $D'_{n,r}$, and which have precisely the same attributes as before, except that $\lambda-\mu$ is substituted for λ.

If we now choose one of these intervals for each successive value of n, so that each lies within the preceding, they will determine a limiting point P or a limiting interval δ. In either case they determine one or more points which, not being *exterior* to any one of the defining intervals $D'_{n,r}$, is certainly *interior* to the corresponding uncurtailed intervals $D_{n,r}$. Q. E. D.

21. Applying this theorem to the results of the preceding article, we see that G will be unclosed; that is to say, *the hypothesis that the limit of the sequence of quantities $R_n(\epsilon_n)$ is other than zero is only compatible with our other conditions if G is unclosed, which proves the theorem of* § 2.

22. It should be noticed that the discussion of § 19 permits us to remove the restriction as to the finiteness of the number of intervals $D_{n,r}$. The more general theorem is as follows:—

If we have a set of non-overlapping intervals Δ_n for every integral value of n such that all the intervals of Δ_n for given n lie inside the intervals of Δ_m for every value of $m < n$, and contain all the intervals of Δ_p for every value of $p > n$, then, if the sum S_n of all the intervals Δ_n for every n is greater than some assignable positive quantity λ, there are points such that each is interior to an interval of Δ_n for every value of n.

23. It remains to deduce Osgood's theorem.* Denote by I and I_n the contents of G and G_n respectively, and by L the length of the finite segment over which they extend; then the contents of the corresponding sets of black intervals are $(L-I)$ and $(L-I_n)$.

By the theorem of § 2,

$$L-I_n-(L-I) = R_n(\epsilon)-R(\epsilon) < R_n(\epsilon) < \sigma \quad (n \geqslant m),$$

which proves the theorem.

* *Amer. Jour. of Math.*, Vol. XIX., 1897.