TUTORIAL 05:
NUMERICAL ASPECT II: CONSISTENCY AND STABILITY

In this tutorial, we will explore the consistency and stability of a numerical scheme. We will focus on a simple case study: the one-dimensional advection problem. Using the Taylor expansion approximation, we will define the order of a numerical scheme and test its stability. We will uncover the famous CFL condition.

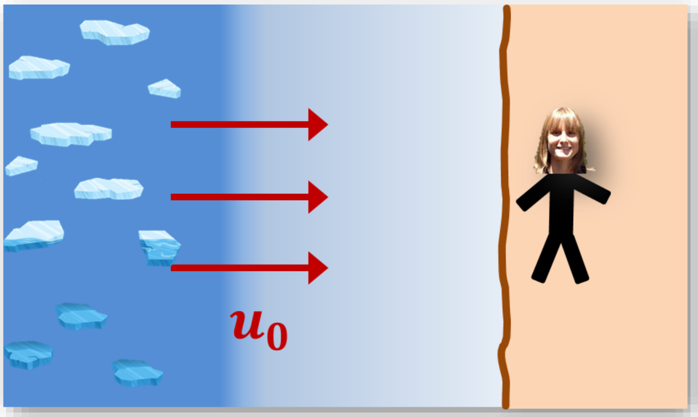
1: The 1D advection equation

→ From CROCO 3D temperature equation:

$$\frac{\partial T}{\partial t} + \boldsymbol{u}\nabla T = \nabla_h(K_{Th}\nabla_h T) + \frac{\partial}{\partial z}\left(K_{Tv}\frac{\partial T}{\partial z}\right)$$

↪ We simplify the processes at work by studying a simple case study, where:

- there is no surface forcing (adiabatic).
- there is a constant current directed toward the shore u_0 (homogeneous in y).
- there is no variation of temperature with depth (barotropic case), i.e. we can cross-out the vertical turbulent diffusion term.
- there is no horizontal diffusion.



→ From the 3D temperature, we need to solve the 1D advection equation:

$$\frac{\partial T}{\partial t} + u_0\frac{\partial T}{\partial x} = 0 \qquad x \in [O, L], \qquad t \in [0, T] \quad (1)$$

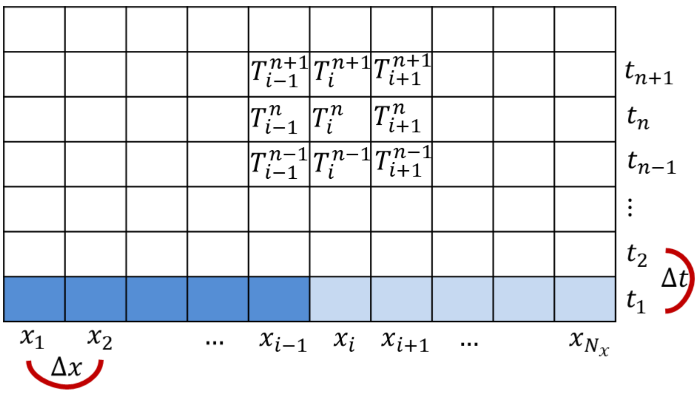
↪ There are only first-order derivatives in time and space.

↪ The initial conditions that portray this temperature front are known. The constant parameter u_0 (the current advecting the cold condition toward the coast) must be given.

2: Consistency of a numerical scheme

→ Same as in **#TUTORIAL03**, we work on a discretized model grid. We replace the continuous domain $[0, L] \times [0, T]$ by a set of **equally spaced mesh points**, such that:

$$x_i = i\Delta x, i = 1, \dots, N_x \quad \text{and} \quad t_n = n\Delta t, n = 1, \dots, N_t$$



→ We need to find a **consistent** approximation for the equation derivatives: $\frac{\partial T}{\partial t}$ and $\frac{\partial T}{\partial x}$ on our model grid. This means that the error between the discretized and the real solution must approach 0.

→ In order to quantify the error we make by solving any equation on a spatial and temporal discretised grid, we use the Taylor series expansion of a continuous function f at a point x_0 close to a reference point x:

$$f(x_0) = f(x) + \frac{f'(x)}{1!}(x_0 - x) + \frac{f''(x)}{2!}(x_0 - x)^2 + \cdots + \frac{f^n(x)}{n!}(x_0 - x)^n + R(x)$$

➥ If x is close to x_0, such that $x_0 = x + \Delta x$ and Δx is small, we can write:

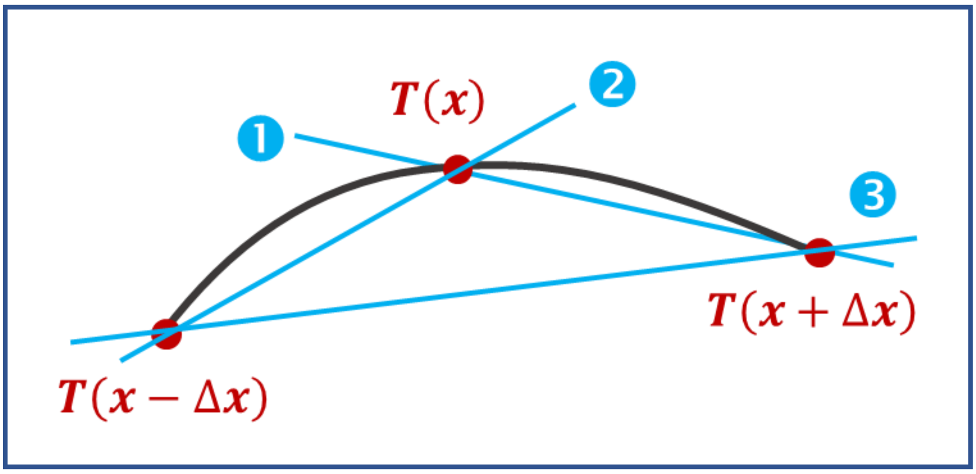
$$f(x + \Delta x) = f(x) + \frac{f'(x)}{1!}\Delta x + \frac{f''(x)}{2!}\Delta x^2 + \cdots + \frac{f^n(x)}{n!}\Delta x^n + R(x)$$

→ Let discretize $\frac{\partial T}{\partial x}$. There are 3 different numerical schemes:

❶ The **downstream** (Euler) scheme: $\frac{\partial T}{\partial x} \simeq \frac{T(x+\Delta x) - T(x)}{\Delta x}$

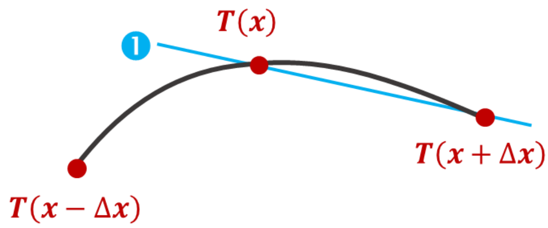
❷ The **upstream** scheme: $\frac{\partial T}{\partial x} \simeq \frac{T(x) - T(x-\Delta x)}{\Delta x}$

❸ The **centered** scheme: $\frac{\partial T}{\partial x} \simeq \frac{T(x+\Delta x) - T(x-\Delta x)}{2\Delta x}$



➢ Estimation of the error we make when we choose the downstream scheme (❶):

$$T(x + \Delta x) = T(x) + \frac{T'(x)}{1!}\Delta x + \frac{T''(x)}{2!}\Delta x^2 + \frac{T'''(x)}{3!}\Delta x^3 + \cdots$$



$$\frac{\partial T}{\partial x} = \frac{T(x+\Delta x) - T(x)}{\Delta x} + \text{Error}$$

$$\frac{\partial T}{\partial x} = \frac{\cancel{T(x)} + \frac{T'(x)}{1!}\cancel{\Delta x} + \frac{T''(x)}{2!}\Delta x^{\cancel{2}} - \cancel{T(x)}}{\cancel{\Delta x}}$$

$$\frac{\partial T}{\partial x} = T'(x) + \frac{T''(x)\Delta x}{2!}$$

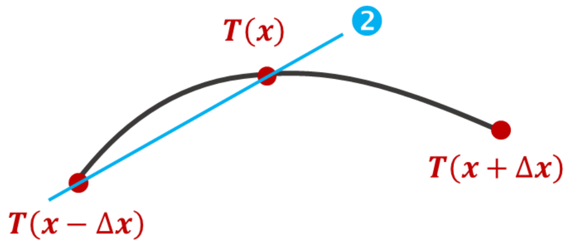
$$\boxed{\text{Error} = \frac{\Delta x\, T''(x)}{2!}}$$

1st order error

➢ Estimation of the error we make when we choose the upstream scheme (❷):

$$T(x-\Delta x) = T(x) - \frac{T'(x)}{1!}\Delta x + \frac{T''(x)}{2!}\Delta x^2 - \frac{T'''(x)}{3!}\Delta x^3 + \cdots$$

$$\frac{\partial T}{\partial x} = \frac{T(x) - T(x-\Delta x)}{\Delta x} + \text{Error !}$$



$$\frac{\partial T}{\partial x} = \frac{\cancel{T(x)} - \left(\cancel{T(x)} - \frac{T'(x)\cancel{\Delta x}}{1!} + \frac{T''(x)\Delta x^{\cancel{2}}}{2!}\right)}{\cancel{\Delta x}}$$

$$\frac{\partial T}{\partial x} = T'(x) - \underbrace{\frac{T''(x)\Delta x}{2!}}_{\text{1st order error}}$$

➢ Estimation of the error we make when we choose the centered scheme (❸):

$$\frac{\partial T}{\partial x} = \frac{T(x+\Delta x) - T(x-\Delta x)}{2\Delta x} + \text{Error}$$

$$T(x+\Delta x) = T(x) + \frac{T'(x)}{1!}\Delta x + \frac{T''(x)}{2!}\Delta x^2 + \frac{T'''(x)}{3!}\Delta x^3 + \cdots$$

$$- \quad T(x-\Delta x) = T(x) - \frac{T'(x)}{1!}\Delta x + \frac{T''(x)}{2!}\Delta x^2 - \frac{T'''(x)}{3!}\Delta x^3 + \cdots$$

$$\text{Error} = \frac{\Delta x^2 T'''(x)}{3!}$$

2nd order error

➥ With the centered scheme, the first-order derivative is better resolved than with the first order schemes.

⇨ The centered scheme is better than upstream and downstream schemes, because the **truncation error** is smaller. To improve it, you can increase your resolution (Δx ↘) or use higher-order schemes.

3: Stability and convergence of a numerical scheme

→ The most important characteristics of a numerical scheme are:

- Its **consistency**, i.e. consistent approximation for the derivative in the equations (truncation error ↘ 0). This is a condition in space.

- Its **stability**, i.e. does the error amplify in time? We do not want that the error increase with time. If this the case, there will be a numerical explosion (a blow-up), and the model will stop.

⇨ If both conditions are respected (consistency and stability) then the discrete solution **converges** toward the real solution.

❶ We will test the stability of a **downstream scheme** for both: $\frac{\partial T}{\partial t}$ and $\frac{\partial T}{\partial x}$, such that:

$$\frac{\partial T}{\partial t} \approx \frac{T(t+\Delta t) - T(t)}{\Delta t} = \frac{T_i^{n+1} - T_i^n}{\Delta t}$$

$$\frac{\partial T}{\partial x} \approx \frac{T(x+\Delta x) - T(x)}{\Delta x} = \frac{T_{i+1}^n - T_i^n}{\Delta x}$$

→ We inject this formulation into the 1D-advection equation. This leads to:

$$\frac{\partial T}{\partial t} + u_0 \frac{\partial T}{\partial x} = 0 \quad \rightarrow \quad \frac{T_i^{n+1} - T_i^n}{\Delta t} + u_0 \frac{T_{i+1}^n - T_i^n}{\Delta x} = 0$$

$$\rightarrow T_i^{n+1} = T_i^n - \frac{\Delta t}{\Delta x} u_0 \left(T_{i+1}^n - T_i^n\right)$$

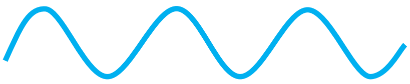
↪ This gives T at time $t + \Delta t$ as a function of T at time t. This is an **explicit method**. It is easy to solve 😉

→ We will perform a **von Neumann** stability analysis of our explicit solution.

↪ For this we use wave-like structure for $T(x)$ using complex form: $T_n = \hat{T}_n e^{ikx}$

- e^{ikx} is a wavy pattern that repeats indefinitely (k provides information about its zonal extension).

- $\hat{T}_n$ is the amplitude of the wavy pattern



→ We rewrite our explicit solution using this new notation.

Von Neumann Stability

C (Courant Number)

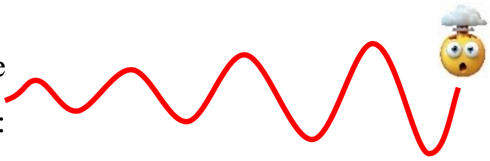
$$\hat{T}^{n+1} e^{ikx} = \hat{T}^n e^{ikx} - \frac{u_0 \Delta T}{\Delta x} \hat{T}^n \left(e^{i(kx+\Delta x)} - e^{ikx}\right)$$

With $C = \frac{u_0 \Delta t}{\Delta x}$, the Courant number. $C > 0$

→ We now define the amplification A, such that:

$$A = \frac{\hat{T}_{n+1}}{\hat{T}_n}$$

➥ We want $A < 1$, because we do not want the amplitude of the oscillation to increase over time, otherwise the solution would explode:



$$\hat{T}_{n+1} = A\,\hat{T}_n = A^2\,\hat{T}_{n-1} = \cdots = A^n\,\hat{T}_0$$

$$A = \frac{\hat{T}_{n+1}}{\hat{T}_n} = 1 - C\left(e^{ik\Delta x} - 1\right)$$

$$A = 1 - C\left(e^{ik\Delta x} - 1\right) = 1 - C(\cos(k\Delta x) - i\,sin(k\Delta x) - 1)$$

$$= 1 + C(1 - \cos(k\Delta x)) - i\,C\,sin(k\Delta x)$$

real part *imaginary part*

$\|A\|^2 =$ *real part²* + *imaginary part²*

$\|A\|^2 = [1 + C(1 - \cos(k\Delta x))]^2 + [C\,sin(k\Delta x)]^2$

$\|A\|^2 = 1 + C^2(1 - \cos(k\Delta x))^2 + 2C(1 - \cos(k\Delta x)) + C^2\,sin^2(k\Delta x)$

$\|A\|^2 = 1 + C^2(1 + \boxed{\cos^2(k\Delta x)} - 2\cos(k\Delta x)) + 2C(1 - \cos(k\Delta x)) + C^2\boxed{sin^2(k\Delta x)}$

$\|A\|^2 = 1 + C^2(2 - 2\cos(k\Delta x)) + 2C(1 - \cos(k\Delta x))$

$\|A\|^2 = 1 + 2C^2(1 - \cos(k\Delta x)) + 2C(1 - \cos(k\Delta x))$

$\|A\|^2 = 1 + (1 - \cos(k\Delta x)) \times 2C \times (1 + C)$

$\|A\|^2 = 1 + (1 - \cos(k\Delta x)) \times 2C \times (1 + C)$

> 0 >0 >0

$\|A\|^2 > 1 \implies$ **Unconditionally unstable scheme!!**

⇨ $\|A\| > 1$. This means that the solution increases over time. This scheme is unstable. The downstream scheme is not a good choice. I will never know if I can go to the beach tomorrow ☹ !

❷ We use the **downstream scheme in space** and the **upstream scheme in time** - the **upwind scheme**:

$$\frac{\partial T}{\partial t} \approx \frac{T(t+\Delta t) - T(t)}{\Delta t} = \frac{T_i^{n+1} - T_i^n}{\Delta t}$$

$$\frac{\partial T}{\partial x} \approx \frac{T(x) - T(x-\Delta x)}{\Delta x} = \frac{\boldsymbol{T_i^n - T_{i-1}^n}}{\Delta x}$$

$$\frac{\partial T}{\partial t} + u_0 \frac{\partial T}{\partial x} = 0 \rightarrow \frac{T_i^{n+1} - T_i^n}{\Delta t} + u_0 \frac{T_i^n - T_{i-1}^n}{\Delta x} = 0$$

➥ This is an explicit scheme: $T_i^{n+1} = T_i^n - \frac{\Delta t}{\Delta x} u_0 \left(T_i^n - T_{i-1}^n\right)$

→ We adopt the complex form: $T_n = \widehat{T_n} e^{ik(x)}$, which leads to $\hat{T}_{n+1} e^{ik(x)} =$

→ Again, we define the amplification A, such that $A = \frac{\hat{T}_{n+1}}{\hat{T}_n}$. In the end, we want $A < 1$, because we do not want the solution to explode !

$$A = \frac{\hat{T}_{n+1}}{\hat{T}_n} = 1 - C\left(1 - e^{-ik\Delta x}\right) = 1 - C\left(1 - \left(\cos(k\Delta x) - i\,sin(k\Delta x)\right)\right)$$

$$= \underbrace{1 - C(1 - \cos(k\Delta x))}_{\textit{real part}} \underbrace{- iC\,sin(k\Delta x)}_{\textit{imaginary part}}$$

$||A||^2 = \textit{real part}^2 + \textit{imaginary part}^2$

$||A||^2 = [1 - C(1 - \cos(k\Delta x))]^2 + [C\,sin(k\Delta x)]^2$

$||A||^2 = 1 + C^2(1 - \cos(k\Delta x))^2 - 2C(1 - \cos(k\Delta x)) + C^2\,sin^2(k\Delta x)$

$||A||^2 = 1 + C^2(1 + \boxed{\cos^2(k\Delta x)} - 2\cos(k\Delta x)) - 2C(1 - \cos(k\Delta x)) + C^2 \boxed{sin^2(k\Delta x)}$

$||A||^2 = 1 + C^2(2 - 2\cos(k\Delta x)) - 2C(1 - \cos(k\Delta x))$

$||A||^2 = 1 + 2C \times C(1 - \cos(k\Delta x) - 2C(1 - \cos(k\Delta x)))$

$||A||^2 = 1 + 2C \times (1 - \cos(k\Delta x)) \times (C - 1)$

$||A||^2 = 1 + 2C \times (1 - C) \times (\cos(k\Delta x) - 1)$

⇨ This upwind scheme is conditionally stable. It is stable only if $C = \frac{u_0 \Delta t}{\Delta x} < 1$. This is the famous CFL condition.