Alpha

The Fine-Structure Constant

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In this paper, I derive the fine-structure constant α as an emergent and parameter-free invariant of Relator theory. The construction uses the Schrödinger equation, classical electrodynamics and mechanics, and Relator geometry with luminal evolution on the $\mathbb{C}$-space and the orthogonal spatial sector $\mathbb{R}^3$, which fixes a minimal phase cavity and the Coulombic baseline. A transverse traceless inductive channel is incorporated through a UV, IR, and OUT decomposition, and a logarithmic closure uniquely determines α with no fitted numbers and with no use of measured constants e, c, and $\hbar$. No quantum electrodynamics is invoked at any stage. The prediction matches the most precise independent measurements at sub-ppt accuracy and remains stable under regulator choices and multipole cutoffs.

I. INTRODUCTION

Nature's geometry? From the Pythagoreans' hymn to number to Feynman's "1/137," physics has carried a suspicion that a single ratio stitches disparate phenomena together. Sommerfeld introduced the fine–structure constant as a universal coupling,[1] Eddington dared that it should be a pure number,[2] Born framed it as the hidden governor of atomic detail,[3] and Dirac argued that dimensionless combinations like α must be explained by structure rather than units.[4] The riddle endured, acquiring almost mythic overtones—an Ariadne's thread promised but never found.

In quantum physics, the fine-structure constant α appears almost everywhere, yet its origin remains arguably the field's most stubborn mystery—after a century of attempts no first-principles derivation has predicted its value even at the percent level; the rare multi-decimal matches have come from numerology or ad-hoc parameter tuning rather than a physical explanation.

In this paper I take that challenge literally. I show that α is *emergent and parameter–free*: its value follows from a purely geometric, gauge–invariant construction rooted in the Relator postulate $R\omega = c$ (luminal internal evolution on $\mathbb{C}$ orthogonal to spatial winding in $\mathbb{R}^3$). No measured dimensional constants are invoked—no e, no c, no $\hbar$—and no fitted numbers appear. A closed root condition fixes α by locking a Coulombic shell functional $\mathcal{D}_{\mathrm{C}}$ to the vector–inductive sector through a universal map,

$$C_{\log} \;\equiv\; \frac{\pi^2}{\mathcal{D}_{\mathrm{C}}}\,\zeta(1+\zeta) \;=\; \frac{1}{3}, \qquad \zeta = \frac{K}{2\pi^2}\Lambda,$$

so that the electromagnetic coupling is set by geometry alone. The construction yields rigid, dimensionless ratios between the Coulomb and Λ–channel sectors; these geometric invariants, not empirical inputs, pin down α.

The same mechanism unifies how "time" flows for quantum phases [5]. In a companion analysis, the electron g–factor appears as an *evolution–rate shift* of the phase clock induced by the large–$\mathcal{D}$ functional on the matching shell—precisely analogous to time dilation in GR —whether momentum or gravity induced— now for the Coulomb field predicted by the Relator [5, 6]. Thus the Relator framework does more than produce a number; it provides a single geometric origin for coupling and for evolution–rate renormalization, turning the century–old riddle of α into a calculable constant and opening a concrete path toward band–like stability structures for leptons within a background–free, gauge–invariant setting.[7]

Our closed pipeline predicts an emergent value $\alpha_{\mathrm{pred}} = 0.007\,297\,352\,564\,326$, agreeing with CODATA 2022 $\alpha = 7.297\,352\,5643(11) \times 10^{-3}$ to 2.16 ppt ($z = 0.0146\,\sigma$), thereby reproducing all certain published digits and predicting subsequent ones.

The numerical outcome—as shown—emerges from a deliberately minimalist formal and computational pathway. While a small background risk of bias toward *overfitting* can never be fully excluded, the relations employed here are grounded in physically meaningful structure and rigorous mathematics rather than ad hoc symbol–play. In principle, the final equation for α can be compressed into a more compact form, but such a reduction strips away its physical content—which I do not advocate.

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Code: All numerical evaluations are fully reproducible from independent scripts in the companion repository — `github.com/pajuhaan/AlphaEmergent`.

II. BACKGROUND

We begin with the time–dependent Schrödinger equation

$$i\hbar\,\partial_t\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + V\,\psi, \qquad \psi(x,t) = R(x,t)\,e^{iS(x,t)/\hbar}. \tag{1}$$

written in Madelung form to expose the phase field S. Under minimal coupling, $\mathbf{A}\to\mathbf{A}+\nabla\chi,\ \phi\to\phi-\frac{1}{c}\partial_t\chi,\ \psi\to e^{iq\chi/(\hbar c)}\psi$, the phase shifts as $S\to S+\frac{q}{c}\chi$, so the *mechanical* momentum

$$\boldsymbol{p}_{\rm mech} = \nabla S - \frac{q}{c}\,\mathbf{A} \tag{2}$$

is gauge–invariant. This is the unique combination of fields and phase that will enter our construction.

The Relator postulate asserts a luminal internal kinematics,

$$R\,\omega = c, \tag{3}$$

with an orthogonal split of the total phase rate into a timelike rotation in the internal $\mathbb{C}$–space and a spatial winding in $\mathbb{R}^3$.

With orthogonal splitting of the internal frequency [5]

$$\begin{cases} \omega_{\mathbb{C}}(\tau) := -\dfrac{1}{\hbar}\,\dfrac{d}{d\tau}\,S(\tau) \ \equiv\ \dfrac{mc^2}{\hbar}\Big|_{t=\tau}, & \text{(phase rotation in } \mathbb{C}\text{)},\\[2ex] \boldsymbol{\omega}_{\mathbb{R}^3}(x,t) := \dfrac{1}{m\,R}\,\boldsymbol{\nabla} S_*(x,t) \ =\ \dfrac{1}{m\,R}\,\underbrace{\Big(\boldsymbol{\nabla} S(x,t) - \dfrac{q}{c}\,\mathbf{A}(x,t)\Big)}_{\text{gauge formalism}}, & \text{(spatial winding in } \mathbb{R}^3\text{)}. \end{cases} \tag{4}$$

Denoting these by $\omega_{\mathbb{C}}$ and $\omega_{\mathbb{R}^3}$, respectively, we have

$$\omega^2 = \omega_{\mathbb{C}}^2 + \omega_{\mathbb{R}^3}^2, \qquad \text{and} \qquad \omega_{\mathbb{R}^3} \text{ is fixed by } \boldsymbol{p}_{\rm mech}, \qquad \omega_{\mathbb{R}^3} = \|\boldsymbol{\omega}_{\mathbb{R}^3}\|. \tag{5}$$

At the level of *ratios*, Eq. (3) eliminates the length scale R: the dimensionless spatial–to–total rate reduces to a pure, gauge–invariant quotient of mechanical momentum and the relativistic scale mc. We therefore introduce the master ratio and its square

$$\Omega_{\mathbb{R}^3} \ \equiv\ \frac{\omega_{\mathbb{R}^3}}{\omega} \ =\ \frac{\big\|\nabla S - \frac{q}{c}\mathbf{A}\big\|}{m\,c}, \qquad \mathcal{D} \ \equiv\ \Omega_{\mathbb{R}^3}^2 \ =\ \frac{\big\|\nabla S - \frac{q}{c}\mathbf{A}\big\|^2}{m^2c^2}\ . \tag{6}$$

All emergent effects will be expressed through $\mathcal{D}$. We henceforth refer to $\mathcal{D}$ as the *Particle–Winding Invariant*: the dimensionless measure of spatial winding relative to the total ω of the Relator. For later use we reserve the notation $\mathcal{D}_{\rm C}$ for the *Coulombic (scalar) baseline* contribution to $\mathcal{D}$, and $\Delta\mathcal{D}_\Lambda$ for the *inductive (vector)* correction obtained. Collecting terms, we obtain

$$\mathcal{D} \ =\ \mathcal{D}_{\rm C} \ +\ \Delta\mathcal{D}_\Lambda \ +\ \mathcal{O}\big(\Delta\mathcal{D}_\Lambda^2\big)\ . \tag{7}$$

In the Relator theory the wavefunction evolves *luminally* on the combined $\mathbb{C}\oplus\mathbb{R}^3$ kinematic split— there exists no intrinsic speed other than c. The postulate $R\omega = c$ fixes the internal rate, and the orthogonal decomposition $\omega^2 = \omega_{\mathbb{C}}^2 + \omega_{\mathbb{R}^3}^2$ ensures that the total evolution speed is always c [5]. Operationally, the $\mathbb{C}$-plane hosts a *luminal swirl* with

$$v_\theta(r) = c, \qquad \omega(r) = \frac{c}{r}. \tag{8}$$

Mass is defined as a coarse-grained continuum of energy dots—we call them *Loopon* (photon-like quanta) on $\mathbb{C}$-space that circulate tangentially at speed c [5]. The maximum-entropy coarse-graining gives the circular Gaussian

$$\rho_m(r) = \frac{m}{\pi\sigma_{\mathbb{C}}^2}\,e^{-r^2/\sigma_{\mathbb{C}}^2}, \qquad \int_0^\infty 2\pi r\,\rho_m(r)\,dr = m, \qquad \sigma_{\mathbb{C}} = \varepsilon R = \frac{R}{\sqrt{\pi}}, \qquad \varepsilon = \frac{1}{\sqrt{\pi}}. \tag{9}$$

while electric charge resides on a thin Dirac ring at the Relator radius R,

$$dq = e\,\delta(r-R)\,dr, \qquad I = \frac{e\,\omega}{2\pi} = \frac{e\,c}{2\pi R}. \tag{10}$$

This minimal ($\mathbb{C}$-Gaussian mass sheet + ring charge at R) structure uniquely reproduces the classical limits for spin and magnetic moment:

$$L_{\mathbb{C}} = \int_0^\infty r\,c\,dm = \frac{1}{2}\,mcR \overset{R=\hbar/(mc)}{=} \frac{\hbar}{2}, \qquad \mu_{\mathbb{C}} = \frac{ecR}{2} \overset{R=\hbar/(mc)}{=} \frac{e\hbar}{2m}, \quad \Rightarrow \quad \frac{\mu_{\mathbb{C}}}{L_{\mathbb{C}}} = \frac{e}{m} \implies g = 2. \tag{11}$$

We therefore adopt (8)–(11) as the physical content used in all vector-channel (Λ-chain) calculations.

III. METHODOLOGY

Imagine approaching an electron from astronomical distances; at first it presents itself as a pointlike source with Coulomb and magnetostatic fields; closer in, at a matching sphere—the "shell", classical field theory runs out of room, and if we insist on crossing the shell, even the usual quantum descriptors—charge, spin, mass—cease to be primary objects. Past cures like uniform-charge-clouds and extended-profiles, conflict with the electron's spin and magnetic moment; extra spatial dimensions only displace the paradox. The Relator move is different; we keep physical space $\mathbb{R}^3$ exactly as is, but introduce a generator space $\mathbb{C}$-space that carries the *source*, while $\mathbb{R}^3$ becomes the *propagation* arena. Once the origin is anchored in $\mathbb{C}$-space, we can pass through the shell without contradiction. The observed attributes; fields, spin, effective mass appear as emergent maps from $\mathbb{C}$ into $\mathbb{R}^3$. The analysis below formalizes this journey—crossing the shell of a fundamental particle—via a closed fixed–point construction for α and its geometric blocks.

In the Relator framework $\mathbb{C}$ is the background space—the generative phase hardware of reality—while $\mathbb{R}^3$ is the exposed measurement space. Information is mirrored between them; what is resolved in $\mathbb{R}^3$ reflects in $\mathbb{C}$, and vice versa. We use ordinary *classical* field equations throughout; the classical/quantum divide dissolves under the single postulate $R\,\omega = c$ [8].

Now we compute the gauge–invariant ratio of rates $\Omega_{\mathbb{R}^3} = \omega_{\mathbb{R}^3}/\omega$ through the scalar phase–gradient and the vector Λ–channel, using the single invariant

$$\mathcal{D} \;=\; \frac{\left\|\nabla S - \frac{q}{c}\mathbf{A}\right\|^2}{m^2c^2}. \tag{12}$$

On the matching shell $r = r_*$ this decomposes as

$$\mathcal{D} = \underbrace{\frac{\|\nabla S\|^2}{m^2c^2}}_{\mathcal{D}_{\mathrm{C}}} + \underbrace{\frac{q^2\,\langle A_\parallel^2\rangle}{m^2c^4}}_{\mathcal{D}_A} - \underbrace{\frac{2q}{m^2c^3}\,\big\langle(\nabla S)\cdot\mathbf{A}_\parallel\big\rangle}_{\mathcal{D}_{\mathrm{cross}}}, \tag{13}$$

where $\mathbf{A}_\parallel$ is the *tangential* vector potential on the shell, fixed by the tangential boundary condition. On the shell, the mechanical momentum expands as $\|\nabla S - \frac{q}{c}\mathbf{A}\|^2 = \|\nabla S\|^2 + \frac{q^2}{c^2}\|\mathbf{A}_\parallel\|^2 - \frac{2q}{c}(\nabla S)\cdot\mathbf{A}_\parallel$, with $\mathbf{A}$ kept *tangential* by the shell projector. Angle brackets $\langle\cdot\rangle$ denote the shell/near average (with solid–angle average on $r = r_*$ and radial weight $|u_0(r)|^2$ normalized in (23)). In practice, the phase–locking at the shell moves the mixed term into the Λ–channel bookkeeping as $\Delta\Lambda_{\mathrm{sync}}$ (Sec. V D); its leading shell average does not survive independently of Λ_{eff}.

We partition space as

$$\mathbb{R}^3 = \{\, r < r_* \,\} \;\cup\; \{\, r = r_* \,\} \;\cup\; \{\, r > r_* \,\}, \tag{14}$$

interpreted as near field, matching shell, and far field. The shell is a natural gauge–invariant interface where the interior phase mode is pinned to the exterior field. In the near field we *model* the phase by the stationary scalar Helmholtz equation with s–wave j_0 (Ansatz consistent with $R\,\omega = c$), and the Dirichlet pinning $S(r_*) = 0$ selects the first node,

$$k\,r_* = \pi, \qquad k = \frac{\omega_{\mathbb{C}}}{c} = \frac{1}{\bar{\lambda}_C} \quad\implies\quad r_* = \pi\,\bar{\lambda}_C\,. \tag{15}$$

Introduce $\eta := \bar{\lambda}_C/r_* = 1/\pi$ and take all shell averages in units of r_*. With $k = 1/\bar{\lambda}_C$ and the $j_0(kr)$ profile, the ratio $\|\nabla S\|/(mc)$ reduces to a function of η only, and magnetic weights enter through the dimensionless Λ. Hence m and $\bar{\lambda}_C$ cancel out of $\mathcal{D}_{\mathrm{C}}$ and the inductive correction $\Delta\mathcal{D}_\Lambda$, leaving purely geometric, dimensionless expressions.

Having fixed the geometry, we proceed in two stacks:

(i) *Coulombic (phase–gradient) stack* $\mathcal{D}_\mathrm{C}(\alpha)$; its exterior ($r > r_*$) contribution equals the factor α/π *in our shell normalization* (from the energy integral of $|\nabla S|$ under (12) and (21)); a uniform near/shell average introduces the dimensionless weight $\xi = 2\alpha\, C_0^{\mathrm{uni}}$; departures from uniformity are encoded by an even–power spectral series with geometry–only coefficients K and L_{2m}. Altogether this yields a closed, geometry–only expression for $\mathcal{D}_\mathrm{C}(\alpha)$, as shown in Sec. IV.

(ii) *Self–magnetic* (Λ*–channel)*; the electron's magnetic dipole is represented by a Dirac ring whose filamentary baseline is the inductive logarithm $\Lambda_{\mathrm{ind}} = \ln(8R/\sigma_{\mathbb{C}}) - 2$ with $\sigma_{\mathbb{C}} = R/\sqrt{\pi}$. A finite Gaussian collar supplies the universal UV increment $c_0^{\mathrm{Gauss}} = \frac{1}{2}(\ln 2 + \gamma)$ together with the IR projector P_{IR}; the free–space exterior is removed by $\Delta\Lambda_{\mathrm{OUT}}$; a small Coulomb–coupled phase locking on the shell enters as $\Delta\Lambda_{\mathrm{sync}}$; and polarization of Loopons on $\mathbb{C}$ contributes a bounded ladder term $\Delta\Lambda^{(\chi)}_{\mathrm{pol}}$. A dynamic remainder $\Delta\Lambda_{\mathrm{dyn}} = O(\zeta^2)$ keeps higher–order magnetic dynamics in the budget. The resulting effective sum Λ_{eff} defines the geometric weight $\zeta = (K/2\pi^2)\,\Lambda_{\mathrm{eff}}$ that drives the inductive contribution $\Delta\mathcal{D}_\Lambda$, as shown in Sec. V.

Crucially, every ingredient above is *dimensionless* and geometric. With the normalization after (21), and using the kinematics $R\omega = c$, the explicit m and the Compton scale cancel out of both $\mathcal{D}_\mathrm{C}$ and $\Delta\mathcal{D}_\Lambda$, leaving mass–independent, geometry–only formulas. The two stacks are then coupled only by a single scalar condition,

$$C_{\log}(\mathcal{D}_\mathrm{C};\zeta) \;=\; \frac{\pi^2}{\mathcal{D}_\mathrm{C}}\,\zeta\,(1+\zeta) \;=\; \frac{1}{3}\,, \tag{16}$$

which fixes α with no fitted numbers and no use of measured dimensional constants in the Sec. VI. The constant 1/3 is a Relator–level *dynamical lock*; as the analytic treatment of higher orders in the two stacks is refined, the solution for α approaches this closure. The same logarithmic structure echoes, at the level of form, the well–known lepton–mass–dependent logarithms in perturbative QED; here, however, the lock arises *independently of QED* from Relator geometry and the shell projection. The upshot is twofold: (a) α is derived as a pure number from invariant geometry, and (b) the same mechanism exhibits a stability property for the particle (charge/mass) consistent with a dynamically sustained lock at 1/3.

IV. COULOMBIC BASELINE

A stationary internal circulation projects to a fixed phase on the shell; we impose the Dirichlet pinning

$$S(r_*) = 0, \tag{17}$$

which enforces the minimal closure that balances the interior phase–gradient energy with the outward Coulomb energy at the shell. In the quasi–static near field we take the phase to satisfy the scalar Helmholtz equation with the s–wave radial mode

$$\nabla^2 S + k^2 S = 0, \qquad S(r) \propto j_0(kr) = \frac{\sin(kr)}{kr}. \tag{18}$$

The boundary condition (17) requires $j_0(kr_*) = 0$. The first zero gives

$$k\,r_* = \pi. \tag{19}$$

The internal clock fixes the wavenumber by the Relator kinematics $R\omega = c$:

$$k = \frac{\omega_{\mathbb{C}}}{c} = \frac{1}{\bar{\lambda}_C}, \tag{20}$$

i.e. one internal wavelength across the reduced Compton length. Combining (19) and (20) yields

$$r_* = \pi\,\bar{\lambda}_C\,. \tag{21}$$

This is the minimal (first–node) choice, independent of α, and it produces the smallest cavity consistent with a single–node closure. Higher–node options $kr_* = n\pi$ ($n = 2, 3, \dots$) enlarge the cavity, increase the interior phase–gradient energy, and do not improve the far–field match.

Uniform self–charge in the near field. Inside the cavity ($0 \le r \le r_*$) we use the electrostatic potential of a uniformly charged sphere; this is the baseline model that feeds the s–wave:

$$V(r) = \frac{e}{8\pi\varepsilon_0\, r_*}\left(3 - \frac{r^2}{r_*^2}\right), \qquad 0 \le r \le r_*. \tag{22}$$

s–wave ground mode and normalization. The interior phase is taken as the ground s–wave of the scalar Helmholtz problem with a node at r_*; the mode and its normalization are

$$u_0(r) = \mathcal{N}\, j_0(kr), \qquad k = \frac{\pi}{r_*}, \qquad 4\pi \int_0^{r_*} r^2\, |u_0(r)|^2\, dr = 1. \tag{23}$$

This supplies the weight with which the near/shell potential (22) is sampled via $|u_0(r)|^2$.
Near/shell energy (uniform part). Averaging the potential over the s–wave gives the near/shell energy and the associated dimensionless weight entering $\mathcal{D}_\mathrm{C}$:

$$U_{\text{near-shell}}(r_*) := \int_0^{r_*} 4\pi r^2\, |u_0(r)|^2\, V(r)\, dr = \langle V \rangle = \alpha\, mc^2\, C_0^{\mathrm{uni}}, \qquad C_0^{\mathrm{uni}} = \frac{1}{\pi}\left(\frac{4}{3} + \frac{1}{4\pi^2}\right), \tag{24}$$

$$\xi = \frac{2\, U_{\text{near-shell}}}{mc^2} = 2\alpha\, C_0^{\mathrm{uni}}. \tag{25}$$

Far–field share. The Coulomb energy stored *outside* the cavity fixes the baseline scalar contribution:

$$U_{\mathrm{far}}(r_*) = \int_{r_*}^{\infty} \frac{\varepsilon_0 E^2}{2}\, dV = \frac{\alpha\, \hbar c}{2\, r_*} \quad \Longrightarrow \quad \mathcal{D}_\mathrm{C}^{(0)} = \frac{2U_{\mathrm{far}}}{mc^2} = \frac{\alpha}{\pi}. \tag{26}$$

This "far" piece sets the overall scalar scale in our shell normalization.
Inhomogeneity inside r_* (even–power spectral series). Deviations from uniformity in the near field mix the ground s–mode with higher cavity modes. With the dimensionless radius $x = r/r_* \in [0,1]$,

$$I_n = \int_0^1 x^2 \sin(\pi x)\sin(n\pi x)\, dx = \frac{(-1)^{n-1}}{[(n-1)\pi]^2} + \frac{(-1)^n}{[(n+1)\pi]^2}, \tag{27}$$

$$I_n^{(2m)} = 2\int_0^1 x^{2m} \sin(\pi x)\sin(n\pi x)\, dx, \qquad m = 1, 2, \ldots, \tag{28}$$

and the geometry–only constants

$$K = \frac{2}{\pi^2}\sum_{n=2}^{\infty} \frac{(2I_n)^2}{n^2-1}, \qquad L_{2m} = \frac{2}{\pi^2}\sum_{n=2}^{\infty} \frac{\left(2I_n^{(2m)}\right)^2}{n^2-1} \;(m \ge 1), \qquad K = L_2. \tag{29}$$

Here K is the quadratic (x^2) coefficient; L_{2m} extend the even–power series.
Coulombic contribution—near/shell, far, and inhomogeneity combined. Factoring out the far–field baseline $\mathcal{D}_\mathrm{C}^{(0)} = \alpha/\pi$, the scalar contribution will be

$$\boxed{\mathcal{D}_\mathrm{C}(\alpha) = \frac{\alpha}{\pi}\left[\underbrace{\sqrt{1-\xi}}_{\text{uniform near/shell}} - \underbrace{\left(\tfrac{\xi}{2}\right)K}_{\text{quadratic inhomogeneity}} - \underbrace{\sum_{m=2}^{\infty}\left(\tfrac{\xi}{2}\right)^m L_{2m}}_{\text{higher even powers}}\right],} \qquad \xi = 2\alpha\, C_0^{\mathrm{uni}}. \tag{30}$$

The roles are now explicit; the *far–field* sets the overall scale, the *uniform near/shell* modifies it multiplicatively through $\sqrt{1-\xi}$, and the *inhomogeneity* adds controlled spectral corrections via K and L_{2m}. All quantities are dimensionless and fixed by the cavity geometry; no external inputs are used. For a compact derivation and proof sketch of Eq. (30), see Appendix A.

Table I lists the numerical inputs and the term-by-term values entering the Coulombic shell shift $\mathcal{D}_\mathrm{C}(\alpha)$ used in the real α solve—just to check scales. Parameters that depend on α (e.g. $\xi = 2\, C_0^{\mathrm{uni}}\, \alpha$) are fixed by the root condition; the numbers shown use the solved α and are reported only to indicate scales—no measured α is fed anywhere.

TABLE I. Coulombic shell shift $\mathcal{D}_{\mathrm{C}}$

Item	Symbol / factor	Value (dimensionless)
Input	ξ	0.00631186064658640935274332
Input	K	0.00223153891653197018640879
Input	L_4 (order 1; m=2)	0.00373155489837063530553899
Input	L_6 (order 2; m=3)	0.00143830013553058946046987
Input	L_8 (order 3; m=4)	0.00060470414931351794924673
Uniform near/shell	$\frac{\alpha}{\pi}\sqrt{1-\xi}$	$2.31547720363493266880874175375 \times 10^{-3}$
Quadratic inhomogeneity	$-\frac{\alpha}{\pi}\left(\frac{\xi}{2}\right)K$	$-1.63586450014266319711045280493 \times 10^{-8}$
Higher even power (order 1; m=2)	$-\frac{\alpha}{\pi}\left(\frac{\xi}{2}\right)^2 L_4$	$-8.63296746995696606371305088316 \times 10^{-11}$
Higher even power (order 2; m=3)	$-\frac{\alpha}{\pi}\left(\frac{\xi}{2}\right)^3 L_6$	$-1.05013998946420859004360859248 \times 10^{-13}$
Higher even power (order 3; m=4)	$-\frac{\alpha}{\pi}\left(\frac{\xi}{2}\right)^4 L_8$	$-1.39337505189658064137375522990 \times 10^{-16}$
Tail $m \geq 5$	aggregate of m≥5	$-2.04693759495022820511072493035 \times 10^{-19}$
Total	$\mathcal{D}_{\mathrm{C}}(\alpha)$	0.00231546075855510300146174

V. VECTOR POTENTIAL BASELINE

The same invariant (12) (decomposed on the shell in (13)) receives a vector contribution through the gauge–projected Λ–channel.

A. Intrinsic core Λ_{ind}

The ring current on $\mathbb{C}$-space gives the *filamentary* tangential potential on the loop,

$$A_{\parallel}^{\mathrm{ring}} = \frac{\mu_0 I}{2\pi}\left[\ln\frac{8R}{\sigma_{\mathbb{C}}} - 2\right], \qquad I = \frac{e\,\omega}{2\pi} = \frac{ec}{2\pi R}, \tag{31}$$

so that the dimensionless amplitude built from the gauge term is

$$\frac{q\,A_{\parallel}^{\mathrm{ring}}}{mc} = \frac{\mu_0}{2\pi}\frac{q}{mc}\,I\left[\ln\frac{8R}{\sigma_{\mathbb{C}}} - 2\right] = \frac{\mu_0 e^2 c}{4\pi^2\hbar}\left[\ln\frac{8R}{\sigma_{\mathbb{C}}} - 2\right] = \frac{\alpha}{\pi}\,\underbrace{\left(\ln\tfrac{8R}{\sigma_{\mathbb{C}}} - 2\right)}_{=:\,\Lambda_{\mathrm{ind}}}, \tag{32}$$

using $\mu_0 e^2/(4\pi) = \alpha\,\hbar/c$ and the Relator relation $R = \hbar/(mc)$ on the shell. This identifies the *loop inductive logarithm*

$$\boxed{\Lambda_{\mathrm{ind}} = \ln\frac{8R}{\sigma_{\mathbb{C}}} - 2 = \ln\!\left(8\sqrt{\pi}\right) - 2} \tag{33}$$

which is the universal near–field UV log of a thin circular loop.

B. UV→IR transfer: $\Delta\Lambda_{\mathrm{UV}\to\mathrm{IR}}$

The filamentary tangential vector potential on a thin circular loop follows from the 2D Green function of the Laplacian, $G(\rho) \propto \ln\rho$ (since $\nabla^2 \ln\rho = 2\pi\delta^{(2)}$). A finite cross–section ("collar") regularizes the logarithmic singularity by averaging $\ln(\cdot)$ over the collar profile. For an isotropic Gaussian collar of width $\sigma_{\mathbb{C}}$ we obtain a *parameter–free* finite constant:

$$\boxed{\Delta\Lambda_{\mathrm{UV}} \;=\; \frac{1}{\pi\sigma_{\mathbb{C}}^2}\int_{\mathbb{R}^2} e^{-\rho^2/\sigma_{\mathbb{C}}^2}\,\ln\frac{\sigma_{\mathbb{C}}}{\rho}\,d^2\rho \;+\; \frac{1}{2}\,\ln 2 \;=\; \frac{1}{2}\,(\ln 2 + \gamma) \;\equiv\; c_0^{\mathrm{Gauss}}.} \tag{34}$$

Provenance of the constants (no arbitrariness). (i) *Radial (collar) average.* In polar coordinates, $d^2\rho = 2\pi\rho\, d\rho$ and with $t = \rho^2/\sigma_{\mathbb{C}}^2$ (so $\rho\, d\rho = \frac{\sigma_{\mathbb{C}}^2}{2}\, dt$),

$$\frac{1}{\pi\sigma_{\mathbb{C}}^2}\int d^2\rho\, e^{-\rho^2/\sigma_{\mathbb{C}}^2} \ln\frac{\sigma_{\mathbb{C}}}{\rho} = \int_0^\infty e^{-t}\Big(-\tfrac{1}{2}\ln t\Big)\, dt = \frac{\gamma}{2}. \tag{35}$$

This is the universal finite part of the 2D Gaussian average of $\ln(\sigma_{\mathbb{C}}/\rho)$.

(ii) *Loop (azimuthal) matching.* The thin–loop kernel is built on the chord $|\mathbf{r} - \mathbf{r}'| = 2R\sin(\theta/2)$. Its pure angular average obeys

$$\frac{1}{2\pi}\int_0^{2\pi} \ln\big[2\sin(\tfrac{\theta}{2})\big]\, d\theta = 0,$$

so the azimuthal averaging contributes no finite constant. The additional $\frac{1}{2}\ln 2$ arises *solely* from matching the collar–averaged local logarithm $\ln(\sigma_{\mathbb{C}}/\rho)$ to the standard slender–loop induced normalization $\Lambda_{\text{ind}}(\varepsilon) = \ln(8/\varepsilon) - 2$ with $\varepsilon = \sigma_{\mathbb{C}}/R$ (see (33)); equivalently, it follows from the small–argument expansion of the exponential integral under the same normalization. Combined with the radial Gaussian finite part $\gamma/2$, this fixes

$$c_0^{\text{Gauss}} = \tfrac{1}{2}(\ln 2 + \gamma),$$

with no free parameter.

Only the fraction of this UV constant that survives inside the finite cavity and in the *tangential, gauge–transverse* dot channel contributes to the near field. That fraction is selected by the TT–χ IR projector $P_{\text{IR}}^{(\chi)}(\ell)$ (defined in (43)), which suppresses normal Poynting flow and keeps the tangential TT content on the collar. Hence,

$$\boxed{\Delta\Lambda_{\text{UV}\to\text{IR}}(\ell_0) \;=\; c_0^{\text{Gauss}}\, P_{\text{IR}}^{(\chi)}(\ell_0) \;=\; \frac{1}{2}\,(\ln 2 + \gamma)\, P_{\text{IR}}^{(\chi)}(\ell_0), \quad \ell_0 := \sigma_{\mathbb{C}}/r_* = \frac{1}{\pi\sqrt{\pi}}.} \tag{36}$$

As detailed in Appendix B, the TT kernel is the *transverse–traceless, tangential* projector on the matching shell; it takes the collar field and returns $\Pi_{\text{TT}} A_\parallel$ (divergence-free, purely tangential—toroidal—component on $\mathbb{S}^2_{r_*}$, implemented as $\Pi_{\text{TT}} = \chi(-\Delta_S)\,\mathcal{P}_{\text{div-free}}\,\mathcal{P}_T$), removing (i) longitudinal gauge pieces $\nabla_\parallel\phi$ and (ii) the normal/radial component $(A\cdot n)\, n$. What remains is a divergence–free, loop–like swirl that carries inductive energy but no normal Poynting flux through the shell. The "IR" label indicates a geometric low–pass set by the collar scale $\ell_0 = \sigma_{\mathbb{C}}/r_*$, so only the large–scale tangential TT content survives inside the finite cavity and couples to the dot (χ) channel: $\Delta\Lambda_{\text{UV}\to\text{IR}}(\ell_0) = c_0^{\text{Gauss}}\, P_{\text{IR}}^{(\chi)}(\ell_0)$.

In this work the terms "loop logarithm" and "UV→IR smoothing" are purely *geometric, magnetostatic* notions tied to the thin–ring shell geometry and its near–shell collar; they are not QED loop effects and do not involve running scales, counterterms, or any renormalization scheme. All quantities are rendered dimensionless by construction, and dimensional constants appear only symbolically and cancel in the final relations. Gauge invariance on the matching sphere is enforced by admitting the scalar–vector cross term solely through the TT–χ projector, which selects tangential, surface–divergence–free content with an IR acceptance window. This clarifies that both the emergent α and the g–factor corollary arise from the Relator geometry itself, independent of QED machinery.

Collar and collar field. By a *collar* we mean the thin toroidal tube $\mathcal{C}_a(S^1)$ around the image of the Dirac ring on the matching shell $r = r_*$, with Gaussian cross–section of width $\sigma_{\mathbb{C}}$. The Dirac ring is the closed internal phase orbit on $\mathbb{C}$; via the Relator map $R\omega = c$ and the shell condition $r_* = \pi R$, internal and spatial arc elements are matched on the shell as $R\, d\theta = r_*\, d\phi$ (i.e. proportional, not postulated equal everywhere). The *collar field* is the tangential vector potential averaged across this tube, $A_\parallel^{\text{coll}} := \langle A_\parallel\rangle_{\text{collar}}$; this regularizes the filamentary logarithm and yields a parameter–free UV constant. Applying the TT projector removes longitudinal and normal pieces and retains the gauge–transverse, divergence–free tangential swirl that stores inductive energy inside the cavity and feeds the Λ–channel. Physically, $\mathbb{C}$ acts as the generative phase background while forces and measurements live in $\mathbb{R}^3$-space; the two are coupled only through the gauge–invariant combination $\|\nabla S - \frac{q}{c}\,\mathbf{A}\|$, so structures formed internally are mirrored on the shell and vice versa.

C. Exterior multipole subtraction: $\Delta\Lambda_{\text{OUT}}(\eta)$

The swirl on $\mathbb{C}$ (a thin circular loop of radius R) produces a magnetostatic field that, outside the matching sphere $r > r_*$, carries a free–space energy that must be removed to avoid double counting. The OUT term subtracts exactly that exterior share.

Multipole route. Project the exact loop field on the shell $r = r_*$ onto the toroidal basis $\mathcal{T}_\ell(x) = (1 - x^2)P'_\ell(x)$ with $x = \cos\theta$,

$$a_\ell(\eta) = \frac{r_*^{\ell+2}}{I_\ell} \int_{-1}^{1} B_\theta(r_*, x)\, \mathcal{T}_\ell(x)\, dx, \qquad I_\ell = \int_{-1}^{1} (1 - x^2)\, [P'_\ell(x)]^2\, dx = \frac{2\,\ell(\ell+1)}{2\ell+1}, \tag{37}$$

where $B_\theta = B_\rho\, x - B_z\sqrt{1 - x^2}$ and only odd ℓ contribute by equatorial symmetry. (with $\mathbf{e}_\rho = \sin\theta\, \mathbf{e}_r + \cos\theta\, \mathbf{e}_\theta$ and $\mathbf{e}_z = \cos\theta\, \mathbf{e}_r - \sin\theta\, \mathbf{e}_\theta$ on $r = r_*$). The scalar potential decays as $r^{-(\ell+1)}$ so the *field* decays as $r^{-(\ell+2)}$ in the exterior. Separation of variables and orthogonality on the sphere give the exterior energy of a pure mode as

$$U^{(\ell)}_{\text{out}} = \frac{1}{2\mu_0} \int_{r>r_*} B^2\, dV = \frac{2\pi}{\mu_0}\, \frac{\ell+1}{2\ell+1}\, a_\ell^2\, r_*^{-(2\ell+1)}. \tag{38}$$

Summing over odd ℓ and using our dimensionless normalization (absorbing μ_0 into the loop weight) leads to

$$\Delta\Lambda_{\text{OUT}}(\eta) = -\,4\pi \sum_{\ell=1,3,\ldots} \frac{\ell+1}{2\ell+1}\, a_\ell^2(\eta)\, r_*^{-(2\ell+1)}, \qquad \eta = \frac{R}{r_*}. \tag{39}$$

Closed form (exact resummation of odd modes). Evaluating $a_\ell(\eta)$ from the *exact* loop field on the shell and resumming the odd series yields the elementary closed form

$$\boxed{\Delta\Lambda_{\text{OUT}}(\eta) = -\,\pi \left[\frac{\ln(1 - \eta^4)}{2\eta} + \operatorname{atanh}\eta - \arctan\eta\right], \qquad (|\eta| < 1).} \tag{40}$$

This is *exact* for a loop strictly inside the matching sphere ($R < r_*$); analytic continuation fixes branches when η approaches unity.

Sketch of the resummation. From the multipole route one obtains the odd–ℓ power series

$$\Delta\Lambda_{\text{OUT}}(\eta) = -\,\pi \sum_{\ell=1,3,5,\ldots} \frac{\eta^{\,2\ell+1}}{(\ell+1)(2\ell+1)}. \tag{41}$$

Using $\dfrac{1}{(\ell+1)(2\ell+1)} = -\dfrac{1}{\ell+1} + \dfrac{2}{2\ell+1}$ and restricting to odd $\ell = 2m+1$,

$$\sum_{m\ge 0} \frac{\eta^{4m+3}}{m+1} = -\frac{1}{\eta}\ln(1 - \eta^4), \qquad \sum_{m\ge 0} \frac{\eta^{4m+3}}{4m+3} = \frac{1}{2}\big(\operatorname{atanh}\eta - \arctan\eta\big),$$

which reproduces (40).

Small–η expansion (consistency). Expanding (40) at $\eta \ll 1$ gives the odd–power tower

$$\Delta\Lambda_{\text{OUT}}(\eta) = -\,\pi\left(\frac{\eta^3}{6} + \frac{\eta^7}{28} + \frac{\eta^{11}}{66} + \frac{\eta^{15}}{120} + \cdots\right), \tag{42}$$

which matches (41) term by term.

D. Scalar–vector phase locking: $\Delta\Lambda_{\text{sync}}$

The gauge–invariant scalar invariant

$$\mathcal{D} \;=\; \frac{\left\|\nabla S - \frac{q}{c}\mathbf{A}\right\|^2}{m^2c^2}$$

contains, on the matching sphere $r = r_*$, a cross term

$$-\;\frac{2q}{m^2c^3}\,\Big\langle (\nabla S)\cdot \mathbf{A}_\parallel \Big\rangle_{r=r_*}.$$

Although the scalar s-mode has a *radial* normal derivative on the shell while the loop field is *tangential*, the finite–cavity projection (curved shell, finite collar) generates a small but coherent overlap in the gauge–χ (dot) channel. Physically this is a *phase locking* between the shell s-mode at the Dirichlet node $j_0(kr_*) = 0$ and the azimuthal swirl on $\mathbb{C}$-space. This coherent, magnetoquasistatic (weakly non-radiative) piece is encoded by a synchronization increment $\Delta\Lambda_{\text{sync}}$ inside the logarithmic loop chain, see Appendix C for the boundary derivation.

Projectors on the shell. We use two TT projectors on $r = r_*$: (i) the toroidal transverse–traceless Λ–channel for admitting the UV collar and for the exterior subtraction $\Delta\Lambda_{\rm OUT}$; (ii) the χ–channel (dot) projector to isolate the scalar–vector overlap. The TT–χ projector is

$$P_{\rm IR}^{(\chi)}(\ell) = \frac{\displaystyle\int_0^1 x^2 \sin^2(\pi x)\left[1 - \tfrac{1}{3} f_{\rm swirl}(x;\ell)\right] e^{-((1-x)/\ell)^2}\,dx}{\displaystyle\int_0^1 x^2 \sin^2(\pi x)\,dx}, \qquad f_{\rm swirl}(x;\ell) = \frac{(1-x)^2}{(1-x)^2 + \ell^2}, \tag{43}$$

which preserves tangential content while suppressing normal Poynting flux in the weighted (magnetoquasistatic) sense.

$$\varepsilon = \frac{1}{\sqrt{\pi}}, \qquad \eta = \frac{R}{r_*} = \frac{1}{\pi}, \qquad \ell = \varepsilon\,\eta = \frac{1}{\pi\sqrt{\pi}}\,. \tag{44}$$

We use three dimensionless parameters on the matching shell, each with a concrete physical role. $\varepsilon := 1/\sqrt{\pi}$ — *"χ-dot Gaussian width"* — sets the unit-normalized Gaussian both for the collar averaging on the Dirac ring and for the Gaussian distribution of Loopons in the Relator internal $\mathbb{C}$ (the mass–associated spread of χ–dots). $\eta := R/r_* = 1/\pi$ — *"Relator aspect ratio"* — is the radius–to–shell ratio fixed by $r_* = \pi R$. $\ell := \varepsilon\,\eta = 1/(\pi\sqrt{\pi})$ — *"IR passband scale"* or *"effective collar thickness"* — is the scale that controls the low-ℓ admission in $P_{\rm IR}^{(\chi)}$ via the weights $e^{-((1-x)/\ell)^2}$ and $f_{\rm swirl}(x;\ell)$, i.e. how much tangential TT content of the collar field is accepted into the finite cavity.

Closed synchronization rule. Projecting the cross term on the matching shell with the exact exterior loop field $B_\theta(r_*, x)$ shows that the scalar–vector overlap is a universal geometric multiple of the already–computed exterior piece $\Delta\Lambda_{\rm OUT}(\eta)$. To make the physics explicit, we first write the order–one synchronization *openly* and then extend it to a ladder valid to all orders.

Order–one (open form). Define the intermediate synchronization gain

$$\Gamma_{\rm sync}^{(1)}(\eta; X) \;=\; \gamma_{\rm geom}^{(1)}(\eta) \;+\; \gamma_{\rm map}^{(1)}(X)\,, \tag{45}$$

with the geometric partition written as a curvature series before its analytic closure,

$$\gamma_{\rm geom}^{(1)}(\eta) \;=\; \frac{1}{2}\Big[1 + \sum_{m=1}^{\infty} \frac{\eta^{2m}}{(2m+1)!}\Big] \;=\; \frac{1}{2}\,\frac{\sinh\eta}{\eta}\,, \tag{46}$$

and the UV$\to$IR transfer across the collar (map) entering linearly with full weight

$$\gamma_{\rm map}^{(1)}(X) \;=\; \Big(\frac{K}{2\,\mathcal{D}_{\rm C}}\Big)\, c_0^{\rm Gauss}\, P_{\rm IR}^{(\chi)}(\ell)\,. \tag{47}$$

Here K is the near–field spectral constant, $\mathcal{D}_{\rm C}$ is the Coulombic budget, $c_0^{\rm Gauss} = \frac{1}{2}(\ln 2 + \gamma)$ is the universal Gaussian UV constant, and $P_{\rm IR}^{(\chi)}(\ell)$ is the IR projector on the collar of thickness ℓ. The order–one synchronization then reads

$$\boxed{\Delta\Lambda_{\rm sync}^{(1)}(\eta,\ell) \;=\; \Gamma_{\rm sync}^{(1)}(\eta; X)\; P_{\rm IR}^{(\chi)}(\ell)\; \Delta\Lambda_{\rm OUT}(\eta)\,.} \tag{48}$$

Higher orders — two complementary ladders. Curvature–induced mode–mixing on the shell introduces

$$\kappa \;\equiv\; \frac{\sinh\eta}{\eta} - 1, \qquad X \;\equiv\; \Big(\frac{K}{2\,\mathcal{D}_{\rm C}}\Big)\, c_0^{\rm Gauss}\, P_{\rm IR}^{(\chi)}(\ell), \tag{49}$$

and generates two resummations:

(i) Map ladder (repeated UV$\to$IR transfers): Each additional transfer multiplies the map amplitude by $(-\kappa)\,X$, so the n-th rung contributes

$$\Delta\Lambda_{\rm sync}^{(n),\rm map} \;=\; (-\kappa)^{\,n-1} X^{\,n}\; P_{\rm IR}^{(\chi)}(\ell)\; \Delta\Lambda_{\rm OUT}(\eta), \qquad n \ge 1, \tag{50}$$

whose closed sum is the geometric series

$$\boxed{\sum_{n=1}^{\infty} \Delta\Lambda_{\rm sync}^{(n),\rm map} \;=\; \underbrace{\frac{X}{1+\kappa X}}_{\gamma_{\rm map}^{\rm ladder}(X,\kappa)} \quad P_{\rm IR}^{(\chi)}(\ell)\; \Delta\Lambda_{\rm OUT}(\eta)\,.} \tag{51}$$

Physically, $\gamma_{\rm map}^{\rm ladder}$ encapsulates the cumulative UV→IR transport through the χ-channel, with the curvature factor κ accounting for toroidal–spherical mixing on the collar.

(ii) Self–feedback ladder (re–inserting the induced χ on the shell): Once χ is induced by the single pass, re–projecting it with the scalar on the shell generates a Dyson–like tower governed by

$$\varepsilon_{\rm Dy} \;\equiv\; \Big(\frac{\alpha}{\pi}\Big)\Big(\frac{K}{2\pi^2}\Big)\,P_{\rm IR}^{(\chi)}(\ell)\,\Lambda, \qquad \varepsilon_{\rm Dy} \ll 1 \;\;, \tag{52}$$

with n-th rung ($n \geq 2$)

$$\Delta\Lambda_{\rm sync}^{(n),{\rm self}} \;=\; -\,\gamma_{\rm geom}(\eta)\;P_{\rm IR}^{(\chi)}(\ell)\;\Lambda\;\big[(-\kappa)^{\,n-2}\,\varepsilon_{\rm Dy}^{\,n-1}\big], \tag{53}$$

resumming to

$$\boxed{\sum_{n=2}^{\infty}\Delta\Lambda_{\rm sync}^{(n),{\rm self}} \;=\; -\,\gamma_{\rm geom}(\eta)\;P_{\rm IR}^{(\chi)}(\ell)\;\Lambda\;\frac{\varepsilon_{\rm Dy}}{1+\kappa\,\varepsilon_{\rm Dy}}\,.} \tag{54}$$

Its leading term reproduces the compact second order $\Delta\Lambda_{\rm sync}^{(2)} = -(\alpha/\pi)\,\gamma_{\rm geom}\,(K/2\pi^2)\,[P_{\rm IR}^{(\chi)}]^2\,\Lambda^2$. The signs are fixed by the overlap: $\gamma_{\rm geom}, X, P_{\rm IR}^{(\chi)}, \Lambda > 0$ and $\Delta\Lambda_{\rm OUT}(\eta) < 0$ imply that both the map ladder and the self ladder reduce Λ; higher self orders are strongly suppressed by $\varepsilon_{\rm Dy}^{n-1}$.

Full sigma–ladder (umbrella). Collecting the order–one rule and both ladders gives a closed, gauge–consistent expression with a single external projector:

$$\boxed{\Delta\Lambda_{\rm sync}(\eta,\ell;\Lambda) \;=\; \underbrace{\Big[\gamma_{\rm geom}(\eta)\;+\;\frac{X}{1+\kappa X}\Big]}_{\text{geometric partition + full map ladder}}\;P_{\rm IR}^{(\chi)}(\ell)\;\Delta\Lambda_{\rm OUT}(\eta)\;-\;\underbrace{\gamma_{\rm geom}(\eta)\;P_{\rm IR}^{(\chi)}(\ell)\;\Lambda\;\frac{\varepsilon_{\rm Dy}}{1+\kappa\,\varepsilon_{\rm Dy}}}_{\text{self–feedback ladder}},} \tag{55}$$

with $X, \kappa, \varepsilon_{\rm Dy}$ given in (49) and (52). The first bracket collects the inside–outside geometric gain $\gamma_{\rm geom}$ and the resummed UV→IR map $\gamma_{\rm map}^{\rm ladder}$; the second term is the closed self–feedback of the synchronized scalar–vector channel. No auxiliary fitting parameters enter: all factors trace back to the shell projection, curvature mixing, and the gauge–invariant energy budget on the collar.

Interpretation and signs. $\gamma_{\rm geom}(\eta) = \frac{1}{2}\,\frac{\sinh\eta}{\eta} > 0$ encodes mixed normal–tangential coupling on the curved shell. $\gamma_{\rm map}$ transports the Gaussian UV increment into the TT–χ channel in proportion to the scalar spectral weight $K/(2\mathcal{D}_{\rm C})$ and the IR population P. Since $\Delta\Lambda_{\rm OUT}(\eta) < 0$ and $P_{\rm IR}^{(\chi)}(\ell) > 0$, the baseline case gives $\Gamma_{\rm sync} > 0$ and hence $\Delta\Lambda_{\rm sync} < 0$ with a small magnitude (few-percent of $|\Delta\Lambda_{\rm OUT}|$).

Avoiding circularity in Computing Alpha. To keep Λ *closed* during the α–solve, evaluate $\gamma_{\rm map}$ at the *lock value* $\mathcal{D}_{\rm C}$ from $C_{\log} = 1/3$. The self–feedback parameter carries an explicit (α/π) through $\varepsilon_{\rm Dy}$; in practice we evaluate $\varepsilon_{\rm Dy}$ at the lock value $\alpha_{\rm lock}$ (its effect is $\ll 10^{-8}$), so the root–find remains geometry–closed.

Bounded dynamical remainder. Higher–order magnetic dynamics (retardation, non-quasistatic effects, odd multipoles beyond the working cutoff) are collected in

$$\Delta\Lambda_{\rm dyn} = \mathcal{O}\big(\zeta^2\big), \tag{56}$$

used only as an explicit uncertainty budget for ppt targets.

E. Full Chain Lambda

Gathering the Λ chain,

$$\boxed{\Lambda \equiv \Lambda_{\rm eff} = \underbrace{\Lambda_{\rm ind}}_{\text{filamentary loop}} \;+\; \underbrace{\Delta\Lambda_{\rm UV\to IR}}_{\text{UV→IR (finite cavity)}} \;+\; \underbrace{\Delta\Lambda_{\rm OUT}(\eta)}_{\text{exterior subtraction}} \;+\; \underbrace{\Delta\Lambda_{\rm sync}}_{\text{phase locking}} \;+\; \underbrace{\Delta\Lambda_{\rm dyn}}_{\text{higher–order dynamics}}} \tag{57}$$

and define the universal spectral overlap

$$\Upsilon_{\rm sp} = \frac{K}{2\pi^2}, \qquad \zeta = \Upsilon_{\rm sp}\,\Lambda_{\rm eff}. \tag{58}$$

The vector contribution to (13) then reads

$$\Delta\mathcal{D}_\Lambda(\alpha) = -\frac{\alpha}{\pi}\,\zeta + \frac{\left(\frac{\alpha}{\pi}\,\zeta\right)^2}{4\,\mathcal{D}_\mathrm{C}} + \mathcal{O}(\zeta^3)\,, \tag{59}$$

and is used here *only as a diagnostic scale* for the Λ–channel—i.e., to gauge the inductive share relative to the Coulomb block. The quadratic term is an $O(\zeta^2)$ correction; the explicit normalization by $\mathcal{D}_\mathrm{C}$ makes it gauge–consistent and prevents double counting with the scalar block. We therefore do *not* employ (59) in the emergent–α solve or any iterative updates; it is reported for intuition and bookkeeping. Its physical role is taken up where it matters—namely in the emergent g–factor discussion—where $\mathcal{D} = \mathcal{D}_\mathrm{C} + \Delta\mathcal{D}_\Lambda + O(\zeta^3)$ enters explicitly.

The following numerical values are used in the inductive (vector) block of the *emergent–α solve*. Parameters that depend on α (e.g. $\gamma_\mathrm{map}(\alpha), \gamma_\mathrm{eff}(\alpha), \Delta\Lambda_\mathrm{sync}(\alpha), \zeta(\alpha)$) are treated as unknown targets determined by the root condition; no initial seeding or experimental α is injected, so there is no hidden α contamination. See Table II.

TABLE II. Inductive (vector) block: inputs, gains, sync decomposition, and term values (latest emergent-α run)

Item	Symbol / factor	Value (dimensionless)
IR projector (TT–χ)	$P_\chi^{(\mathrm{IR})}(\ell_0)$	0.0857791925845556011097469
UV→IR contribution	$\Delta\Lambda^{\mathrm{UV}\to\mathrm{IR}}$	0.0544853495865520906532491
Exterior subtraction	$\Delta\Lambda_\mathrm{OUT}(\eta_0, L_\mathrm{max})$	-0.0139671580625860254655225
Baseline inductive sum	Λ_base	0.692324676128502080511137
Curvature series	$\mathrm{curv}(\eta)$	0.0169726204644520534264891
Geometric gain	γ_geom	0.508486310232226026713245
Map gain	$\gamma_\mathrm{map}(\alpha)$	0.0262552879667700470311315
Effective gain	$\gamma_\mathrm{eff}(\alpha)$	0.534741598198996073744376
Sync decomposition: $\Delta\Lambda_\mathrm{sync} =$	$\underbrace{\Delta\Lambda_\mathrm{base}}_{\Delta\Lambda_\mathrm{geom}+\Delta\Lambda_\mathrm{map}} + \Delta\Lambda^{(\mathrm{ladder})}_\mathrm{extra} + \Delta\Lambda^{(\mathrm{ladder})}_\mathrm{self}$	
Geometric component	$\Delta\Lambda_\mathrm{geom} = \gamma_\mathrm{geom} P^{(\chi)}_\mathrm{IR}\Delta\Lambda_\mathrm{OUT}$	-0.0006092131471609058062469770856494138
Map component	$\Delta\Lambda_\mathrm{map} = \gamma_\mathrm{map} P^{(\chi)}_\mathrm{IR}\Delta\Lambda_\mathrm{OUT}$	-0.0000314562384276321673051921626849368
Sync base	$\Delta\Lambda_\mathrm{base} = \gamma_\mathrm{eff} P^{(\chi)}_\mathrm{IR}\Delta\Lambda_\mathrm{OUT}$	-0.0006406693855885379735521686492885799
χ-ladder addendum	$\Delta\Lambda^{(\mathrm{ladder})}_\mathrm{extra}$	$1.40113178768037025305128 \times 10^{-8}$
Self-ladder	$\Delta\Lambda^{(\mathrm{ladder})}_\mathrm{self}$	$-4.70926333239101999626022 \times 10^{-10}$
Sync shift (total)	$\boldsymbol{\Delta\Lambda_\mathrm{sync}(\alpha)}$	**-0.00064065584519699440895164**
Effective inductive sum	$\Lambda_\mathrm{eff}(\alpha)$	0.691684020283305086102185
Lock diagnostic	$\zeta(\alpha)$	0.0000781956270220474977416816
Lock target	$C_\mathrm{log}(\alpha)$	0.3333333333333632128055046
Target	1/3	0.3333333333333333333333333
Lock deviation*	$C_\mathrm{log} - \frac{1}{3}$	$2.98794721712449115518761 \times 10^{-13}$

*Note**. Lock deviation uses the CODATA2022 α and reflects the residual analytical error in Λ.

VI. EMERGENT ALPHA

We posit a *Coulomb–inductive equilibrium* on the matching shell. The Coulomb block $\mathcal{D}_\mathrm{C}$, the inductive Λ–channel produced by the Dirac ring, and the polarization of χ–dots on $\mathbb{C}$-space conspire to a single, universal consistency ratio C_log that does *not* depend on charge or mass parameters, even though it manifests through them in observables. Within the Relator framework this equilibrium appears as a species–independent coefficient linking the scalar and vector sectors; we establish it by two complementary geometric paths and then identify them, thereby fixing α without injected measurements.

Two complementary paths. Path I (loop–polarization route). Starting from the Dirac–ring baseline and its inductive logarithm, we assemble the effective loop weight via Λ_eff (UV collar with Gaussian constant, IR TT projector $P^{(\chi)}_\mathrm{IR}$, exterior subtraction, small sync term, bounded dynamics), which defines $\zeta = (K/2\pi^2)\,\Lambda_\mathrm{eff}$. The Stage–$\chi$ mass–log built from the scalar–vector overlap then carries the coefficient $C^\mathrm{I}_\mathrm{log} = (\pi^2/\mathcal{D}_\mathrm{C})\,\zeta(1+\zeta)$ (higher–order pieces are explicit and controlled). *Path II (in–plane geometric flow).* Working purely on the $\mathbb{C}$–space, scale invariance and shell pinning yield an RG–like drift of the effective coupling with a fixed geometric ratio $C^\mathrm{II}_\mathrm{log} = \frac{1}{3}$, independent of particle–specific constants.

Equating the two routes, $C^\mathrm{I}_\mathrm{log} = C^\mathrm{II}_\mathrm{log}$, gives the closed root

$$\mathcal{D}_\mathrm{C}(\alpha) = 3\pi^2\,\zeta\,\big(1+\zeta\big),$$

and hence α as an emergent, parameter–free solution. All steps use only universal numbers and Relator geometry; no e, m, c, or $\hbar$ are seeded. This universal ratio is what imprints the scalar–vector balance across species, while its consequences appear concretely in Coulombic and magnetic observables.

A. Path I — Geometric derivation of $C_{\log}$ from Dirac-ring loop polarization

We probe the *geometric sensitivity* of the shell–collar structure; how the internal χ–dot Gaussian population on $\mathbb{C}$ (Loopons—mass–associated energy dots) and the Dirac–ring radius R respond to a slow, radial rescaling. Heavier leptons (e.g. the muon) do not enter through measured masses here; rather, they *geometrically* differ by a smaller Relator radius and a tighter χ–dot spread around the ring, which increases the tangential TT polarization admitted by the shell projector. The question is; what universal long–distance coefficient multiplies this geometric stretch?

Radial stretch and the single logarithm. We proceed *geometrically*, without particle inputs. A slow radial rescaling on the internal plane $\mathbb{C}$, $r \to s\,r$ with $s > 0$, yields the single logarithm

$$\int_{\rho=r}^{\rho=sr} d\ln\rho \;=\; \ln s\,. \tag{60}$$

Inductive stack and geometric weight. The collar–shell construction fixes the near–field loop weight once and for all. We assemble the effective inductive logarithm Λ_{eff} (Dirac–ring filament baseline, universal Gaussian UV increment through the TT–χ IR projector, exterior subtraction, thin–shell sync, bounded dynamics). The universal spectral overlap then defines the geometric weight

$$\Upsilon_{\text{sp}} := \frac{K}{2\pi^2}, \qquad \varepsilon := \frac{1}{\sqrt{\pi}}\ \ (\chi\text{–dot Gaussian width}), \qquad \eta := \frac{R}{r_*} = \frac{1}{\pi}\ \ (\text{Relator aspect ratio}), \qquad \ell := \varepsilon\,\eta, \tag{61}$$

$$\zeta \;=\; \Upsilon_{\text{sp}}\,\Lambda_{\text{eff}}(\ell)\,. \tag{62}$$

Here ℓ sets the IR passband of $P_{\text{IR}}^{(\chi)}$; Λ_{eff} is the *inductive log stack*; and $A_\parallel$ is the *tangential vector potential* on the shell (used elsewhere via $\mathcal{D} = \|\nabla S - \frac{q}{c}\,\mathbf{A}\|^2/(m^2c^2)$).

TT population and scalar response. For a uniform Dirac–ring swirl (constant energy density per internal angle $d\theta$ on $\mathbb{C}$), the TT projector on the shell $r = r_*$ selects the azimuthal (toroidal) component. The Relator pinning relates internal and shell azimuths by

$$R\,d\theta \;=\; r_*\,d\phi \quad\Longrightarrow\quad \frac{d\theta}{d\phi} \;=\; \frac{r_*}{R} \;=\; \pi, \tag{63}$$

using $r_* = \pi R$. The TT population accumulated around one full shell turn is therefore

$$c_\chi \;=\; \int_0^{2\pi} \frac{d\theta}{d\phi}\,d\phi \;=\; \int_0^{2\pi} \pi\,d\phi \;=\; 2\pi^2. \tag{64}$$

This is the *minimal* toroidal TT weight (higher toroidal harmonics increase it by Cauchy–Schwarz). Hence the scale–invariant population accumulated across a stretch s is

$$(\text{log accumulation}) \;=\; c_\chi\,\zeta\,\ln s \;+\; \mathcal{O}(\zeta^2)\,. \tag{65}$$

We probe the long response with a linear observable $a(\mathcal{D}_{\text{C}})$ that, to leading order, measures the TT channel; its dressed slope is

$$\Big(\frac{\partial a}{\partial \mathcal{D}_{\text{C}}}\Big)_{\text{dressed}} = \tfrac{1}{2}\qquad \underbrace{(1+\zeta)}_{\text{single–channel dressing}} \qquad + \mathcal{O}(\zeta^2). \tag{66}$$

The factor $(1+\zeta)$ originates from the single–channel TT dressing embedded in $K_{\text{TT+sync}}$; it is not a perturbative expansion assumption. Also, here a always denotes the anomalous magnetic moment, $a \equiv (g-2)/2$, a dimensionless Dirac reporter. It is distinct from the tangential potential $A_\parallel$ on the shell and from any energy/overlap quantity on the internal circle $\mathbb{C}$.

Under a logarithmic rescaling by a factor s, the toroidal TT population then induces

$$\Delta a_{\log} \;=\; \tfrac{1}{2}\,(1+\zeta)\,\mathcal{D}_{\text{C}}\,\big(c_\chi\,\zeta\,\ln s\big) \;+\; \mathcal{O}(\zeta^2\mathcal{D}_{\text{C}})\,. \tag{67}$$

Matching the canonical long operator,

$$\Delta a_{\log} \;=\; \mathcal{D}_{\rm C}^2\, C_{\log}\, \ln s \;+\; \mathcal{O}(\mathcal{D}_{\rm C}^3)\,, \tag{68}$$

determines the one–log constant purely from Relator geometry,

$$\boxed{C_{\log}^{\rm I} \;=\; \frac{c_\chi}{2\,\mathcal{D}_{\rm C}}\,\zeta\,(1+\zeta) \;=\; \frac{\pi^2}{\mathcal{D}_{\rm C}}\,\zeta\,(1+\zeta)\,.} \tag{69}$$

All steps use only universal numbers and Relator geometry; no *measured* e, m, c, or $\hbar$ are injected.

From geometric stretch to a species ratio (interpretation only). One may *afterwards* identify the stretch with a ratio of Relator radii across shells,

$$s \;\equiv\; \frac{R_e}{R_\ell} \quad\Longrightarrow\quad \ln s \;=\; \ln\frac{R_e}{R_\ell} \;=\; \ln\frac{m_\ell}{m_e} \quad (\text{since } R \propto \bar{\lambda}_C \propto 1/m)\,. \tag{70}$$

Hence the long piece across a lepton pair takes the form

$$\Delta a_{\log} \;=\; \mathcal{D}_{\rm C}^2\, C_{\log}\, \ln\frac{m_\ell}{m_e} \;+\; \mathcal{O}(\mathcal{D}_{\rm C}^3)\,, \tag{71}$$

with the *universal* coefficient $C_{\log}$ fixed by (69). In Sec. VI B an independent in–plane geometric flow on $\mathbb{C}$ yields $C_{\log} = 1/3$; equating the two routes closes the root for α via $\mathcal{D}_{\rm C}(\alpha)$ and $\zeta(\alpha)$.

If α is fundamentally geometric, it must *lock* the internal degrees of freedom of elementary particles in a path–independent way. Distinct constructions—our Coulombic shell control, TT acceptance on the thin shell, and the variational/response/inductive routes—converge to the same long–log coefficient and close the same root for α (Appendix D), indicating that the lock does not depend on how one approaches it.

B. Path II — Geometric origin of the single–log coefficient $\frac{1}{3}$

Goal. Path I fixed the *form* of the long piece,

$$\Delta a_{\log} \;=\; \mathcal{D}_{\rm C}^2\, C_{\log}\, \ln s \;+\; \mathcal{O}(\mathcal{D}_{\rm C}^3), \tag{72}$$

with s a geometric stretch on the internal plane $\mathbb{C}$. Path II shows, using only shell geometry and TT kinematics (no mass or Λ–channel input), that

$$C_{\log} \;=\; \frac{1}{3}. \tag{73}$$

TT acceptance of a logarithmic annulus. At a point on the matching shell let $\hat{\boldsymbol n}$ be the outward normal, and $P_T(\hat{\boldsymbol n}) := I - \hat{\boldsymbol n}\hat{\boldsymbol n}^\top$ the projector onto the tangential plane. The tangent plane admits a 2D Helmholtz split into a *solenoidal* (toroidal) and a *gradient–like* tangent direction. The TT projector selects the solenoidal *rank–1* subspace; averaging over the in–plane phase ψ yields $\langle P_{\rm TT}(\hat{\boldsymbol n},\varphi)\rangle_\varphi = \frac{1}{2}\,P_T(\hat{\boldsymbol n})$. Therefore the local acceptance operator for a marginal (logarithmic) annulus is

$$\mathcal{A}(\hat{\boldsymbol n}) \;=\; \frac{1}{2}\,P_T(\hat{\boldsymbol n}) \;=\; \frac{1}{2}\big(I - \hat{\boldsymbol n}\hat{\boldsymbol n}^\top\big). \tag{74}$$

If $\hat{\boldsymbol u}$ denotes the unit direction carried by the scalar channel on the collar, the overlap weight is

$$\mathcal{W}(\hat{\boldsymbol u},\hat{\boldsymbol n}) \;:=\; \hat{\boldsymbol u}^\top\,\mathcal{A}(\hat{\boldsymbol n})\,\hat{\boldsymbol u} \;=\; \tfrac{1}{2}\big(1 - (\hat{\boldsymbol u}\cdot\hat{\boldsymbol n})^2\big). \tag{75}$$

Isotropic shell average (trace argument). In the thin–shell limit the collar samples normals uniformly on S^2. Rotational invariance gives

$$\Big\langle \hat{\boldsymbol n}\hat{\boldsymbol n}^\top \Big\rangle_{S^2} \;=\; \frac{1}{3}\,I \qquad\Longrightarrow\qquad \Big\langle P_T(\hat{\boldsymbol n}) \Big\rangle_{S^2} = \frac{2}{3}\,I. \tag{76}$$

Hence the TT acceptance averaged over the shell is

$$\Big\langle \mathcal{A}(\hat{\boldsymbol n}) \Big\rangle_{S^2} = \tfrac{1}{2}\,\Big\langle P_T(\hat{\boldsymbol n}) \Big\rangle_{S^2} = \frac{1}{3}\,I, \tag{77}$$

and therefore

$$\left\langle \mathcal{W} \right\rangle_{S^2} = \hat{\boldsymbol{u}}^{\top}\left(\tfrac{1}{3}I\right)\hat{\boldsymbol{u}} = \frac{1}{3}. \tag{78}$$

Equivalently, writing $\hat{\boldsymbol{u}}{\cdot}\hat{\boldsymbol{n}} = \cos\theta$ and averaging on S^2 gives $\langle \sin^2\theta\rangle_{S^2} = 2/3$, whence $\langle \frac{1}{2}(1-\cos^2\theta)\rangle_{S^2} = 1/3$. *Remark.* Using the full tangent projector P_T (instead of TT) would wrongly yield $2/3$ and double–count the gradient–like component.

Locking the coefficient. A marginal annulus contributes to the long piece as "source × response" weighted by the purely geometric factor $\langle \mathcal{W}\rangle_{S^2}$. Because the scalar long piece is quadratic (cf. (72)), the shell–isotropic TT acceptance (78) fixes the universal coefficient:

$$C_{\log} \;=\; \left\langle \mathcal{W} \right\rangle_{S^2} \;=\; \frac{1}{3}\,, \tag{79}$$

independent of the collar kernel, micro–population of χ–dots, or any measured parameter. Corrections from finite collar width or mild anisotropy enter only beyond the thin–shell limit as subleading terms and do not affect the universal value $\frac{1}{3}$.

Outcome. Path I provides $C_{\log}^{\mathrm{I}} = (\pi^2/\mathcal{D}_{\mathrm{C}})\,\zeta(1+\zeta)$; Path II yields $C_{\log}^{\mathrm{II}} = \frac{1}{3}$ from TT geometry alone. Equating $C_{\log}^{\mathrm{I}} = C_{\log}^{\mathrm{II}}$ closes the root for α through $\mathcal{D}_{\mathrm{C}}(\alpha)$ and $\zeta(\alpha)$, with no injected masses or fitted constants.

Connection to QED and lepton–mass logs. The universal value $C_{\log} = \frac{1}{3}$ obtained here is the same single–log coefficient that multiplies the lepton–mass–dependent term in QED's g-factor analysis (the familiar one–log piece $\propto \ln(m_\ell/m_e)$). In the Relator framework this weight does not come from diagrammatics but from pure geometry: the TT acceptance of a logarithmic annulus on the shell fixes the factor $\frac{1}{3}$, while the *origin* of the logarithm is the redistribution of the internal χ–dot density on $\mathbb{C}$ around the Dirac ring under the particle's own Coulomb field (polarization of Loopons). Thus the same numerical coefficient that QED attributes to field–theoretic structure appears here as a gauge–invariant, shell–geometric invariant, linking the mass–dependent g-factor log to the Coulomb–induced polarization of the χ–dot ensemble.

C. Alpha Lock Point: ALP geometric closure of $C_{\log}$

Combining the Path I master relation with the shell–flow result of Path II yields the algebraic lock

$$\frac{c_\chi}{2\,\mathcal{D}_{\mathrm{C}}}\,\zeta\left(1+\zeta\right) \;=\; \frac{1}{3}, \qquad \zeta = \Upsilon_{\mathrm{sp}}\,\Lambda_{\mathrm{eff}} \quad \text{(see (57))}, \qquad \Upsilon_{\mathrm{sp}} = \frac{K}{2\pi^2}. \tag{80}$$

Solving for the scalar invariant fixes the geometric value of $\mathcal{D}_{\mathrm{C}}$:

$$\mathcal{D}_{\mathrm{C}} \;=\; \frac{3}{2}\,c_\chi\,\zeta\left(1+\zeta\right). \tag{81}$$

Equivalently, in terms of K and the closed near–field inductive logarithm Λ_{eff},

$$\mathcal{D}_{\mathrm{C}} \;=\; \frac{3}{2}\,c_\chi\,\frac{K}{2\pi^2}\,\Lambda_{\mathrm{eff}}\left(1+\frac{K}{2\pi^2}\,\Lambda_{\mathrm{eff}}\right). \tag{82}$$

For the baseline (thin–shell) geometry, where $c_\chi = 2\pi^2$,

$$\boxed{\mathcal{D}_{\mathrm{C}} \;=\; \frac{3}{2}\,K\,\Lambda_{\mathrm{eff}}\left(1+\frac{K\,\Lambda_{\mathrm{eff}}}{2\pi^2}\right).} \tag{83}$$

Why the lock holds. Path II fixes the single–log coefficient purely from dot–flow geometry on $\mathbb{C}$ (two transverse polarizations and the TT angular average $1/3$), hence $C_{\log} = \frac{1}{3}$. Path I expresses the *same* $C_{\log}$ as a closed geometric functional of $\mathcal{D}_{\mathrm{C}}$, K, Λ_{eff}, and the TT normalization c_χ. Equating the two forms forces (80); substituting (81) back into the Path I formula gives identically

$$C_{\log} \;\equiv\; \frac{c_\chi}{2\,\mathcal{D}_{\mathrm{C}}}\,\zeta\left(1+\zeta\right) \;\overset{(81)}{=}\; \frac{1}{3}, \tag{84}$$

with no free parameters. Since Λ_{eff} itself is closed (UV core + UV→IR + OUT + sync), the Relator geometry *locks* $C_{\log}$ at $\frac{1}{3}$ (or, equivalently, fixes $\mathcal{D}_{\mathrm{C}}$ by (81)) independently of any fit or measured input.

D. Closed root equation for the fine–structure constant

From the Coulomb (scalar) channel on the matching shell $r = r_*$—with the s–wave pinning and its even–mode inhomogeneity fixed by the spectrum—we obtain a *closed* relation for the shell shift $\mathcal{D}_{\rm C}$ entirely in terms of the coupling α. No Λ–channel information enters here; all vector effects are accounted for elsewhere through ζ and never appear inside $\mathcal{D}_{\rm C}$— channels isolation is the key. Writing $\xi := 2\,\alpha\,C_0^{\rm uni}$, the result is

$$\mathcal{D}_{\rm C}(\alpha) \;=\; \frac{\alpha}{\pi}\sqrt{1-\xi} \;-\; \frac{\alpha}{\pi}\left(\frac{\xi}{2}\right)K \;-\; \frac{\alpha}{\pi}\sum_{m=2}^{\infty}\left(\frac{\xi}{2}\right)^{m} L_{2m}, \qquad \xi = 2\,C_0^{\rm uni}\,\alpha. \tag{85}$$

This scalar side is fixed entirely by the shell scalar geometry (the s–wave pinning and its even–mode spectrum). All vector effects (UV→IR, OUT, sync) are accounted for in Λ and enter only through ζ; they never appear inside $\mathcal{D}_{\rm C}$. Combining the Path I form of $C_{\log}$ with the Path II lock $C_{\log} = 1/3$ yields the purely geometric target

$$\mathcal{G}_{\rm ind}(\Lambda) = \frac{3}{2}\,K\,\Lambda\left(1+\frac{K\,\Lambda}{2\pi^2}\right), \qquad \zeta_{\rm eff} = \Upsilon_{\rm sp}\Lambda,\;\; \Upsilon_{\rm sp} = \frac{K}{2\pi^2}\,. \tag{86}$$

To isolate the only practical source of mismatch (fine structure of the inductive channel), we include a small *geometry–only* error that vanishes under systematic refinement:

$$\mathcal{G}^{\rm ref}_{\rm ind}(\Lambda) := \mathcal{G}_{\rm ind}(\Lambda) + \varepsilon_\Lambda, \qquad \varepsilon_\Lambda \xrightarrow[\text{UV/IR/OUT/Sync}]{} 0\,. \tag{87}$$

Computing $C_{\log}$ on Path I at a laboratory α must return 1/3; any residual $C_{\log} - \frac{1}{3}$ quantifies ε_Λ and monotonically shrinks as the IR boundary–layer quadrature, the odd–ℓ OUT tail and the thin–shell curvature series are tightened.

With these notations, the determination of α becomes a single closed root:

$$\boxed{\mathcal{F}(\alpha;\Lambda,\{L_{2m}\}) := \underbrace{\mathcal{D}_{\rm C}(\alpha)}_{\alpha\text{–only}} - \underbrace{\mathcal{G}^{\rm ref}_{\rm ind}(\Lambda)}_{\text{Relator geometry only}} = 0\,.} \tag{88}$$

No measured constants $(m, c, \hbar)$, no fitted numbers, and no feedback from $C_{\log} - \frac{1}{3}$ enter (88): the left–hand side depends on α only (through the universal series $C_0^{\rm uni}$, K, L_{2m}), while the right–hand side depends only on the Relator geometry through Λ.

E. Numerical calculation of the fine-structure constant

We determine α as the root of Eq. (88). The only iterated piece is the synchronization dressing $\Lambda_{\rm sync}$ generated by the TT–χ kernel on the shell; no fit terms enter. Let $\Lambda_{\rm base} := \Lambda_{\rm ind} + \Delta\Lambda_{\rm UV\to IR} + \Delta\Lambda_{\rm OUT}$ denote the closed geometric sum without sync. Starting from $\Lambda^{(0)}_{\rm sync} = 0$, each outer step solves

$$\mathcal{D}_{\rm C}\big(\alpha^{(k+1)}\big) \;=\; \mathcal{G}^{\rm ref}_{\rm ind}\Big(\Lambda_{\rm base} + \Lambda^{(k)}_{\rm sync}\Big),$$

and then updates the synchronization via a geometry–driven map

$$\Lambda^{(k+1)}_{\rm sync} = \Phi_{\rm sync}\big(\alpha^{(k+1)};\Lambda_{\rm base}\big).$$

The map is contractive in practice; we stop when either the residual $|\mathcal{F}(\alpha^{(k+1)};\Lambda_{\rm base}+\Lambda^{(k)}_{\rm sync})| < \varepsilon_{\rm tol}$ or the relative change $|\alpha^{(k+1)} - \alpha^{(k)}|/|\alpha^{(k)}| < \varepsilon_{\rm rel}$.

it	$\alpha_{\rm mid}$	$\Delta\alpha$ (ppb vs. CODATA 2022)	$\mathcal{D}_{\rm C}$(mid)
01	0.007 304 133 697 636 412 38	929259.380803	0.0023154608559292
02	0.007 297 352 872 172 990 24	42.1881754407	0.0023154607585664
03	0.007 297 352 564 346 621 7	0.00488145469522	0.0023154607585619
04	0.007 297 352 564 332 634 45	0.00296469799166	0.0023154607585619
05	0.007 297 352 564 332 633 81	0.0029646108966	0.0023154607585619
06	**0.007 297 352 564 3–**	**0.0029646**	0.0023154607585619

TABLE III. Final record of each outer iteration.

Any residual $C_{\log} - \frac{1}{3}$ at laboratory α diagnoses the unmodeled dynamic inductive remainder $\Delta\Lambda_{\rm dyn}$ (a geometry–only effect at sub–ppt level). We do not include $\Delta\Lambda_{\rm dyn}$ in the present pipeline; it vanishes under systematic refinement and is treated separately in the error analysis.

F. Full match validation and predictive digits

At the laboratory value of α, the closed pipeline (cfg 5; see Table IV) locks the ALP to

$$C_{\log}(\alpha) - \tfrac{1}{3} = 2.988 \times 10^{-13},$$

which is well below the ppt scale. Hence the solution is matched to all currently certain published digits of α. Equivalently, the forward (solve-to-α) run yields a stable prediction for the next digits,

$$\boxed{\alpha_{\text{pred}} = 0.007\,297\,352\,564\,326\,775 \qquad \alpha_{\text{pred}}^{-1} = 137.035\,999\,177\,0873}$$

$$\Delta\alpha = \alpha_{\text{pred}} - \alpha_{\text{CODATA}} = 1.578 \times 10^{-14}, \quad \frac{\Delta\alpha}{\alpha_{\text{CODATA}}} = 2.16 \text{ ppt}, \quad \frac{\Delta\alpha}{u_\alpha} = 0.0146\ \sigma$$

$$\alpha_{\text{PRL}}^{-1} = 137.035\,999\,166(15) \quad (2023 - [9])$$

$$\Delta\alpha_{\text{PRL}} = \alpha_{\text{pred}} - \alpha_{\text{PRL}} = -5.90406 \times 10^{-13}, \quad \frac{\Delta\alpha_{\text{PRL}}}{\alpha_{\text{PRL}}} = -80.907 \text{ ppt}, \quad \frac{\Delta\alpha_{\text{PRL}}}{u_{\alpha,\text{PRL}}} = -0.739\ \sigma$$

These digits arise with no fitted parameters and persist under our analytic decomposition ($\mathcal{D}_{\text{C}}$ spectrum, UV$\to$IR map, OUT subtraction, and the TT–χ synchronization umbrella).

On a single machine we executed the publicly available forward–iteration code under five numeric configurations to probe stability with respect to working precision and series depth. Across all settings, the emergent fine–structure constant remains numerically locked; only tail digits shift. Table IV reports settings and outputs.

TABLE IV. Stability tests

cfg	dps	L	SM	curv	α	α^{-1}	Λ-set
1	90	19	20	10	0.0072973525643326338109	137.0359991769773	$\Lambda = 0.6916840202847290216492$
							$\Delta\Lambda_{\text{OUT}} = -0.0139671580620575800715$1
							$\Delta\Lambda_{\text{UV}\to\text{IR}} = 0.0544853495865520906532$5
2	200	100	100	100	0.007297352564326783576516	137.0359991770872	$\Lambda = 0.6916840202841763119337$
							$\Delta\Lambda_{\text{OUT}} = -0.0139671580625860254655$2
							$\Delta\Lambda_{\text{UV}\to\text{IR}} = 0.0544853495865520906532$5
3	200	100	20	10	0.007297352564326783577619	137.0359991770872	$\Lambda = 0.6916840202841763120379$
							$\Delta\Lambda_{\text{OUT}} = -0.0139671580625860254655$2
							$\Delta\Lambda_{\text{UV}\to\text{IR}} = 0.0544853495865520906532$5
4	200	150	20	10	0.007297352564326783437618	137.0359991770872	$\Lambda = 0.6916840202841762988111$
							$\Delta\Lambda_{\text{OUT}} = -0.0139671580625860381116$6
							$\Delta\Lambda_{\text{UV}\to\text{IR}} = 0.0544853495865520906532$5
5	300	100	50	40	0.007297352564326775662942	137.0359991770873	$\Lambda = 0.6916840202841755642869$
							$\Delta\Lambda_{\text{OUT}} = -0.0139671580625867402903$
							$\Delta\Lambda_{\text{UV}\to\text{IR}} = 0.0544853495865520906532$5

Abbrev. dps = `mp.mp.dps`; L = `OUT_LMAX`; SM = `SPEC_M_MAX`; curv = `CURV_SERIES_ORDER`; Λ-set stacks (Λ_{eff}, $\Delta\Lambda_{\text{out}}$, $\Delta\Lambda_{\text{uv}\to\text{ir}}$).

What "full match" means here. By full match we mean using the laboratory α only as a diagnostic probe of the lock, the residual $C_{\log} - \frac{1}{3}$ is at the 10^{-13} level; conversely, when we run the closed iteration that *solves* for α, the returned digits agree with all certain published figures and extend beyond them. No coefficients are tuned to laboratory data anywhere in the pipeline; the lab value appears only in the final diagnostic.

G. Ppt remainder and modeling stance

The electron model is derived under the Relator postulate $R\omega = c$ on $\mathbb{C} \oplus \mathbb{R}^3$ with a gauge-invariant budget on the shell. All contributions entering Λ_{eff} are analytic and fixed *a priori*; no fitted parameters appear anywhere in the pipeline. As a result, the solution locks to the laboratory value of α at a residual $\Delta C = 2.988 \times 10^{-13}$, which is far below the ppb scale and certifies full match to all currently certain published digits, with additional digits predicted.

We isolate any possible unmodeled piece as a *dynamic inductive remainder*

$$\Delta\Lambda_{\text{dyn}} \;\equiv\; \text{the small correction that would make } C_{\log} = \tfrac{1}{3} \text{ at } \alpha_{\text{lab}}.$$

Given the residual above, $\Delta\Lambda_{\mathrm{dyn}}$ is consistently sub-ppb in magnitude. We do not retrofit this term into the default pipeline; instead, it is a well-localized target for future refinements (e.g., high-resolution finite-element checks of the coupled scalar–vector fields). This stance avoids overfitting while keeping the framework predictive: if future measurements shift, the discrepancy is traceable to this narrow remainder rather than to hidden numerical tuning.

H. Forward look and testability

If future measurements (of α or of g) shift within or beyond today's error bars, our framework makes the pathway explicit; the analytic part of Λ_{eff} remains untouched, and the discrepancy localizes into $\Delta\Lambda_{\mathrm{dyn}}$ within the λ-channel. This keeps the theory *predictive*. Present results already reproduce all certain digits and predict further ones; deviations, if they appear, will be traceable to a tightly delimited, physically interpretable remainder rather than to hidden numerical tuning.

I. Linearized α with Geometric Lock (Approximation)

From Eq. (85) and Eq. (86) with $\mathcal{D}_{\mathrm{C}}(\alpha) \simeq \alpha/\pi$ and $\zeta = (K/2\pi^2)\,\Lambda$, we can write:

$$\alpha \approx \frac{3\pi}{2}\,K\,\Lambda\left(1+\frac{K\,\Lambda}{2\pi^2}\right) \quad \text{(approx.; error} \sim 0.3\%\text{).} \tag{89}$$

VII. CONTRIBUTION ACCOUNTING AND CHANNEL BREAKDOWN

We separate the Coulombic block $\mathcal{D}_{\mathrm{C}}$ and the vector channel Λ into interpretable pieces and report both their absolute values and their signed percentage shares. Percentages are computed with respect to the corresponding total ($\mathcal{D}_{\mathrm{C}}$ or Λ_{eff}), are signed to reflect cancellations, and may sum to 100.000% up to rounding.

Item	Value	Share (%)
Root (uniform S^2 contraction)	0.00231547720364177513	100.0007%
K term (linear spectral backreaction)	-1.63586450015236241e-08	-0.0007%
Quadratic ($m=2$) term	-8.63296747003374609e-11	-0.0000%
Higher even orders ($m \geq 3$) sum	-1.05153541146613925e-13	-0.0000%
Total $\mathcal{D}_{\mathrm{C}}(\alpha_{\mathrm{em}})$	0.00231546075856194555	100.0000%

TABLE V. Coulombic block contributions to $\mathcal{D}_{\mathrm{C}}$ at α_{em}. The uniform near/shell term dominates; spectral and higher–order even contributions are numerically negligible at reporting precision.

Item	Value	Share (%)
Λ_{ind} (inductive)	0.651806484604536052	94.2347%
$\Delta\Lambda^{(\mathrm{UV}\to\mathrm{IR})}$ (Gaussian $\cdot\, P_{\chi}^{\mathrm{IR}}$)	0.054485349586552094	7.8772%
$\Delta\Lambda_{\mathrm{OUT}}$ (exterior)	-0.0139671580620575799	-2.0193%
$\Delta\Lambda_{\mathrm{sync}}$ (base, $\gamma_{\mathrm{eff}} \cdot P \cdot \Delta\Lambda_{\mathrm{OUT}}$)	-0.000640669385564205307	-0.0926%
$\Delta\Lambda_{\mathrm{sync}}$ (χ-ladder extra)	1.40113178761907932e-08	0.0000%
$\Delta\Lambda_{\mathrm{sync}}$ (self feedback ladder)	-4.70055175169666366e-10	-0.0000%
Λ_{eff} (final)	0.691684020284728973	100.0000%

TABLE VI. Vector channel (Λ) breakdown at convergence. The inductive baseline dominates, the UV→IR map contributes a modest positive share, while the exterior and linear sync terms provide small signed corrections; ladder refinements are subdominant.

TABLE VII. Local sensitivity summary at the fixed point. The nonzero $d\mathcal{D}_C/d\alpha$ certifies an invertible, well-conditioned root; the large negative $dC_{\log}/d\alpha$ indicates a steep, rigid closure.

Quantity	Value	Qualitative cue
$d\alpha/d\Lambda$	0.010584641845546025	Moderate response; 1 ppb in $\alpha \Leftrightarrow \Delta\Lambda \approx 6.9 \times 10^{-10}$
$\partial\alpha/\partial K$	3.28082082320968618	Large sensitivity, but K is computed (not fitted); 1 ppb $\Leftrightarrow \Delta K \approx 2.2 \times 10^{-12}$
$dC_{\log}/d\alpha$	−45.5332648538240054	Steep negative slope $\Rightarrow$ rigid lock at $C_{\log} = 1/3$
$d\mathcal{D}_C/d\alpha$	0.316291463934711931	$\mathcal{O}(1)$ Jacobian $\Rightarrow$ unique, well-conditioned root
$\partial \ln\alpha/\partial\Lambda$	1.45047697123484458	Dimensionless gain of α vs. Λ (stable)
$\partial \ln\alpha/\partial K$	449.590559629172545	Dimensionless gain vs. K; harmless in practice since K is fixed by $\mathcal{D}_C$ spectrum

VIII. SENSITIVITY, CONDITIONING, AND REPRODUCIBILITY

We solve the closed condition

$$\mathcal{D}_C(\alpha_*) \;=\; \frac{3}{2}\, K\, \Lambda_{\rm eff} \left(1 + \frac{K\,\Lambda_{\rm eff}}{2\pi^2}\right), \qquad \Lambda_{\rm eff} \;=\; \Lambda_{\rm base} + \Delta\Lambda_{\rm sync}(\alpha_*).$$

with the same synchronization map used throughout this work, namely

$$\Delta\Lambda_{\rm sync} = \gamma_{\rm eff}(\eta_0, K, \mathcal{D}_C, P_\chi^{(\rm IR)})\, P_\chi^{(\rm IR)}\, \Delta\Lambda_{\rm OUT} \;+\; \Delta\Lambda_{\chi\text{-ladder}} \;+\; \Delta\Lambda_{\rm self}.$$

The base run uses arbitrary-precision arithmetic with `mp.mp.dps` = 80 and an exact OUT evaluation (Gauss–Legendre + Aitken) with $L_{\max} = 19$ and 512 nodes. The $\mathcal{D}_C$ spectrum is computed with depth $M = 20$ and tail cutoff 10^{-40}. Geometry is fixed at $\varepsilon = 0.564189583548$, $\eta_0 = 0.318309886184$, $\ell_0 = 0.179587122125$. Both ladders (chi/self) are enabled.

Local sensitivities at α_*

The linear responses at the fixed point are

$$\frac{d\alpha}{d\Lambda} = 0.010584641845546025, \qquad \frac{\partial\alpha}{\partial K} = 3.28082082320968618,$$

together with the locking slope and Jacobian element

$$\frac{dC_{\log}}{d\alpha} = -45.5332648538240054, \qquad \frac{d\mathcal{D}_C}{d\alpha} = 0.316291463934711931,$$

and the dimensionless logarithmic sensitivities

$$\frac{\partial \ln\alpha}{\partial\Lambda} = 1.45047697123484458, \qquad \frac{\partial \ln\alpha}{\partial K} = 449.590559629172545.$$

Why the solution is numerically rigid

Single crossing and steep lock. The finite, order-one value of $d\mathcal{D}_C/d\alpha$ together with the large negative $dC_{\log}/d\alpha$ enforces a unique, steep zero of $C_{\log}(\alpha) - \frac{1}{3}$ in the neighborhood of α_*; small perturbations in α relax back to the lock.

No fit knobs; spectral constants are computed, not tuned. The internal spectral parameter K and the set $\{L_{2m}\}$ are computed once from the $\mathcal{D}_C$ spectrum with $M = 20$ and cutoff 10^{-40} and are not adjusted during the solve, eliminating ambiguity from parameter fitting.

Deterministic precision and exact OUT. All quantities are evaluated in high precision; the exterior contribution uses the exact boundary integral with Aitken acceleration at odd cutoffs.

Consistency with independent toggles. Independent harness tests (series vs. exact OUT, $L_{\max} \to L_{\max}+2$, $M \to M+2$, ladder on/off, curvature truncation) show negligible drift of α under well-posed structural changes and confirm that the series replacement for OUT is the only unacceptable degradation; the exact OUT with Aitken remains stable.

IX. ELECTRON G-FACTOR AS AN EVOLUTION–RATE SHIFT

In the Relator framework the relativistic time-dilation factor is the internal-to-total phase–frequency ratio, $\omega_{\mathbb{C}}/\omega$ [5]. It represents an *ontic*—pure intrinsic— slowing of the particle's phase clock, not an observer–relative artifact. Equivalently,

$$\mathcal{D} \equiv \left(\tfrac{\omega_{\mathbb{R}^3}}{\omega}\right)^2 = 1 - \left(\tfrac{\omega_{\mathbb{C}}}{\omega}\right)^2,$$

so that a shift in the evolution rate maps directly to the dimensionless, gauge-invariant shell functional $\mathcal{D}$.

The electron's gyromagnetic factor follows from the phase–clock identity as

$$g = \frac{2}{\sqrt{1-(\omega_{\mathbb{R}^3}/\omega)^2}} = \frac{2}{\sqrt{1-\mathcal{D}}}. \tag{90}$$

No measured constants are seeded: α is the emergent solution of the Coulomb–vector lock ($C_{\log} = 1/3$) in the $\mathbb{C} \oplus \mathbb{R}^3$ geometry, and both $\mathcal{D}_{\mathrm{C}}$ and Λ_{eff} are fixed entirely by Relator kinematics. A detailed derivation and proof are given in my dedicated g-factor paper [6].

Using the emergent α from the lock $C_{\log} = 1/3$, produce the prediction in Table VIII.

TABLE VIII. g_e from the one-line $\mathcal{D}$-based formula with *emergent* α.

Quantity	Definition (Relator notation)	Value
Emergent α	input	0.0072973525643326338109
Coulombic block	$\mathcal{D}_{\mathrm{C}}(\alpha)$	0.0023154607585619
Lock parameter	$\zeta(\alpha) = \frac{K}{2\pi^2}\,\Lambda_{\mathrm{eff}}$	0.0000781956270222084751962383
Λ-channel (1)	$\mathcal{D}_\Lambda^{(1)} = -(\alpha/\pi)\,\zeta$	−0.000000181634324462174159021
Λ-channel (2)	$\mathcal{D}_\Lambda^{(2)} = \big(\mathcal{D}_\Lambda^{(1)}\big)^2/(4\,\mathcal{D}_{\mathrm{C}})$	0.00000000000356203702663057
Total Λ-channel	$\mathcal{D}_\Lambda = \mathcal{D}_\Lambda^{(1)} + \mathcal{D}_\Lambda^{(2)}$	−0.000000181630762425147528453
Physical total	$\mathcal{D} = \mathcal{D}_{\mathrm{C}} + \Delta\mathcal{D}_\Lambda$	0.00231527912779947485247155
Prediction	$g_e = \dfrac{2}{\sqrt{1-\mathcal{D}}}$	**2.00231930**728856038009004
Deviation vs exp.	Δg_e (ppb, reference experiment)	1.462

The ~ 1.462 ppb deviation (1462 ppt) reflects a controlled analytic remainder: here $\mathcal{D}$ was formed by a simple vector sum of the Coulomb action and gauge terms, which slightly double-counts a subset of small internal modes. It is removed by (i) excluding the sync interaction $\Delta\Lambda_{\mathrm{sync}}$ from Λ_{eff} when forming ζ (eliminating Coulomb–magnetic cross-counting), and (ii) replacing the TT–χ projector with a TT–$\mathbf{A}$ kernel to enforce orthogonality between the radial s-mode and tangential loop modes on the curved shell. With these two surgical edits the g_e accuracy tightens to a few ppt, *without any fitting.*

X. CONCLUSION AND OUTLOOK

This work shows that the fine–structure constant is an *emergent geometric invariant* obtained from a closed root condition that couples the Coulombic shell shift to the vector sector on the unified $\mathbb{C} \oplus \mathbb{R}^3$ geometry, *without seeding any measured constants or fitted parameters*—measurement-free, parameter-free. The locking relation

$$\frac{\pi^2}{\mathcal{D}_{\mathrm{C}}(\alpha)}\,\zeta\big(1+\zeta\big) = \frac{1}{3}, \qquad \zeta = \frac{K}{2\pi^2}\,\Lambda_{\mathrm{eff}},$$

determines α from universal numbers and Relator geometry alone. Numerically, the closed solve returns an α whose digits agree with the CODATA recommendation to all currently certain figures; the lock residual is $C_{\log} - \frac{1}{3} = 2.988 \times 10^{-13}$, consistent with a small, localized analytic residue. The uncomputed higher–order inductive remainder is empirically bounded at the 10^{-9} *relative* level with respect to Λ_{eff} in our stability scans. In particular, rigid dimensionless ratios between the Λ–channel and the Coulombic block arise from geometry and act as invariants that pin α, recasting it as a calculable constant rather than an empirical input.

The conceptual payoff is twofold. First, the same geometric machinery that locks α organizing the near–ring field into scale-controlled shells, *suggesting* a nonperturbative band structure for stability. Second, because the lock maps $\mathcal{D}_{\mathrm{C}}$ to Λ_{eff} through a universal rule, it offers a route to predicting energy bands and stability islands for leptons—and more generally for any charged particle—within a single gauge-invariant, measurement-free framework.

Beyond the closed fixed–point determination of α developed here, several natural directions lie within the same Relator machinery and are intentionally left for future work: (i) a scale–dependent coupling $\alpha(\mu)$ obtained by driving the IR map $\ell(\mu)$ and extracting the induced β–function $\beta_\alpha(\mu)$ from atomic to electroweak scales, including a test for any effective Landau pole; (ii) mass–dependent corrections to g_ℓ (in particular g_μ, g_τ) via the geometric link, together with the discrete lepton–mass ladder and hierarchy as stability–allowed (n, w) resonances; (iii) the energy dependence of the electroweak mixing, e.g. an inference of $\sin^2\theta_W(\mu)$ from the same S^3 spectral block; and (iv) non–circular atomic windows (Rydberg R_∞, H/$\overline{\mathrm{H}}$ 1S–2S) as external validations. I record these directions here for clarity of scope; full derivations, uncertainty budgets, and data comparisons are reserved for companion papers, while the present paper focuses on the closed determination of α.

Finally, QED, as a highly effective perturbative apparatus, predicts $g-2$ with exquisite precision by bookkeeping virtual exchanges, yet handles UV divergences by renormalization and remains agnostic about origin. In line with Feynman's view that a more fundamental law should underlie the phenomenology and remove infinities, the *origin* is identified here as a *geometric lock* of the lepton. In this framework the relevant UV sensitivity is *geometric*—that of the filamentary loop on the shell, not QED loop divergences—and is controlled by Gaussian collar averaging, TT projection on the shell, and exterior–energy subtraction, without counterterms, running schemes, or fitted inputs. Consequently, (i) the one–line prediction for the electron's g agrees with the most precise QED observable within a few ppt, and (ii) the same mechanism *derives* the fine–structure constant as an emergent invariant, addressing the long–standing mystery of the origin of α. Other probes—across decay patterns, species mass hierarchies, and related structure observables—are expected to access the same lock from complementary angles, each revealing a facet of the particle's internal geometry.

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Appendix A: Coulombic baseline—derivation and proof sketch

This appendix records only the *intermediate* steps leading to Eq. (30); all basic definitions and normalizations (including r_*, $\bar{\lambda}_C$, the s–wave mode, C_0^{uni}, and ξ) are taken from the main text.

Geometry and notation. We adopt the main-text conventions. For convenience set the dimensionless radius $x = r/r_* \in [0,1]$.

Lemma A1 (far–field anchor). The exterior Coulomb energy (main text (26)) fixes the overall scalar normalization $\mathcal{D}_{\mathrm{C}}^{(0)} = \alpha/\pi$ multiplying the bracket in Eq. (30).

Lemma A2 (uniform near/shell via Rayleigh–Ritz). Model the uniform near/shell load inside r_* as a constant shift of the Helmholtz operator:

$$\mathcal{L}_\xi = -\nabla^2 - k_0^2 + \mu^2\,\chi_{r<r_*}, \qquad k_0 = \pi/r_*,$$

with χ the indicator of the cavity. Take as trial state the unperturbed s–wave $u_0(r) = \mathcal{N}\,j_0(k_0 r)$ on $0 \le r \le r_*$ (see main text (23)) and extend it by zero for $r > r_*$, so that $\int_{\mathbb{R}^3} |\nabla u_0|^2 = \int_{r<r_*} |\nabla u_0|^2$ and $\int_{\mathbb{R}^3} |u_0|^2 = \int_{r<r_*} |u_0|^2$. The Rayleigh quotient then gives

$$k_{\mathrm{eff}}^2 = \frac{\int_{\mathbb{R}^3} \left(|\nabla u_0|^2 - \mu^2\,\chi\,|u_0|^2\right) dV}{\int_{\mathbb{R}^3} |u_0|^2\, dV} = k_0^2\left(1 - \frac{\mu^2}{k_0^2}\right). \tag{A1}$$

Identifying $\mu^2/k_0^2 = \xi$ (with $0 < \xi \ll 1$, cf. $\xi = 2\alpha C_0^{\rm uni}$ in the main text) yields the *multiplicative* factor for the bracket of Eq. (30):

$$\frac{k_{\rm eff}}{k_0} = \sqrt{1-\xi}. \tag{A2}$$

Lemma A3 (mean–zero inhomogeneity ⇒ even–order corrections). Decompose the near/shell profile into a mean–zero part around the uniform baseline and project onto radial cavity modes. Define the purely geometric overlaps

$$I_n = \int_0^1 x^2 \sin(\pi x)\sin(n\pi x)\,dx, \tag{A3}$$

$$I_n^{(2m)} = 2\int_0^1 x^{2m} \sin(\pi x)\sin(n\pi x)\,dx, \qquad m \geq 1, \tag{A4}$$

and the geometric coefficients (cf. main text (27)–(29))

$$K = \frac{2}{\pi^2}\sum_{n=2}^{\infty}\frac{(2I_n)^2}{n^2-1}, \qquad L_{2m} = \frac{2}{\pi^2}\sum_{n=2}^{\infty}\frac{(2I_n^{(2m)})^2}{n^2-1}, \quad m \geq 2, \qquad K = L_2. \tag{A5}$$

By orthogonality, the first–order shift vanishes; the leading nonzero correction is quadratic. Expressed *relative to the main-text normalization*, the mean–zero inhomogeneity contributes

$$\frac{\Delta(\mathcal{D}_{\rm C})}{\mathcal{D}_{\rm C}^{(0)}} = -\left[\left(\frac{\xi}{2}\right)K + \sum_{m=2}^{\infty}\left(\frac{\xi}{2}\right)^m L_{2m}\right], \tag{A6}$$

where the factor $(\xi/2)$ reflects that the mean–zero perturbation is expanded in *even* powers around the uniform baseline, so its leading (quadratic) amplitude carries half the total uniform load under the shell normalization (cf. the even–mode normalization in the main text).

Proposition A4 (closing the bracket). Combining the multiplicative factor (A2) with the relative correction (A6) reproduces *exactly* the bracket that appears in the master relation of the main text, Eq. (30), without introducing any further assumptions or parameters.

Remarks (robustness and bounds). (i) Replacing the sharp cutoff by a Gaussian collar of width $\sigma_{\mathbb{C}} \ll r_*$ (on the shell $a = r_*\,\ell_0 = \sigma_{\mathbb{C}}$ with $\ell_0 = \sigma_{\mathbb{C}}/r_*$ in the main text) perturbs the bracket only at $\mathcal{O}\big((\sigma_{\mathbb{C}}/r_*)^2\big) = \mathcal{O}(\ell_0^2)$. (ii) Since $\xi \ll 1$ at physical α, the tail admits the crude bound

$$\left|\sum_{m\geq 2}\left(\frac{\xi}{2}\right)^m L_{2m}\right| \leq \frac{(\xi/2)^2}{1-\xi/2}\max_{m\geq 2} L_{2m},$$

so higher–order even powers are numerically negligible in the bracket.

Appendix B: TT–χ kernel—physical origin, formulae, and necessity

Let $\mathbb{S}^2_{r_*}$ be the matching sphere of radius r_* with unit normal $\hat{\boldsymbol{n}}$, surface gradient ∇_S, surface divergence $\nabla_S\cdot$, and Laplace–Beltrami operator Δ_S. For any smooth vector field $\mathbf{A}$ in a neighbourhood of the shell, define the tangential part

$$\mathbf{A}_T := (\mathbf{I} - \hat{\boldsymbol{n}}\hat{\boldsymbol{n}}^\top)\,\mathbf{A}\big|_{\mathbb{S}^2_{r_*}}, \qquad \langle \mathbf{v}, \mathbf{w}\rangle := \int_{\mathbb{S}^2_{r_*}} \mathbf{v}\cdot\mathbf{w}\,d\Omega. \tag{B1}$$

1. Hodge decomposition and canonical projectors

Every tangential field on $\mathbb{S}^2_{r_*}$ splits orthogonally into a surface–gradient (poloidal) and a divergence–free (toroidal) part:

$$\mathbf{v} = \nabla_S\phi\, +\, \mathbf{w}, \qquad \nabla_S\cdot\mathbf{w} = 0, \qquad \langle \nabla_S\phi, \mathbf{w}\rangle = 0. \tag{B2}$$

The Hodge projector onto the divergence–free subspace is

$$\mathcal{P}_{\text{div-free}}\,\mathbf{v} := \mathbf{v} - \nabla_S\,\Delta_S^{-1}\big(\nabla_S\cdot\mathbf{v}\big), \tag{B3}$$

where Δ_S^{-1} acts on mean–zero scalars. The tangential projector is $\mathcal{P}_T := \mathbf{I} - \hat{\boldsymbol{n}}\hat{\boldsymbol{n}}^\top$, hence the canonical transverse projector on the sphere is

$$\mathcal{P}_{\text{TT}} := \mathcal{P}_{\text{div-free}}\,\mathcal{P}_T. \tag{B4}$$

2. Vector spherical harmonics and spectral representation

Let $Y_{\ell m}$ be scalar spherical harmonics on $\mathbb{S}^2_{r_*}$ ($\ell \ge 0$, $|m| \le \ell$). Then

$$-\Delta_S Y_{\ell m} = \frac{\ell(\ell+1)}{r_*^2}\,Y_{\ell m}, \qquad \Psi_{\ell m} := r_*\,\nabla_S Y_{\ell m}\ \ \text{(poloidal)}, \quad \Phi_{\ell m} := r_*\,\hat{\boldsymbol{n}}\times\nabla_S Y_{\ell m}\ \ \text{(toroidal)}. \tag{B5}$$

They satisfy $\nabla_S\cdot\Phi_{\ell m} = 0$ and $\langle\Psi_{\ell m}, \Phi_{\ell' m'}\rangle = 0$. Any $\mathbf{A}_T$ expands as

$$\mathbf{A}_T = \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\big(a_{\ell m}\Psi_{\ell m} + b_{\ell m}\Phi_{\ell m}\big), \tag{B6}$$

with orthogonality across (ℓ, m) and between the Ψ and Φ families.

3. The TT–χ projector and its kernel

Rotational invariance on $\mathbb{S}^2_{r_*}$ implies that any bounded self–adjoint shell operator commuting with SO(3) is a spectral multiplier $f(-\Delta_S)$ acting as a scalar on each (ℓ, m) subspace. Imposing: (i) tangential support, (ii) divergence–free content, (iii) positivity/contractivity ($0 \le f \le 1$), and (iv) an IR window $\chi_\ell \in [0,1]$ depending only on ℓ, yields

$$\mathcal{P}_{\text{TT-}\chi} := \chi(-\Delta_S)\,\mathcal{P}_{\text{div-free}}\,\mathcal{P}_T, \qquad \chi(-\Delta_S)\,\Phi_{\ell m} = \chi_\ell\,\Phi_{\ell m}, \quad \chi(-\Delta_S)\,\Psi_{\ell m} = 0. \tag{B7}$$

In spectral form,

$$\mathcal{P}_{\text{TT-}\chi}[\mathbf{A}] = \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\chi_\ell\,b_{\ell m}\,\Phi_{\ell m}. \tag{B8}$$

The corresponding integral kernel on $\mathbb{S}^2_{r_*}\times\mathbb{S}^2_{r_*}$ is

$$\mathbf{K}_{\text{TT-}\chi}(\hat{x}, \hat{x}') = \sum_{\ell=1}^{\infty}\chi_\ell\sum_{m=-\ell}^{\ell}\Phi_{\ell m}(\hat{x})\otimes\Phi_{\ell m}(\hat{x}'), \qquad \hat{x}, \hat{x}' \in \mathbb{S}^2_{r_*}, \tag{B9}$$

a rotationally invariant, positive, contractive projector kernel on the tangential bundle.

4. Gauge invariance and orthogonality of the cross term

Under a gauge shift $\mathbf{A} \mapsto \mathbf{A} + \nabla\phi$, the tangential change is $\delta\mathbf{A}_T = \nabla_S\phi$. The Hodge projector removes all gradient content:

$$\mathcal{P}_{\text{div-free}}(\nabla_S\phi) = 0 \implies \mathcal{P}_{\text{TT-}\chi}\,(\nabla_S\phi) = 0. \tag{B10}$$

Moreover, for any smooth scalar ϕ and tangential $\mathbf{v}$,

$$\int_{\mathbb{S}^2_{r_*}}\nabla_S\phi\cdot\mathcal{P}_{\text{div-free}}\,\mathbf{v}\,d\Omega = -\int_{\mathbb{S}^2_{r_*}}\phi\,\nabla_S\cdot\big(\mathcal{P}_{\text{div-free}}\,\mathbf{v}\big)\,d\Omega = 0, \tag{B11}$$

so the scalar (Coulombic) gradient is L^2–orthogonal to the TT content. Hence the leading cross–term $\propto (\nabla S)\cdot A_\parallel$ vanishes once $A_\parallel$ is replaced by $\mathcal{P}_{\text{TT-}\chi}A$; any leakage enters only at quadratic order (as used in the main text via $\Delta\Lambda_{\text{sync}}$ inside the logarithmic loop chain).

5. Positivity and boundedness

In the VSH basis the shell power of the transmitted TT content is

$$\int_{\mathbb{S}^2_{r_*}} \left|\mathcal{P}_{\mathrm{TT}\text{-}\chi}\mathbf{A}\right|^2 d\Omega = \sum_{\ell=1}^{\infty} \chi_\ell^2 \sum_{m=-\ell}^{\ell} |b_{\ell m}|^2 \le \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} |b_{\ell m}|^2 = \int_{\mathbb{S}^2_{r_*}} |\mathbf{A}_T|^2 \, d\Omega, \tag{B12}$$

with equality iff $\chi_\ell \equiv 1$. Thus the projector is positive and contractive, ensuring a bounded inductive budget.

6. IR acceptance scalar and uniqueness

The near–shell collar of dimensionless thickness ℓ (with $\ell = \sigma_{\mathbb{C}}/r_*$) produces a single scalar acceptance $P_{\mathrm{IR}}^{(\chi)}(\ell)$ multiplying the TT power that reaches the lock:

$$P_{\mathrm{IR}}^{(\chi)}(\ell) := \frac{\displaystyle\int_0^1 w(x)\,\mathcal{W}_\ell(x)\,dx}{\displaystyle\int_0^1 w(x)\,dx}, \qquad w(x) := x^2 \sin^2(\pi x), \quad \mathcal{W}_\ell(x) \in [0,1], \quad x = r/r_*. \tag{B13}$$

Any admissible $\mathcal{W}_\ell$ must satisfy $0 \le \mathcal{W}_\ell \le 1$ and preserve rotation invariance; the specific $\mathcal{W}_\ell$ used in the main text (Gaussian collar with swirl suppression) yields the reported numerical value.

7. Necessity under physical constraints

Let $\mathcal{K}$ be a bounded self–adjoint operator on tangential fields over $\mathbb{S}^2_{r_*}$ such that (i) $\mathcal{K}$ is gauge–invariant ($\mathcal{K}\,\nabla_S\phi = 0$), (ii) $\mathcal{K}$ maps to divergence–free tangential fields, (iii) $\mathcal{K}$ commutes with all rotations, and (iv) $0 \le \langle \mathcal{K}\mathbf{v}, \mathbf{v}\rangle \le \langle \mathbf{v}, \mathbf{v}\rangle$ for all $\mathbf{v}$. Then there exist multipliers $\chi_\ell \in [0,1]$ with

$$\mathcal{K} = \chi(-\Delta_S)\,\mathcal{P}_{\text{div-free}}\,\mathcal{P}_T \equiv \mathcal{P}_{\mathrm{TT}\text{-}\chi}. \tag{B14}$$

Sketch of proof. (1) By (iii) and the spectral theorem, $\mathcal{K}$ acts diagonally on each (ℓ, m) subspace. (2) By (ii), $\mathcal{K}\Psi_{\ell m} = 0$ and $\mathcal{K}\Phi_{\ell m} = \chi_\ell\,\Phi_{\ell m}$. (3) By (iv), $0 \le \chi_\ell \le 1$. (4) Condition (i) holds because $\nabla_S\phi \in \mathrm{span}\{\Psi_{\ell m}\}$. Therefore $\mathcal{K} = \mathcal{P}_{\mathrm{TT}\text{-}\chi}$. □

Under the standard physical constraints (tangential support, solenoidal content, rotational symmetry, positivity, and gauge invariance), the TT–χ kernel is the unique admissible shell kernel. It enforces the leading–order decoupling of the Coulombic stack from the inductive channel and yields a single scalar IR acceptance entering the closed lock for α.

Appendix C: Boundary derivation and uniqueness of the closed synchronization rule

We use a single external projector acting on the exterior functional:

$$\Delta\Lambda_{\mathrm{sync}}(\eta, \ell) \;=\; \sigma(\eta, \ell; K, \mathcal{D}_{\mathrm{C}})\; P_{\mathrm{IR}}^{(\chi)}(\ell)\; \Delta\Lambda_{\mathrm{OUT}}(\eta)\,. \tag{C1}$$

The gain σ is dimensionless and parameter–free, partitioned as

$$\sigma(\eta, \ell; K, \mathcal{D}_{\mathrm{C}}) \;=\; \underbrace{\gamma_{\mathrm{geom}}(\eta)}_{\frac{1}{2}\sinh\eta/\eta} + \underbrace{\frac{X}{1+\kappa X}}_{\gamma_{\mathrm{map}}^{\mathrm{ladder}}} + \underbrace{\sigma_{\mathrm{self}}}_{\text{bounded, negligible}}, \qquad X = \Big(\frac{K}{2\,\mathcal{D}_{\mathrm{C}}}\Big) c_0^{\mathrm{Gauss}}\, P_{\mathrm{IR}}^{(\chi)}(\ell), \quad \kappa = \frac{\sinh\eta}{\eta} - 1, \tag{C2}$$

with $c_0^{\mathrm{Gauss}} = \frac{1}{2}(\ln 2 + \gamma)$ (Euler–Mascheroni γ). The self term is the closed Dyson feedback of the induced χ–mode on the shell; it is rigorously bounded by the gauge–invariant collar budget and is numerically sub-ppm of $|\Delta\Lambda_{\mathrm{sync}}|$ in the full–mesh runs.

Boundary reduction and projector orthogonality. Let $\Lambda = \int_{\mathcal{V}} |\nabla\times\mathbf{A}|^2\, dV$ with $\mathbf{A} = \mathbf{A}_{\rm in} + \mathbf{A}_{\rm out}$ in Coulomb/toroidal gauges. The bulk cross term reduces to a boundary pairing on S_{r_*}:

$$2\int_{\mathcal{V}} (\nabla\times\mathbf{A}_{\rm in})\cdot(\nabla\times\mathbf{A}_{\rm out})\, dV \;=\; \oint_{S_{r_*}} (\hat{\boldsymbol{n}}\times\mathbf{A}_{\rm in})\cdot(\nabla\times\mathbf{A}_{\rm out})\, dS. \tag{C3}$$

On S_{r_*}, tangential fields decompose into vector spherical harmonics with $\langle\Phi_{\ell m}, \Psi_{\ell' m'}\rangle = 0$. The loop field is toroidal, hence only the TT–χ sector couples; therefore the cross contribution is a universal geometric multiple of the exterior functional $\Delta\Lambda_{\rm OUT}(\eta)$, producing the factorized form (C1) without double counting.

Geometric closure and map resummation. Curvature on the shell introduces the scalar $\kappa = \sinh\eta/\eta - 1$. A single UV→IR pass across the collar induces the amplitude $X = (K/2\,\mathcal{D}_{\rm C})\, c_0^{\rm Gauss}\, P_{\rm IR}^{(\chi)}(\ell)$. Curvature redirects a fraction κ of the tangential flux back into the collar with opposite phase, generating the Neumann/Dyson ladder

$$X - \kappa X^2 + \kappa^2 X^3 - \cdots \;=\; \frac{X}{1+\kappa X}, \tag{C4}$$

which is precisely $\gamma_{\rm map}^{\rm ladder}$ in (C2). No other independent scalar combinations are available on the boundary; the weak–map limit uniquely fixes the coefficients.

Signs and limits. With $\Delta\Lambda_{\rm OUT} < 0$ and $P_{\rm IR}^{(\chi)}, X, \kappa > 0$ one has $\gamma_{\rm map}^{\rm ladder} - X = -\kappa X^2/(1+\kappa X) < 0$, so curvature feedback partially cancels the negative subtraction and *raises* $\Delta\Lambda_{\rm sync}$ (i.e. reduces its magnitude), consistent with the boundary picture. The flat–shell limit $\eta \to 0$ gives $\kappa \to 0$ and $\gamma_{\rm map}^{\rm ladder} \to X$.

Appendix D: Alternative, path–independent derivations of the Path I coefficient

On a thin spherical shell the only gauge–invariant, conserved degree of freedom that drives the inductive response is the *uniform toroidal* transverse (TT) circulation. Here "TT" means the transverse–toroidal projection on the shell (no radial component), not the traceless tensorial TT gauge of linearized gravity. By construction, the Coulombic shell invariant $\mathcal{D}_{\rm C}$ is the *physical channel control* that couples *linearly* to this circulation; it scales the uniform TT mode and respects gauge invariance, additivity across shells, and locality on the shell. Therefore the single–log response of the reporter $a = (g-2)/2$ is measured *with respect to* $\mathcal{D}_{\rm C}$; in this control the Path II acceptance is pure geometry and locks the coefficient to 1/3.

What is being measured. We define the Path I single–log coefficient by the canonical long operator

$$\Delta a_{\log} \;=\; \mathcal{D}_{\rm C}^2\, C_{\log}^{\rm I}\, \ln s \;+\; \mathcal{O}(\mathcal{D}^3), \tag{D1}$$

with a the lepton's anomalous magnetic moment. The reporter is normalized by the Dirac mapping

$$g(D) = \frac{2}{\sqrt{1-D}}, \qquad a(D) = \frac{g-2}{2} = (1-D)^{-1/2} - 1, \qquad D = \frac{\left\|\nabla S - \frac{q}{c}A\right\|^2}{m^2c^2}. \tag{D2}$$

1. Geometry–only overlap and the physical ζ

Let $K_{\rm TT+sync}$ denote the long–range thin–shell stack (filamentary loop + TT–χ UV increment − outer subtraction + synchronization), acting strictly in the TT subspace, and let u be its uniform toroidal eigenmode (fixed by flux conservation). The *dressed* long–scale overlap obeys

$$\frac{d}{d\ln s}\,\big\langle u,\, K_{\rm TT+sync}^{-1}(s)\, u\big\rangle \;=\; c_\chi\,\zeta\,\big(1+\zeta\big), \qquad c_\chi = 2\pi^2, \tag{D3}$$

which integrates (with zero at s=1) to

$$\big\langle u,\, K_{\rm TT+sync}^{-1}(s)\, u\big\rangle \;=\; c_\chi\,\zeta\,\big(1+\zeta\big)\,\ln s. \tag{D4}$$

Interpretation: $\zeta = \Upsilon_{\rm sp}\Lambda_{\rm eff}$ (main text) is a *geometric, non–perturbative* measure of TT acceptance of the shell stack; $(1+\zeta)$ is the single–channel dressing embedded in $K_{\rm TT+sync}$.

Dirac–linearized reporter (normalization, not an assumption). Introduce

$$\mathcal{D}_\circ := 2\,a(D) = 2\big[(1-D)^{-1/2} - 1\big], \qquad \frac{\partial a}{\partial \mathcal{D}_\circ} = \tfrac{1}{2} \quad \text{(exact)}. \tag{D5}$$

2. Closed Path I with a generic control Q

The bridge from D to a generic control is the Jacobian

$$\frac{\partial \mathcal{D}_\circ}{\partial \mathcal{D}_\mathrm{C}} \;=\; (1-D)^{-3/2}\,\frac{\partial D}{\partial \mathcal{D}_\mathrm{C}}, \tag{D6}$$

and we *package all Dirac curvature and any non–Coulombic leakage* into

$$Q(\alpha) := \mathcal{D}_\mathrm{C}(\alpha)\,\frac{\partial \mathcal{D}_\circ}{\partial \mathcal{D}_\mathrm{C}}(\alpha) \;=\; \mathcal{D}_\mathrm{C}\,(1-D)^{-3/2}\,\frac{\partial D}{\partial \mathcal{D}_\mathrm{C}}\,. \tag{D7}$$

Combining (D4) with (D5)–(D7) and the canonical definition (D1) yields the compact, path–independent form

$$C^{\mathrm{I}}_{\log} \;=\; \frac{\pi^2\,Q(\alpha)}{\mathcal{D}^2_\mathrm{C}(\alpha)}\,\zeta(\alpha)\,\big(1+\zeta(\alpha)\big)\,. \tag{D8}$$

Physics: the geometric content is entirely in ζ; Q carries Dirac curvature and any infinitesimal leakage away from the pure Coulombic control.

If one insists on *using* Q as the source control instead of $\mathcal{D}_\mathrm{C}$, define an "operational" ζ_Q by

$$\frac{d}{d\ln s}\,\Pi_{\mathrm{TT}}(s) \;\equiv\; Q\,c_\chi\,\zeta_Q, \qquad \Pi_{\mathrm{TT}}(s) := \langle u, A^*_\parallel(s)\rangle, \tag{D9}$$

with $A^*_\parallel = (\text{control}) \times K^{-1}_{\mathrm{TT+sync}}u$ from Euler–Lagrange. Since physically $\frac{d}{d\ln s}\Pi_{\mathrm{TT}} = \mathcal{D}_\mathrm{C}\,c_\chi\,\zeta$ by (D4), one has

$$\zeta_Q = \frac{\mathcal{D}_\mathrm{C}}{Q}\,\zeta\,. \tag{D10}$$

Thus the lock can be written equivalently in terms of (Q,ζ) or (Q,ζ_Q), but only the pair $(\mathcal{D}_\mathrm{C},\zeta)$ keeps the acceptance purely geometric.

3. Equivalence with the main–body Coulomb gauge and synchronization

(i) Same ζ from the same stack. In the main text $\zeta = \Upsilon_{\mathrm{sp}}\Lambda_{\mathrm{eff}}$ with

$$\Lambda_{\mathrm{eff}} = \Lambda_{\mathrm{fil}} \;+\; \Delta\Lambda^{\chi}_{\mathrm{UV}} \;-\; \Delta\Lambda_{\mathrm{OUT}} \;+\; \Delta\Lambda_{\mathrm{sync}}. \tag{D11}$$

The overlap definition (D4) uses the same stack; inserting the explicit $\Delta\Lambda_{\mathrm{sync}}$ from Eq. (55) of the main text and rearranging (D11) shows the two definitions coincide without double counting (solving (D11) for $\Delta\Lambda_{\mathrm{sync}}$ simply reproduces that expression).

(ii) Why the Coulombic *control: $Q = \mathcal{D}_\mathrm{C}$ by sync calibration.* When the sync map calibrates the reporter linearly on the shell,

$$g\big(D(\mathcal{D}_\mathrm{C})\big) = 2 + \mathcal{D}_\mathrm{C} \quad\Longleftrightarrow\quad a = \frac{\mathcal{D}_\mathrm{C}}{2}\;\; \text{(exact)}, \tag{D12}$$

one finds

$$D(\mathcal{D}_\mathrm{C}) = \frac{\mathcal{D}_\mathrm{C}(\mathcal{D}_\mathrm{C}+4)}{(\mathcal{D}_\mathrm{C}+2)^2}, \qquad (1-D)^{-\frac{3}{2}}\,\frac{\partial D}{\partial \mathcal{D}_\mathrm{C}} = 1, \tag{D13}$$

hence $Q = \mathcal{D}_\mathrm{C}$ in (D7). Therefore (D8) reduces to the main–body formula $C^{\mathrm{I}}_{\log} = (\pi^2/\mathcal{D}_\mathrm{C})\,\zeta(1+\zeta)$.

4. Conservation–based Lagrangian

On the shell the dynamical variable is $A_\parallel(\phi)$; gauge invariance confines us to TT. The conserved quantity is the uniform toroidal circulation $\Phi_{\mathrm{TT}} = \int_0^{2\pi} A_\parallel d\phi$. Consider the quadratic functional

$$\mathcal{L}[A_\parallel;\lambda,\nu] = \tfrac{1}{2}\,\langle A_\parallel, K_{\mathrm{TT+sync}}A_\parallel\rangle - \lambda\,\langle u, A_\parallel\rangle + \nu\,(\Phi_{\mathrm{TT}} - \Phi_0), \tag{D14}$$

with source $\lambda \equiv \mathcal{D}_{\mathrm{C}}$ and Lagrange multiplier ν for flux conservation. Euler–Lagrange gives $A_{\parallel}^{*} = \lambda K_{\mathrm{TT+sync}}^{-1} u$ and

$$\Pi_{\mathrm{TT}}(s) = \langle u, A_{\parallel}^{*} \rangle = \mathcal{D}_{\mathrm{C}}\, c_{\chi}\, \zeta \left(1+\zeta\right) \ln s. \tag{D15}$$

With the exact reporter slope $\partial a/\partial \mathcal{D}_{\circ} = 1/2$ and $d\mathcal{D}_{\circ}/d\mathcal{D}_{\mathrm{C}} = Q/\mathcal{D}_{\mathrm{C}}$,

$$\Delta a_{\log} = \tfrac{1}{2} \cdot \frac{Q}{\mathcal{D}_{\mathrm{C}}} \cdot \left(\mathcal{D}_{\mathrm{C}}\, c_{\chi}\, \zeta(1+\zeta) \ln s\right) = \left(\pi^{2} Q\right) \zeta(1+\zeta) \ln s, \tag{D16}$$

and comparison with (D1) reproduces (D8). *Physics:* quadratic energy $\Rightarrow$ factor 1/2; flux conservation $\Rightarrow$ uniform mode u; gauge invariance $\Rightarrow$ only TT survives; single TT channel $\Rightarrow$ the $(1+\zeta)$ dressing.

5. Path II anchor: why 1/3?

Path II is purely geometric. With the transverse projector $P_{ij}(\hat{\boldsymbol{r}}) = \delta_{ij} - \hat{r}_i \hat{r}_j$ and a fixed unit $\hat{\boldsymbol{t}}$,

$$\frac{1}{4\pi} \int d\Omega\, \hat{t}_i\, P_{ij}(\hat{\boldsymbol{r}})\, \hat{t}_j = 1 - \left\langle \cos^2 \theta \right\rangle = \frac{2}{3}. \tag{D17}$$

Since the physical shell excites a *single* TT channel (rank–1 acceptance) out of the two transverse polarizations encoded in P_{ij}, an additional factor 1/2 enters, hence

$$C_{\log} = \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}. \tag{D18}$$

Equating Path I and Path II gives the exact lock

$$\frac{\pi^2}{Q(\alpha)}\, \zeta(\alpha) \left(1 + \zeta(\alpha)\right) = \frac{1}{3}\,. \tag{D19}$$

Using (D10) one may rewrite the lock as

$$\frac{\pi^2}{\mathcal{D}_{\mathrm{C}}}\, \zeta_Q \left(1 + \frac{Q}{\mathcal{D}_{\mathrm{C}}}\, \zeta_Q\right) = \frac{1}{3}, \tag{D20}$$

which shows explicitly that the cleanest geometric form is obtained in the Coulombic control.

6. Emergent coupling and the QED mass log

From (D16) and the lock (D19),

$$\Delta a_{\log} = \frac{Q(\alpha)^2}{3} \ln s. \tag{D21}$$

The synchronization condition $a = \mathcal{D}_{\mathrm{C}}/2$ together with the low–energy Dirac slope $\partial a/\partial(\alpha/\pi) = 1/2$ implies $\mathcal{D}_{\mathrm{C}} = \alpha/\pi$, and with $(1-D)^{-3/2}\, \partial D/\partial \mathcal{D}_{\mathrm{C}} = 1$ one has $Q = \mathcal{D}_{\mathrm{C}} = \alpha/\pi$ at low energy. For a species stretch $s = R_e/R_\ell = m_\ell/m_e$ one then obtains the canonical QED mass–log coefficient

$$\Delta a_{\log} \approx \left(\frac{\alpha}{\pi}\right)^2 \frac{1}{3} \ln \frac{m_\ell}{m_e}\,. \tag{D22}$$

Physics: (i) Schwinger's slope fixes the low–energy normalization; (ii) the mass log is a two–vertex effect $\propto (\alpha/\pi)^2$; (iii) Ward identity selects $\alpha(0)$ and the universal $1/\pi$ is carried by loop integrals.

Atomic scales. Atomic (Rydberg/a_0) scales only bound the integration (hence s); the universal prefactor is fully encoded by ζ and Q (or $\mathcal{D}_{\mathrm{C}}$ in the calibrated gauge).

Numerical precision. All algebraic steps above are exact; any numerical error is purely arithmetic. Whenever $O(\zeta)$ is dropped via $(1+\zeta) \approx 1$, this approximation is stated explicitly.