Discussion of Key Points and Critical Aspects of the Theoretical Model

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Abstract

We introduce the potential criticalities of the model, demonstrating and verifying them mathematically, thereby reinforcing what has already been published. This discussion makes the theory of decomposition and extension of the Einstein tensor defining the spacetime structure even more robust.

1 Verification of Mathematical Consistency: Conservation of the Energy-Momentum Tensor

In the model, the modified field equation takes the form:

$$G_{\mu\nu} = G^{(s)}_{\mu\nu} + G^{(t)}_{\mu\nu} + \varphi(x^\alpha)\, G^{(c)}_{\mu\nu}, \tag{1}$$

where $\varphi(x^\alpha)$ is a dynamic scalar field, and $G^{(c)}_{\mu\nu}$ represents the geometric correlation tensor.

For the theory to be compatible with standard differential geometry, the **conservation of energy-momentum** requires:

$$\nabla^\mu G_{\mu\nu} = 0 \quad \Rightarrow \quad \nabla^\mu G^{\text{eff}}_{\mu\nu} = 0, \tag{2}$$

where the effective tensor is:

$$G^{\text{eff}}_{\mu\nu} := G^{(s)}_{\mu\nu} + G^{(t)}_{\mu\nu} + \varphi(x^\alpha)\, G^{(c)}_{\mu\nu}. \tag{3}$$

Let us now compute the covariant derivative of the corrective term:

$$\nabla^\mu \left(\varphi\, G^{(c)}_{\mu\nu}\right) = (\nabla^\mu \varphi)\, G^{(c)}_{\mu\nu} + \varphi\, \nabla^\mu G^{(c)}_{\mu\nu}. \tag{4}$$

To ensure **energy-momentum conservation**, we must impose that:

$$\nabla^\mu G^{(s)}_{\mu\nu} + \nabla^\mu G^{(t)}_{\mu\nu} + \nabla^\mu \left(\varphi\, G^{(c)}_{\mu\nu}\right) = 0. \tag{5}$$

Therefore, the mathematical consistency condition reduces to:

$$\nabla^\mu G^{(s)}_{\mu\nu} + \nabla^\mu G^{(t)}_{\mu\nu} + (\nabla^\mu \varphi)\, G^{(c)}_{\mu\nu} + \varphi\, \nabla^\mu G^{(c)}_{\mu\nu} = 0. \tag{6}$$

Sufficient Conditions for Conservation

For equation (6) to be satisfied, it is sufficient that at least one of the following holds:

(i) $\nabla^\mu G^{(c)}_{\mu\nu} = 0$: the correlation tensor is divergence-free;

(ii) $G^{(c)}_{\mu\nu} \propto \nabla_\mu \varphi$: coupling consistent with the gradient of the scalar field;

(iii) The terms $\nabla^\mu G^{(s)}_{\mu\nu}$ and $\nabla^\mu G^{(t)}_{\mu\nu}$ exactly compensate the correlation term.

Conclusion

The formulation remains **mathematically coherent** if the tensor $G_{\mu\nu}^{(c)}$ is constructed to be compatible with geometric symmetries and if the field $\varphi(x^\alpha)$ ensures, together, the **covariant conservation of the total energy-momentum tensor**.

In the particular case where $\varphi \to 0$, the theory exactly reduces to General Relativity, preserving formal coherence even in the classical limit.

1. Structural Definition of the Model

Let the decomposition of the extended Einstein tensor be as follows:

$$G_{\mu\nu} = G_{\mu\nu}^{(s)} + G_{\mu\nu}^{(t)} + \varphi(x^\alpha) \cdot G_{\mu\nu}^{(c)}, \tag{7}$$

where:

- $G_{\mu\nu}^{(s)}$: spatial contribution;
- $G_{\mu\nu}^{(t)}$: temporal contribution;
- $G_{\mu\nu}^{(c)}$: mixed space-time contribution;
- $\varphi(x^\alpha)$: dynamic scalar field (correlation factor).

2. Energy-Momentum Conservation Condition

General Relativity imposes:

$$\nabla^\mu G_{\mu\nu} = 0, \tag{8}$$

which must also hold in the modified theory. Let's insert the decomposition:

$$\nabla^\mu G_{\mu\nu} = \nabla^\mu \left(G_{\mu\nu}^{(s)} + G_{\mu\nu}^{(t)} + \varphi \cdot G_{\mu\nu}^{(c)} \right) \tag{9}$$

$$= \nabla^\mu G_{\mu\nu}^{(s)} + \nabla^\mu G_{\mu\nu}^{(t)} + \nabla^\mu (\varphi \cdot G_{\mu\nu}^{(c)}). \tag{10}$$

Suppose the spatial and temporal components satisfy:

$$\nabla^\mu G_{\mu\nu}^{(s)} + \nabla^\mu G_{\mu\nu}^{(t)} = 0.$$

Then:

$$\nabla^\mu G_{\mu\nu} = \nabla^\mu (\varphi \cdot G_{\mu\nu}^{(c)}) = (\nabla^\mu \varphi) G_{\mu\nu}^{(c)} + \varphi \cdot \nabla^\mu G_{\mu\nu}^{(c)}. \tag{11}$$

Consistency condition:
To ensure energy-momentum conservation is guaranteed, it is sufficient to require:

$$(\nabla^\mu \varphi) G_{\mu\nu}^{(c)} + \varphi \cdot \nabla^\mu G_{\mu\nu}^{(c)} = 0. \tag{12}$$

This condition can be satisfied in two cases:

- **(a) Constant field:** $\nabla^\mu \varphi = 0$ (GR limit), we have $\nabla^\mu G_{\mu\nu}^{(c)} = 0$
- **(b) Dynamic field:** If $\varphi \neq \text{const}$, then we impose:

$$\nabla^\mu G_{\mu\nu}^{(c)} = -\frac{1}{\varphi} (\nabla^\mu \varphi) G_{\mu\nu}^{(c)}.$$

Observation: This relation defines a *controlled covariant divergence* of the tensor $G^{(c)}_{\mu\nu}$ which can be regular if φ is smooth and never zero (i.e., infinitely differentiable: $\varphi(x^\alpha) \neq 0$).

3. Consistency with the Tensorial Structure

The tensor $G^{(c)}_{\mu\nu}$ is defined on a subspace of spacetime, where the geometric interaction between the temporal surface and the three-dimensional spatial manifold is active.

Let:

$$G^{(c)}_{\mu\nu} = \begin{bmatrix} 0 & * & * & * \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{bmatrix},$$

where the asterisks indicate mixed (space-time) components. The symmetry $G_{\mu\nu} = G_{\nu\mu}$ is preserved.

Note: This type of structure is similar to that of optical birefringence in anisotropic crystals, with locally privileged directions but not globally orientable (analogy with the Moebius surface).

4. Consistency with Observational Data

The observable effect of the term $\varphi \cdot G^{(c)}_{\mu\nu}$ is a modulation of the phase and amplitude of gravitational waves (QNM), in particular:

- Systematic shifts in frequencies f_n and damping times τ_n
- Variation of the phase ϕ_n
- Appearance of gravitational birefringence (differential polarization)

These effects have been tested in data from events like GW190412, GW190521, etc., through QNM fits with two modes. The fit residuals indicate that adding the term $\varphi \cdot G^{(c)}_{\mu\nu}$ significantly improves the model's efficiency.

5. Conclusion

The proposed mathematical structure is:

- **Covariant:** thanks to the differential form of the interaction between φ and $G^{(c)}_{\mu\nu}$
- **Physically compatible:** with the 4-dimensional structure of the metric
- **Observationally testable:** with GW signals and cosmological data

Therefore, the decomposition:

$$G_{\mu\nu} = G^{(s)}_{\mu\nu} + G^{(t)}_{\mu\nu} + \varphi(x^\alpha) G^{(c)}_{\mu\nu}$$

is formally well-defined, free of divergences, and consistent with energy-momentum conservation provided the derivative coupling conditions are respected.

2 Uniqueness of the Einstein Tensor Decomposition

The decomposition proposed in the theory takes the form:

$$G_{\mu\nu} = G_{\mu\nu}^{(s)} + G_{\mu\nu}^{(t)} + \phi(x^\alpha) \cdot G_{\mu\nu}^{(c)}, \tag{13}$$

where the terms $G_{\mu\nu}^{(s)}$, $G_{\mu\nu}^{(t)}$, and $G_{\mu\nu}^{(c)}$ represent spatial, temporal, and mixed correlation contributions, respectively.

However, this structure **is not unique**: it represents a **specific constructive choice**, coherent with the geometric formalism and the phenomenological objectives of the model. Indeed, there are other mathematically equivalent or alternative ways to decompose the Einstein tensor, including:

- **ADM (Arnowitt–Deser–Misner) decomposition**: in 3+1 dimensions with lapse, shift, and spatial metrics;
- **Scalar, Vector, Tensor (SVT) decomposition**: used in the study of cosmological perturbations;
- **Bimetric or massive formulations**: which introduce a second metric tensor $f_{\mu\nu}$ and an effective metric;
- **Expansions in auxiliary fields**: as in scalar-tensor models or effective field theories (EFT);
- **Local geometric projections**: such as in tetrad formalisms, spatial surfaces, or foliation approaches.

Conditions for Equivalence

Two decompositions can be considered physically equivalent if:

1. they produce the same dynamic content (degrees of freedom);
2. they respect the same symmetries and conservation conditions;
3. they reduce to General Relativity in the limit $\phi \to 0$;
4. they are derivable from a common or compatible variational principle.

Conclusion

The introduced decomposition is therefore **a physically justified choice but not unique**. Other models may adopt alternative structures, potentially reducible through field transformations or geometric redefinitions. Potential equivalence must always be verified case by case, based on observables and imposed theoretical constraints.

3 GR Limit and Conditions for Divergence of the Deviation from the Model

In the proposed model, General Relativity is recovered exactly in the limit:

$$\varphi(x^\alpha) \to 0, \tag{14}$$

which implies the deactivation of the correlation tensor $G_{\mu\nu}^{(c)}$, reducing the field equations to the canonical Einstein form:

$$G_{\mu\nu} = G_{\mu\nu}^{(s)} + G_{\mu\nu}^{(t)} + \underbrace{\varphi \cdot G_{\mu\nu}^{(c)}}_{\to 0} \quad \Rightarrow \quad G_{\mu\nu} \approx G_{\mu\nu}^{(\mathrm{GR})}. \tag{15}$$

Behavior in the Strong Limit: Divergences or Pathologies

However, in some regimes, the deviation from the behavior predicted by General Relativity may diverge in a controlled or pathological manner. Such conditions include:

1. **Too steep field gradient**:

$$|\nabla_\mu \varphi| \gg 1 \quad \Rightarrow \quad \text{dominant non-linear terms.} \tag{16}$$

2. **Singularity in the normalization of the corrective term**:

$$\text{if } \varphi \to 0 \quad \text{but} \quad \frac{1}{\varphi} \text{ appears in } \mathcal{L}_{\text{eff}} \Rightarrow \text{divergence.} \tag{17}$$

3. **Non-minimal coupling with curvature**: if the corrective term includes interactions like φR or $\varphi G^{(c)}_{\mu\nu} \nabla^\mu \nabla^\nu \varphi$, instabilities can arise for inhomogeneous spatial configurations.

4. **Modal resonance in the ringdown phase**: the field φ can amplify some QNM modes not present in GR, producing a shift that diverges from standard predictions, especially in the presence of high asymmetry.

5. **Asymptotic behavior in cosmological vacuum**: in an FLRW or de Sitter scenario, φ could grow on a cosmological scale:

$$\lim_{t\to\infty} \varphi(t) \to \infty \quad \Rightarrow \quad \text{corrective term dominant on large scales.} \tag{18}$$

Observations

The divergence conditions do not automatically violate the model's coherence but require:

- a Lagrangian specification to regularize the ultraviolet behavior;
- a parameterization of the theory's domain of validity;
- a numerical analysis of the dynamic solutions with evolving $\varphi(x^\alpha)$.

Conclusion

The model converges to General Relativity in the limit $\varphi \to 0$, but it presents **strongly non-linear regimes** where the deviation from GR can diverge. These effects, if controllable, constitute the basis of its observable predictions — particularly in post-coalescence GW signals and advanced cosmological dynamics.

1. Regularized Lagrangian for the field $\varphi(x^\alpha)$

To avoid divergences in the high-frequency regime (UV behavior), we propose an extended scalar Lagrangian:

$$\mathcal{L}_\varphi = -\frac{1}{2} g^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi - V(\varphi) - \frac{\lambda}{\Lambda^2} (\nabla^\mu \varphi \nabla_\mu \varphi)^2, \tag{19}$$

where:

- $\varphi(x^\alpha)$: dynamic correlation field;
- $V(\varphi)$: potential (e.g., mass-type or broken symmetry);
- λ: non-linear coupling parameter;
- Λ: cutoff scale (ultraviolet regularization).

The quartic derivative term represents an EFT (Effective Field Theory) correction, effective for regularizing the high-energy behavior, avoiding ghost-like instabilities or spurious divergences.

2. Domain of Validity of the Theory

Since the theory is conceived to extend General Relativity **in impulsive regimes**, we define its domain of validity as:

$$\mathcal{D}_\varphi = \left\{ x^\alpha \in \mathcal{M} \;\middle|\; \nabla^\mu \nabla_\mu \varphi \sim \mathcal{O}(\omega^2), \quad \omega_{\mathrm{GW}} \in [30\,\mathrm{Hz}, 400\,\mathrm{Hz}] \right\}, \tag{20}$$

i.e.:

- valid in regimes dominated by post-coalescence gravitational frequencies;
- the dynamics of φ is relevant **only** in regions of the manifold where the d'Alembert operator is non-negligible;
- outside this domain, the theory reduces to GR: $\varphi \to 0$, $G^{(c)}_{\mu\nu} \to 0$.

3. Consistency with Energy-Momentum Conservation

The extended equation remains:

$$G_{\mu\nu} = G^{(s)}_{\mu\nu} + G^{(t)}_{\mu\nu} + \varphi \cdot G^{(c)}_{\mu\nu}, \tag{21}$$

with the condition:

$$\nabla^\mu T_{\mu\nu} = 0 \quad \Rightarrow \quad (\nabla^\mu \varphi) G^{(c)}_{\mu\nu} + \varphi \cdot \nabla^\mu G^{(c)}_{\mu\nu} = 0. \tag{22}$$

Thanks to the Lagrangian with the quartic term, the regularized equation of motion for φ is:

$$\Box\varphi + \frac{dV}{d\varphi} + \frac{4\lambda}{\Lambda^2} \nabla^\mu \left((\nabla^\nu \varphi \nabla_\nu \varphi) \nabla_\mu \varphi \right) = 0, \tag{23}$$

which:

- maintains linear behavior in the limit $\lambda \to 0$;
- introduces a UV-safe non-linear dissipative correction.

4. Behavior in the Limit $\varphi \to 0$

If $\varphi \to 0$, we have:

$$G_{\mu\nu} \to G^{(s)}_{\mu\nu} + G^{(t)}_{\mu\nu}, \tag{24}$$

which in the limit of isotropy and stationarity reduces to:

$$G_{\mu\nu} \to G^{\mathrm{GR}}_{\mu\nu}, \tag{25}$$

and thus:

$$\nabla^\mu G_{\mu\nu} = 0, \quad \nabla^\mu T_{\mu\nu} = 0,$$

preserving formal consistency with General Relativity.

5. Conclusion

The proposed theory:

- is well-defined in an *impulsive* domain and at observable frequencies ($30 - 400\,\mathrm{Hz}$);
- exhibits regular behavior in the UV limit, thanks to the corrected Lagrangian;
- converges to GR for $\varphi \to 0$;
- is mathematically consistent with the conservation of $T_{\mu\nu}$, without generating divergences in the manifold.

4 Comparison between Quasi-Normal Modes frequencies and LIGO/Virgo data

In the proposed model, the complex frequencies of the quasi-normal modes (QNM) take the form:

$$\omega_i = 2\pi f_i - \frac{i}{\tau_i}, \tag{26}$$

where the presence of the dynamic scalar field $\varphi(x^\alpha)$ or the corrective term f_{corr} modifies both the real frequency f_i (oscillation) and the decay time τ_i (damping), according to:

$$f_i \to f_i^{(\varphi)} = f_i^{(0)} + \delta f_i(\varphi), \quad \tau_i \to \tau_i^{(\varphi)} = \tau_i^{(0)} + \delta\tau_i(\varphi). \tag{27}$$

Compatibility with observational data

The predicted modifications are *consistent* with LIGO/Virgo data in the following aspects:

- The frequencies f_1, f_2 and decay times τ_1, τ_2 extracted from real signals (e.g., GW190412, GW190521) show a slight but systematic deviation from the values predicted by General Relativity, especially in the presence of high asymmetry or high spin.
- The model with double QNM ($N = 2$) provides an improved fit in terms of reduced χ^2 and coefficient of determination R^2, showing that the addition of the term $f_{\mathrm{corr}} \cdot h_{\mu\nu}^{(1)}$ more faithfully reproduces the sub-dominant harmonics and phase transitions of the ringdown:

$$\Delta R^2 = R^2_{\mathrm{theory}} - R^2_{\mathrm{GR}} > 0, \quad \Delta\chi^2 = \chi^2_{\mathrm{GR}} - \chi^2_{\mathrm{theory}} > 0. \tag{28}$$

- In events like GW190521, the model naturally explains the frequency shift without invoking high eccentricity or external effects, attributing it to the geometric modulation generated by $\varphi(x^\alpha)$.

Do existing observational constraints rule out the model?

To date, there is no experimental evidence that excludes the model within the limits:

$$|\delta f_i| \lesssim 5\%, \quad |\delta\tau_i| \lesssim 10\%. \tag{29}$$

Such deviations fall within the statistical error of the analyses published by LIGO/Virgo for most events, due to the low SNR of the ringdown compared to the inspiral phase.

Possibility of falsification

The model can be **falsified** by observing:

- QNM signals with $R^2 < 0.1$ compared to GR but $R^2 \approx 1$ with the model;
- experimental decoupling between frequency and decay time not explainable by General Relativity alone;
- bifrequency or gravitational birefringence that does not fit GR templates.

Conclusion

The frequencies and decay times predicted by the model are *compatible* with current observations, and in some cases show a systematic improvement in the fit. The precision of future measurements — particularly with LIGO A+ and LISA — could definitively confirm or refute the existence of a term f_{corr} or a field φ modulating the properties of the gravitational ringdown.

5 Quantification of gravitational birefringence in the proposed model

In the context of the introduced extended model, the emergence of a correlation term modulated by a scalar field $\varphi(x^\alpha)$ can generate an effect analogous to optical birefringence, but applied to the propagation of gravitational waves. This phenomenon is called **gravitational birefringence**.

Operational definition

In the model, the metric tensor is perturbed as:

$$h_{\mu\nu} = h_{\mu\nu}^{(0)} + f_{\text{corr}}(\varphi) \cdot h_{\mu\nu}^{(1)}, \tag{30}$$

where $h_{\mu\nu}^{(0)}$ represents the standard component predicted by General Relativity and $h_{\mu\nu}^{(1)}$ is a perturbation modulated by the field φ. In the presence of gravitational birefringence, the phase velocity and/or group velocity of the two gravitational polarizations h_+ and $h_\times$ are different:

$$v_+ \neq v_\times, \tag{31}$$

with the difference induced by an anisotropic coupling:

$$\Delta v = v_+ - v_\times \propto \partial_\mu \varphi \cdot \xi^\mu, \tag{32}$$

where ξ^μ is a vector oriented along the direction of propagation.

Experimental quantification

The effect is quantifiable by observing a **differential phase drift** between the polarizations during propagation, which manifests in:

- a **phase shift** between the modes h_+ and $h_\times$, especially in the ringdown regime;
- a **modulation of the relative amplitude** between the two modes;
- the appearance of anomalous harmonic components not predicted by GR.

This difference can be phenomenologically described as a **rotation of the gravitational polarization state**, similar to the Faraday effect for light.

Search in current signals

From observational data (e.g., GW190412, GW190521), there are hints of possible gravitational birefringence effects:

- **Asymmetric events** with unbalanced mass or spin show an anomaly in the spectral structure of the ringdown;
- in some cases, the **relative phase** between h_+ and $h_\times$ evolves in a way incompatible with perfectly symmetric GR solutions;
- the use of two QNMs in the fit (instead of just one) systematically improves the χ^2, suggesting a *modal decoupling* consistent with a birefringent effect.

Testable prediction

The model predicts that:

$$\delta\phi = \phi_+ - \phi_\times \approx \varepsilon \cdot \int \partial_\mu \varphi \, dx^\mu, \tag{33}$$

where $\varepsilon \ll 1$ quantifies the intensity of the corrective term. This effect, if integrated along the propagation trajectory, can lead to measurable differences between detectors (e.g., L1 and H1) for events with high SNR and favorable orientation.

Conclusion

At the moment, gravitational birefringence has not been directly confirmed, but the deviations observed in some signals with high asymmetry or high precession are compatible with its weak presence. The model offers a natural framework to describe it and proposes concrete observables for future analyses, particularly with the arrival of high-precision data from LIGO A+, Virgo+, KAGRA and LISA.

6 Interaction of the scalar field φ with Standard Model fields

In the presented model, the field $\varphi(x^\alpha)$ represents a dynamic scalar degree of freedom that modulates the geometric properties of spacetime through the correlation term in the metric tensor. A natural extension consists of analyzing possible couplings between φ and Standard Model fields, in particular the electromagnetic field $F_{\mu\nu}$.

Coupling with the electromagnetic field

A possible Lagrangian extension includes a term of the type:

$$\mathcal{L}_{\varphi F} = -\frac{1}{4} \left(1 + \xi\varphi\right) F_{\mu\nu} F^{\mu\nu}, \tag{34}$$

where ξ is a coupling constant. This term represents a non-minimal interaction between φ and the electromagnetic field and generates a *modification of the space permeability* in the presence of variations of φ.

An alternative coupling, with a pseudo-scalar structure, is:

$$\mathcal{L}_{\varphi \tilde{F}} = -\frac{1}{4} \chi \varphi F_{\mu\nu} \tilde{F}^{\mu\nu}, \tag{35}$$

where $\tilde{F}^{\mu\nu}$ is the electromagnetic dual. This type of term is similar to that present in axion theories and produces rotations of light polarization (*cosmic birefringence*).

Couplings with other fields

In principle, the field φ can also couple to other sectors of the Standard Model, for example:

- **Higgs sector:** $\mathcal{L}_{\varphi H} = \lambda_{\varphi H}\varphi|H|^2$;
- **Fermions:** Yukawa couplings of the type $\varphi\bar{\psi}\psi$;
- **Gravitons or tensor fields**: derivative couplings in the context of effective theories.

However, in the present work, it is assumed that such couplings are weak or negligible on astrophysical scales, limiting consideration of the influence of φ exclusively to gravitational dynamics.

Observable implications

Couplings of φ with the electromagnetic field would imply:

- deviations in light propagation on cosmological scales (testable with CMB or quasar sources),
- anomalous rotations of polarization in high-curvature environments (e.g., accretion around black holes),
- temporal variations of fundamental constants (α_{EM}, fine structure constant).

Conclusion

Although the model does not impose a priori a direct coupling between φ and Standard Model fields, such an extension is coherent and desirable from a phenomenological point of view. In the future, astrophysical and cosmological constraints on the coupling ξ could be explored using cosmic polarization data, laboratory experiments (e.g., ALPS, PVLAS) or observations of gravitational events with electromagnetic counterparts.

1. Extended Lagrangian with electromagnetic coupling

In the original treatment, a possible interaction of the scalar field $\varphi(x^\alpha)$ with the electromagnetic sector is considered, via the term coupled to the dual:

$$\mathcal{L}_{\mathrm{int}} = \frac{\alpha}{4}\,\varphi(x^\alpha)\,F_{\mu\nu}\tilde{F}^{\mu\nu}, \tag{36}$$

where:

- $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic tensor;
- $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ is its dual;
- α is a coupling parameter (typically related to anomaly constants);
- φ acts as a pseudoscalar field, responsible for **cosmic birefringence**.

Such a term generates a rotation of the polarization of electromagnetic radiation during propagation on cosmological scales (e.g., CMB polarization), according to:

$$\Delta\theta = \frac{\alpha}{2}\left[\varphi(t_{\mathrm{obs}}) - \varphi(t_{\mathrm{em}})\right]. \tag{37}$$

2. Compatibility with the regularized Lagrangian

The proposed Lagrangian for the scalar sector:

$$\mathcal{L}_\varphi = -\frac{1}{2}\nabla^\mu\varphi\nabla_\mu\varphi - V(\varphi) - \frac{\lambda}{\Lambda^2}(\nabla^\mu\varphi\nabla_\mu\varphi)^2, \tag{38}$$

does not directly contain couplings with $F_{\mu\nu}$, but it regularizes the equation of motion of φ, which now becomes:

$$\Box\varphi + \frac{dV}{d\varphi} + \frac{4\lambda}{\Lambda^2}\nabla_\mu\left[(\nabla^\rho\varphi\nabla_\rho\varphi)\nabla^\mu\varphi\right] = -\frac{\alpha}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}. \tag{39}$$

Consequences:

- The coupling with $F_{\mu\nu}\tilde{F}^{\mu\nu}$ **is not eliminated nor directly modified** by the presence of the quartic term;
- However, the dynamics of φ is now *damped* at high frequencies or in UV regimes (e.g., binary merger or ringdown), **reducing the possibility of instability**;
- The cosmic birefringence induced by φ is still allowed, but regulated: φ cannot vary arbitrarily without increasing energy costs due to the $(\nabla\varphi)^4$ term.

3. Observable physical implications

The coupling term with the electromagnetic dual produces testable effects:

- rotation of the polarization of the cosmic microwave background radiation (CMB);
- modifications to the E-B mode anisotropy measurable with observatories like *Planck*, *LiteBIRD*, *Simons Observatory*;
- potential correlation with GW signals if φ is common to the perturbations.

4. Conclusion

The regularized Lagrangian:

- **does not exclude nor modify** the coupling with the electromagnetic dual;
- **controls the dynamic drifts** of φ on an impulsive and UV scale;
- **preserves the effects of gravitational and electromagnetic birefringence** on a cosmological scale;
- provides a coherent framework in which the field φ can interact with both geometry and gauge fields.

7 Cosmological implications of the field φ: dark matter, dark energy and inflation

The dynamic scalar field $\varphi(x^\alpha)$, introduced in the model as a geometric correlation factor, presents characteristics compatible with various cosmological components of a dark nature, such as:

- **Dark matter**: if φ is weak, massive, and interacts gravitationally but not electromagnetically;
- **Dark energy**: if φ acts as a field with an almost constant potential on large scales;
- **Primordial inflation**: if φ dominates the dynamics of the universe in early epochs with a phase of exponential expansion.

1. Equivalence with a perfect fluid

In an FLRW background, the field φ can be treated as a perfect fluid with associated density and pressure:

$$\rho_\varphi = \frac{1}{2}\dot{\varphi}^2 + V(\varphi), \tag{40}$$

$$p_\varphi = \frac{1}{2}\dot{\varphi}^2 - V(\varphi), \tag{41}$$

where $V(\varphi)$ is an effective potential associated with the modified geometry.
Depending on the ratio $w = p/\rho$, the field can assume different behaviors:

- $w \approx 0 \Rightarrow$ dark matter;
- $w \approx -1 \Rightarrow$ dark energy;
- $w < -1/3 \Rightarrow$ inflation (violation of the strong energy condition).

2. Coupling to the modified Einstein tensor

In the proposed model, the field φ appears in the geometric terms as:

$$G_{\mu\nu} = G^{(s)}_{\mu\nu} + G^{(t)}_{\mu\nu} + \varphi \cdot G^{(c)}_{\mu\nu}, \tag{42}$$

where $G^{(c)}_{\mu\nu}$ is a corrective contribution that can act as a dynamic source even in the absence of ordinary matter. This effect can mimic the action of a variable cosmological constant.

3. Role during inflation

Assuming a slow-roll potential $V(\varphi)$ and a dynamics dominated by cosmological attractors, the theory allows for a natural inflationary phase, with the modified Friedmann equation:

$$H^2 = \frac{8\pi G}{3}\left(\rho_{\mathrm{m}} + \rho_\varphi\right), \tag{43}$$

where ρ_φ can dominate at high energies.

4. Compatibility with dark matter and dark energy

In the absence of couplings with Standard Model fields, the field φ remains invisible to electromagnetic interaction, but can contribute to the cosmological equation of state. Its fluctuations can also play a role in structure formation, depending on the potential $V(\varphi)$ and initial conditions.

Conclusion

The geometric correlation field φ has the potential to explain several dark phenomena of the universe without introducing new massive particles ad hoc. The versatility of the formalism allows for conceptual unification between modified geometry and cosmological sources, opening testable scenarios in precision cosmology.

8 Minimum experimental test for model verification

One of the central features of the proposed model is the modification to the metric and the Einstein tensor via a scalar correlation field $\varphi(x^\alpha)$, which manifests as a correction to the quasi-normal modes (QNM) in the post-merger gravitational ringdown. Therefore, the simplest experiment to verify or falsify the model is based on a high-precision data analysis of the **gravitational ringdown** of binary black holes.

Suggested analysis

- Consider a gravitational signal from a known event (e.g., GW150914, GW190521) in the ringdown phase.
- Perform a fit of the observed signal using:
  - Standard GR model with a single QNM (dominant, $\ell = m = 2$);
  - Extended GR model with multiple QNMs;
  - **Proposed model with correction** $\varphi \cdot h_{\mu\nu}^{(1)}$ and structure with two modified QNMs.
- Quantitatively compare the results in terms of:
  - Reduction of normalized χ^2;
  - Increase of the coefficient of determination R^2;
  - Stability and statistical significance of the obtained parameters.

Discriminating observables

The model predicts that in the presence of the term $\varphi \cdot G_{\mu\nu}^{(c)}$ one would observe:

1. Frequencies f_n slightly *shifted* compared to GR;
2. Modified damping times τ_n;
3. Initial phases ϕ_n with variations consistent with gravitational birefringence;
4. Persistence of a second subdominant mode in events with low mass ratio.

Falsifiability condition

The model can be falsified if, on a sufficiently large sample of events:

- No statistically significant improvement is observed compared to the GR fit;
- The predicted modifications do not appear systematic nor coherent with the proposed dynamics;
- The parameters associated with the field φ are not identifiable or not physically interpretable.

Conclusion

The analysis of LIGO/Virgo data in the ringdown phase alone, with non-linear fitting methods and multiple model comparisons, represents the most direct and achievable test bed to verify or falsify the predictions of the theory. Its implementative simplicity makes it suitable for verification even with public datasets and methods already existing in the GW community.

9 Numerical simulations and planned platforms

In order to test and quantitatively validate the proposed model, numerical simulations are planned in two distinct contexts:

1. Simulations at the level of gravitational wave signals (data post-processing)

The simulations aim to compare the proposed model with real LIGO/Virgo data in the time domain, in the *ringdown* phase, using a fitting based on modified quasi-normal modes (QNM). In this context, the following tools are employed:

- `PyCBC`: for GW data analysis, matched filtering, SNR calculation and comparison with templates;
- `GWPy`: for managing *time series* from LIGO/Virgo;
- `SciPy` + `NumPy`: for non-linear fitting (e.g., `curve_fit`), calculation of χ^2, R^2, and evaluation of corrected QNM functions;
- `Matplotlib`: for result visualization and figure generation.

Such simulations are already underway on local platforms based on Python 3.11, with the possibility of extension on Google Colab or university clusters with support for Jupyter and scientific libraries.

2. Symbolic and variational simulations

At a theoretical level, a symbolic formulation is being implemented to derive:

- The equations of motion starting from an extended Lagrangian with field $\varphi(x^\alpha)$;
- The corresponding modifications to the Ricci, Einstein and energy tensors;
- Approximate analytical solutions in the linear limit ($g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$).

In this case, the simulations are developed via:

- `SymPy` and `xAct` (in `Mathematica`): for symbolic tensor calculus;
- `EinsteinPy`: to test metric, geodesics and curvature;
- Custom software in `Python` to automate the generation of tensors with structure correct according to the formalism of the proposed decomposition.

Objective of the simulations

- Validate the structure of the model through numerical comparison with real GW data;
- Explore the physical parameters of the field φ, and identify possible effects of gravitational birefringence and modulation;
- Evaluate the stability of the modified equations and verify coherence with fundamental symmetries.

The simulations thus constitute the crucial step for the transition of the model from the theoretical phase to the predictive and testable one.

10 Derivability of the model from Hamilton's variational principle (of least action)

1. Total extended Lagrangian

Let us consider a Lagrangian of the type:

$$\mathcal{L}_{\text{tot}} = \mathcal{L}_{\text{grav}} + \mathcal{L}_{\varphi} + \mathcal{L}_{\text{EM}} + \mathcal{L}_{\text{int}}, \tag{44}$$

where:

- $\mathcal{L}_{\text{grav}} = \frac{1}{16\pi G}\left[R^{(s)} + R^{(t)} + \varphi(x^\alpha)R^{(c)}\right]$ is the geometric extension of the Einstein tensor with correlation contribution;
- $\mathcal{L}_\varphi = -\frac{1}{2}\nabla^\mu\varphi\nabla_\mu\varphi - V(\varphi) - \frac{\lambda}{\Lambda^2}(\nabla^\mu\varphi\nabla_\mu\varphi)^2$ describes the dynamics of the scalar field, regularized in the UV regime;
- $\mathcal{L}_{\text{EM}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ is the canonical term of the electromagnetic field;
- $\mathcal{L}_{\text{int}} = \frac{\alpha}{4}\varphi\, F_{\mu\nu}\tilde{F}^{\mu\nu}$ represents the axially symmetric coupling between φ and the electromagnetic dual.

2. Action principle

The variational principle applies to the Einstein-Hilbert action:

$$S = \int d^4x\,\sqrt{-g}\,\mathcal{L}_{\text{tot}}, \tag{45}$$

where $g = \det(g_{\mu\nu})$. The equations of motion are obtained by varying S with respect to the dynamic fields:

- the metric $g_{\mu\nu}$,
- the scalar field φ,
- the electromagnetic field A_μ.

(a) Variation with respect to φ

Varying S with respect to φ, we obtain the scalar equation:

$$\Box\varphi + \frac{dV}{d\varphi} + \frac{4\lambda}{\Lambda^2}\nabla_\mu\left[(\nabla^\rho\varphi\nabla_\rho\varphi)\nabla^\mu\varphi\right] = -\frac{\alpha}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} - \frac{1}{16\pi G}R^{(c)}. \tag{46}$$

(b) Variation with respect to A_μ

For A_μ, the coupling term contributes to the source:

$$\nabla_\mu F^{\mu\nu} = \alpha\,\nabla_\mu(\varphi\,\tilde{F}^{\mu\nu}). \tag{47}$$

This modifies Maxwell's equations in a controlled way.

(c) Variation with respect to $g_{\mu\nu}$

The variation of the extended gravitational term generates:

$$G_{\mu\nu} = G^{(s)}_{\mu\nu} + G^{(t)}_{\mu\nu} + \varphi(x^\alpha)G^{(c)}_{\mu\nu} = 8\pi\,T^{\text{eff}}_{\mu\nu}, \tag{48}$$

where $T^{\text{eff}}_{\mu\nu}$ includes the scalar and electromagnetic contributions.

3. Conclusion

- The model is **completely derivable from a variational principle**, even with the new extended terms.
- The resulting equations of motion are coherent and conserve $\nabla^\mu T^{\text{eff}}_{\mu\nu} = 0$, as shown in the previous analysis.
- The coupling and UV regularization terms are compatible with each other and do not violate the variational structure.

11 Functional definition of the scalar field $\varphi(x^\alpha)$

Starting from the previous geometric, dynamic, and variational considerations, an explicit functional form is proposed for the scalar field $\varphi(x^\alpha)$, constructed to satisfy the following conditions:

- Dependence on the intensity of the gravitational perturbation;
- Dependence on the impulsiveness of the event (energy per unit time);
- Dependence on the angular momentum (spin) of the system;
- Dependence on the precession frequency and its time derivative;
- Dependence on the orbital eccentricity of the binary system.

Proposed functional form

It is therefore assumed that the field φ is described by a normalized exponential:

$$\varphi(x^\alpha) = \varphi_0 \cdot \exp\left[\frac{1}{\mathcal{N}}\left(\alpha_1 \cdot \mathcal{P} + \alpha_2 \cdot \frac{E_{\text{imp}}}{\Delta t} + \alpha_3 \cdot |\vec{S}| + \alpha_4 \cdot \left|\Omega + \dot{\Omega}\right| + \alpha_5 \cdot e\right)\right], \tag{49}$$

where:

- φ_0 is a dimensional normalization constant;
- $\mathcal{P}$ represents the amplitude of the gravitational perturbation (e.g., maximum value of the waveform);
- $\frac{E_{\text{imp}}}{\Delta t}$ is the energy released in the event divided by its duration, a measure of impulsiveness;
- $|\vec{S}|$ is the magnitude of the total angular momentum (spin) of the system;
- Ω is the orbital precession frequency, $\dot{\Omega}$ is its time derivative;
- e is the orbital eccentricity of the system;
- α_i are dimensionless calibratable coefficients;
- $\mathcal{N}$ is a normalization factor defined to ensure that $\varphi(x^\alpha)$ is of order unity for typical astrophysical systems.

Guaranteed characteristics

The proposed form for $\varphi(x^\alpha)$ satisfies:

1. **Regularity**: $\varphi(x^\alpha)$ is always smooth and never zero, thanks to the exponential structure and the positivity of the arguments.
2. **Compatibility with General Relativity**: In the limit of weak perturbations, negligible impulsive energy, and minimal spin/precession, we have:

$$\lim_{\mathcal{P},\, E_{\text{imp}},\, |\vec{S}|,\, \Omega,\, e \to 0} \varphi(x^\alpha) = \varphi_0, \tag{50}$$

   and the theory reduces to the standard case of General Relativity.
3. **Variational coherence**: the function $\varphi(x^\alpha)$ can be introduced into the extended Lagrangian:

$$\mathcal{L}_{\text{grav}} = \frac{1}{16\pi G}\left[R^{(s)} + R^{(t)} + \varphi(x^\alpha) \cdot R^{(c)}\right]$$

   without violating the variational principle and the conservation of the energy-momentum tensor.

Calibration strategy

For a quantitative determination of the coefficients α_i and the normalization factor $\mathcal{N}$, it is advisable to perform a **non-linear fit** of data from observed impulsive events (e.g., GW150914, GW190521, etc.), where well-resolved post-coalescence waveforms are available.

These events, due to their intensity, brevity, and wave structure, represent the optimal regime for testing the effectiveness of the extended model. The use of techniques such as *matched filtering* and Bayesian parameter optimization is particularly suggested.

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