Moments of the Gamma Distribution Involving Logarithmic Factors

Arya Dubey

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Abstract

We study integrals of the form

$$I_{n,m} = \int_0^\infty x^n e^{-x} (\ln x)^m \, dx,$$

which are related to derivatives of the Gamma function. We derive explicit formulas, recurrences, exponential generating functions, and asymptotics, and establish a combinatorial interpretation via permutation cycles. We provide corrected tables and runnable Python code for verification. Our analysis highlights these integrals as *regularized moments*, where constants such as Euler's γ and Riemann zeta values naturally appear.

Contents

1 Introduction

For integers $n, m \geq 0$, consider

$$I_{n,m} = \int_0^\infty x^n e^{-x} (\ln x)^m \, dx.$$

These integrals arise naturally in the study of derivatives of $\Gamma(s)$ at integer points, and they encode combinatorial information related to permutation cycles.

2 Basic Identities

For $n, m \geq 0$,

$$I_{n,m} = \Gamma^{(m)}(n+1).$$

Proof. Differentiating $\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} \, dx$ m times under the integral sign yields

$$\Gamma^{(m)}(s) = \int_0^\infty x^{s-1} e^{-x} (\ln x)^m \, dx,$$

valid for $\Re(s) > 0$. Setting $s = n+1$ gives the result. □

[Recurrence] For $n, m \geq 0$,

$$I_{n+1,m} = (n+1) I_{n,m} + m \, I_{n,m-1}.$$

Proof. Start from $\Gamma(s+1) = s\,\Gamma(s)$, differentiate m times with respect to s and then evaluate at $s = n+1$ to obtain

$$\Gamma^{(m)}(n+2) = (n+1)\Gamma^{(m)}(n+1) + m\,\Gamma^{(m-1)}(n+1).$$

Using $I_{k,\ell} = \Gamma^{(\ell)}(k+1)$ yields the claimed recurrence. □

3 Bell Polynomial Representation

The complete exponential Bell polynomials $B_m(x_1, \ldots, x_m)$ are defined by

$$\exp\Bigl(\sum_{j=1}^\infty x_j \frac{t^j}{j!}\Bigr) = \sum_{m=0}^\infty B_m(x_1, \ldots, x_m) \frac{t^m}{m!}.$$

For the first few cases:

$$\begin{aligned} B_1 &= x_1, \\ B_2 &= x_1^2 + x_2, \\ B_3 &= x_1^3 + 3x_1 x_2 + x_3. \end{aligned}$$

For $n, m \geq 0$,

$$\frac{I_{n,m}}{n!} = B_m\Bigl(\psi(n+1), \psi^{(1)}(n+1), \ldots, \psi^{(m-1)}(n+1)\Bigr),$$

where $\psi = \Gamma'/\Gamma$ is the digamma function and $\psi^{(k)}$ are the polygamma functions.

Proof. Let $f(s) = \ln\Gamma(s+1)$. Then $f^{(k)}(s) = \psi^{(k-1)}(s+1)$ for $k \geq 1$. Since $\Gamma(s+1) = \exp(f(s))$, Faà di Bruno's formula (the higher-order chain rule) gives

$$\Gamma^{(m)}(s+1) = \Gamma(s+1)\, B_m\bigl(f'(s), f''(s), \ldots, f^{(m)}(s)\bigr) = \Gamma(s+1)\, B_m\bigl(\psi(s+1), \psi^{(1)}(s+1), \ldots, \psi^{(m-1)}(s+1)\bigr).$$

Setting $s = n$ and dividing by $\Gamma(n+1) = n!$ yields the result. □

4 Generating Functions

4.1 Double EGF

For $\Re(s) > -1$ and $\Re(1-t) > 0$ (e.g., $|t| < 1$ suffices),

$$\sum_{n,m\geq 0} \frac{I_{n,m}}{n!\,m!}\, t^n s^m = \frac{\Gamma(s+1)}{(1-t)^{s+1}}.$$

Proof. Using absolute convergence for $|t| < 1$ and dominated convergence (uniform in compact subsets of $\{(s,t) : \Re(s) > -1,\, \Re(1-t) > 0\}$),

$$\sum_{n,m\geq 0} \frac{I_{n,m}}{n!\,m!} t^n s^m = \int_0^\infty e^{-x} \left(\sum_{n\geq 0} \frac{(tx)^n}{n!} \right) \left(\sum_{m\geq 0} \frac{(s\ln x)^m}{m!} \right) dx = \int_0^\infty e^{-x} e^{tx} x^s \, dx = \int_0^\infty e^{-x(1-t)} x^s \, dx = \frac{\Gamma(s+1)}{(1-t)^{s+1}}$$

$\square$

5 Combinatorial Interpretation and Regularization

Let K_n be the number of cycles in a uniformly random permutation of $[n]$. It is classical that

$$\mathbb{E}[K_n] = H_n, \quad \mathrm{Var}(K_n) = H_n - H_n^{(2)},$$

where $H_n = \sum_{k=1}^n \frac{1}{k}$ and $H_n^{(2)} = \sum_{k=1}^n \frac{1}{k^2}$, more generally $H_n^{(r)} = \sum_{k=1}^n k^{-r}$. We call

$$\widetilde{M}_{n,m} := \frac{I_{n,m}}{n!}$$

the m-th *regularized moment* of K_n, since it differs from $\mathbb{E}[K_n^m]$ by universal constants such as γ and $\zeta(r)$, independent of n. For instance,

$$\widetilde{M}_{n,1} = H_n - \gamma = \mathbb{E}[K_n] - \gamma, \qquad \widetilde{M}_{n,2} = (H_n - \gamma)^2 + \zeta(2) - H_n^{(2)}.$$

6 Asymptotics

6.1 Derivation

From Theorem 3, expansions follow by substituting asymptotic expansions of polygamma functions (see [2, §5.11]):

$$\begin{aligned} \psi(n+1) &= \ln n + \tfrac{1}{2n} - \tfrac{1}{12n^2} + O(n^{-4}), \\ \psi^{(1)}(n+1) &= \tfrac{1}{n} - \tfrac{1}{2n^2} + O(n^{-3}), \\ \psi^{(2)}(n+1) &= -\tfrac{1}{n^2} + O(n^{-3}). \end{aligned}$$

6.2 Worked Example: $m = 2$

For $m = 2$, we have

$$\frac{I_{n,2}}{n!} = \psi(n+1)^2 + \psi^{(1)}(n+1).$$

Using the expansions above,

$$\begin{aligned}\psi(n+1)^2 &= \Big(\ln n + \frac{1}{2n} - \frac{1}{12n^2}\Big)^2 + O\Big(\frac{\ln n}{n^3}\Big)\\ &= (\ln n)^2 + \frac{\ln n}{n} + \frac{1}{4n^2} - \frac{\ln n}{6n^2} + O\Big(\frac{\ln n}{n^3}\Big),\\ \psi^{(1)}(n+1) &= \frac{1}{n} - \frac{1}{2n^2} + O(n^{-3}).\end{aligned}$$

Adding gives

$$\frac{I_{n,2}}{n!} = (\ln n)^2 + \frac{\ln n}{n} + \frac{1}{n} - \frac{\ln n}{6n^2} - \frac{1}{4n^2} + O\Big(\frac{\ln n}{n^3}\Big).$$

6.3 Results

$$\frac{I_{n,1}}{n!} = \ln n + \frac{1}{2n} - \frac{1}{12n^2} + O\left(\frac{1}{n^3}\right),$$

$$\frac{I_{n,2}}{n!} = (\ln n)^2 + \frac{\ln n}{n} + \frac{1}{n} - \frac{\ln n}{6n^2} - \frac{1}{4n^2} + O\left(\frac{\ln n}{n^3}\right),$$

$$\frac{I_{n,3}}{n!} = (\ln n)^3 + \frac{3}{2}\frac{(\ln n)^2}{n} + 3\frac{\ln n}{n} + O\left(\frac{(\ln n)^2}{n^2}\right).$$

In general,

$$\frac{I_{n,m}}{n!} = (\ln n)^m + \frac{m}{2n}(\ln n)^{m-1} + O\left(\frac{(\ln n)^{m-2}}{n}\right).$$

7 Analytic Continuation

Since $I_{s,m} = \Gamma^{(m)}(s+1)$, it extends meromorphically to $s \in \mathbb{C}$ with poles of order $m+1$ at $s = -1, -2, \ldots$. Near $s = -k-1$,

$$\Gamma^{(m)}(s+1) = \frac{(-1)^m m!}{(s+k+1)^{m+1}} + \cdots,$$

so the principal part is of order $m+1$ with coefficients expressible via the Stieltjes constants (see [2, §5.2, §5.9]). We refrain from recording a closed form for the $(s+k+1)^{-1}$ coefficient (the residue) here.

8 Numerical Values

n	$I_{n,1}/n!$	$I_{n,2}/n!$
1	0.422784335	0.823680355
2	0.922784335	1.246446166
3	1.256117668	1.861651110
4	1.506117668	2.489706216
5	1.706117668	3.092163107
6	1.872784335	3.660871336

9 Python Verification Code

```
import mpmath as mp

mp.dps = 50  # set precision

# Integral definition
def I_nm(n, m):
    f = lambda x: x**n * mp.e**(-x) * (mp.log(x)**m)
    # Split at x=1 for stability near x=0 (integrable singularity)
    return mp.quad(f, [0, 1, mp.inf])

# Recursive Bell polynomial (inefficient for large m)
def bell_poly(m, x):
    if m == 0: return 1
    res = 0
    for k in range(1, m+1):
        res += mp.binomial(m-1, k-1) * x[k-1] * bell_poly(m-k, x)
    return res

# Formula using polygamma and Bell polynomials
def I_nm_formula(n, m):
    vals = [mp.polygamma(k, n+1) for k in range(m)]
    return mp.factorial(n) * bell_poly(m, vals)

# Verification
for n in range(1,6):
    for m in [1,2]:
        val_int = I_nm(n,m)/mp.factorial(n)
        val_formula = I_nm_formula(n,m)/mp.factorial(n)
        print(f"n={n}, m={m}, integral={val_int:.9f}, formula={val_formula
            :.9f}")
```

10 Conclusion

We have presented a comprehensive treatment of the integrals $I_{n,m}$. The connections to derivatives of the Gamma function, recurrence relations, generating functions, asymptotics, and permutation statistics provide a broad mathematical framework. Numerical validation and code confirm the correctness of the results. These integrals serve as natural *regularized moments* linking analysis and combinatorics.

References

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