SYNTHETIC ADAMS-NOVIKOV CHARTS

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1. Instructions and legends

This document and the accompanying charts are intended to be paired with our paper, "Stable comodule deformations and the synthetic Adams-Novikov spectral sequence." The paper can be found on the arxiv here. These charts contain computations of the $\mathbb{F}_2$-synthetic Adams-Novikov spectral sequence for the sphere ($\mathbb{S}_{\mathbb{F}_2}$) and the cofiber of lambda ($\mathbb{S}_{\mathbb{F}_2}/\lambda$). Charts are provided in HTML format and can be viewed with any modern web browser.

1.1. Charts for $\mathbb{S}_{\mathbb{F}_2}/\lambda$.

Chart 1 displays the E_2-page of the Adams-Novikov spectral sequence for $\mathbb{S}_{\mathbb{F}_2}/\lambda$ with differentials. Chart 2 displays the E_∞-page of the Adams-Novikov spectral sequence for $\mathbb{S}_{\mathbb{F}_2}/\lambda$ with hidden extensions. For each fixed stem and filtration, the synthetic Adams-Novikov spectral sequence for $\mathbb{S}_{\mathbb{F}_2}/\lambda$ is a module over $\mathbb{F}_2[h]$.

Analogous to the charts in [IKLRZ23], we plot classes not only according to their Adams-Novikov filtration but also with respect to a filtration by powers of h. An extra convention like this is required because h lives in filtration 0. We indicate the associated graded of this h-filtration as follows:

1.1.1. *Classes.*

- An open box □ indicates a copy of $\mathbb{F}_2[h]$ in the associated graded object.
- A solid dot ● indicates a copy of $\mathbb{F}_2[h]/h \cong \mathbb{F}_2$ in the associated graded object.
- A solid box $\boxed{k}$ with an inscribed number k indicates a copy of $\mathbb{F}_2[h]/h^k$ in the associated graded object.
- A dot or box is colored gray if it is in the image of the inclusion of the bottom cell.
- A dot or box is colored red if it is detected in projection to the top cell.

1.1.2. *Differentials.*

Cyan lines of negative slope indicate Adams-Novikov differentials. The length of a differential corresponds to the page on which it occurs.

1.1.3. *Extensions.*

- Short vertical gray lines indicate h-multiplications that are in the image of inclusion of the bottom cell.
- Short vertical red lines indicate h-multiplications that are detected by projection to the top cell.
- Gray lines of slope 1 and $1/3$ indicate α_1 and $\alpha_{2/2}$ multiplications that are in the image of inclusion of the bottom cell.
- Red lines of slope 1 and $1/3$ indicate α_1 and $\alpha_{2/2}$ multiplications that are detected by projection to the top cell.

Date: August 29, 2025.

- Yellow lines of slope 1 and 1/3 indicate α_1 and $\alpha_{2/2}$ multiplications that are hidden in the sense that their sources are detected by the top cell but their targets are detected by the bottom cell.
- Arrows of slope 1 indicate infinite towers of α_1 extensions.
- Orange lines indicate hidden extensions by h, η, and ν.

1.2. **Charts for $\mathbb{S}_{\mathbb{F}_2}$.** Chart 3 displays the E_2-page of the Adams-Novikov spectral sequence for $\mathbb{S}_{\mathbb{F}_2}$ with differentials. Chart 4 displays the E_∞-page of the Adams-Novikov spectral sequence for $\mathbb{S}_{\mathbb{F}_2}$ with hidden extensions. For each fixed stem and filtration, the Adams-Novikov spectral sequence for $\mathbb{S}_{\mathbb{F}_2}$ is a module over $\mathbb{Z}_{(2)}[\lambda, h]/(\lambda h = 2)$. Similar to the charts for $\mathbb{S}_{\mathbb{F}_2}/\lambda$, we indicate classes by a filtration by powers of h as follows:

1.2.1. *Classes.*

- An open box □ indicates a copy of
$$\mathbb{Z}_{(2)}[\lambda, h]/(\lambda h = 2)$$
in the associated graded object.
- A solid gray dot ● indicates a copy of
$$\mathbb{F}_2[\lambda] \cong \mathbb{Z}_{(2)}[\lambda, h]/(\lambda h = 2, h)$$
in the associated graded object.
- A solid gray box $\boxed{k}$ with an inscribed number k indicates a copy of
$$\mathbb{Z}/2^k[\lambda, h]/(\lambda h = 2, h^k) \cong \mathbb{Z}_{(2)}[\lambda, h]/(\lambda h = 2, h^k)$$
in the associated graded object.
- A solid colored dot indicates a copy of
$$\mathbb{F}_2[\lambda]/\lambda^r \cong \mathbb{Z}_{(2)}[\lambda, h]/(\lambda h = 2, h, \lambda^r)$$
in the associated graded object. The value of r is encoded in the color of the dot, as indicated below.
- A solid colored box with an inscribed k number indicates a copy of
$$\mathbb{Z}/2^{\min\{k,r\}}[\lambda, h]/(\lambda h = 2, h^k, \lambda^r) \cong \mathbb{Z}_{(2)}[\lambda, h]/(\lambda h = 2, h^k, \lambda^r)$$
in the associated graded object. The value of r is encoded in the color of the box, as indicated below.
- We use the following colors for classes:
  - Gray dots correspond to λ-free classes.
  - Red dots correspond to λ-torsion classes.
  - Blue dots correspond to λ^2-torsion classes.
  - Dark green dots correspond to λ^3-torsion classes.

1.2.2. *Differentials.* Lines of negative slope indicate Adams-Novikov differentials. The length of the differential corresponds to the page on which it occurs. The color of a differential indicates which λ-multiple of the target is hit.

- Teal lines hit $\lambda^0 = 1$ times a generator.
- Magenta lines hit λ times a generator.
- Lime green lines hit λ^2 times a generator.
- Purple lines hit λ^3 times a generator.

1.2.3. *Extensions.* Vertical lines and lines of positive slope indicate extensions. Lines that stay in the same stem indicate multiplication by λ or h. Lines that change stem by 1 indicate multiplication by α_1 (η in homotopy). Lines that change stem by 3 indicate multiplication by $\alpha_{2/2}$ (ν in homotopy). The curvature of an extension has no mathematical meaning and is there for aesthetic reasons. The color of an extension indicates something particular about the extension:

- Gray, red, blue, and dark green lines indicate non-hidden extensions which hit $\lambda^0 = 1$ times a generator. The color corresponds to the λ-torsion (or freeness) of the target.
- Orange indicates hidden extensions which hit $\lambda^0 = 1$ times a generator.
- Magenta lines hit λ times a generator. These can be non-hidden or hidden extensions.
- Lime green lines hit λ^2 times a generator. These can be non-hidden or hidden extensions.
- Purple lines hit λ^3 times a generator. In this range, these are only non-hidden.
- Cyan indicate hidden λ-extensions. These all hit $\lambda^0 = 1$ times a generator.

Example 1.2.4. In the 14-stem of the ANSS for $\mathbb{S}_{\mathbb{F}_2}$ we have the module displayed in Figure 1. This notation indicates the $\mathbb{Z}_{(2)}[\lambda, h]/(\lambda h = 2)$-module with generators $\beta_{4/4}$ and β_3 and relations

$$\lambda h \beta_{4/4} = 0, \qquad h^2 \beta_{4/4} = \lambda h \beta_3, \qquad h^3 \beta_3 = 0.$$

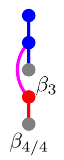

FIGURE 1. $E_2^{2,14}(\mathbb{S}_{\mathbb{F}_2})$

Example 1.2.5. In the 30-stem of the ANSS for $\mathbb{S}_{\mathbb{F}_2}$ we have the module displayed in Figure 2. This notation indicates the $\mathbb{Z}_{(2)}[\lambda, h]/(\lambda h = 2)$-module with generators $\beta_{8/8}$, $\beta_{6/2}$, and $P^2\beta_3$ and relations

$$\lambda h \beta_{8/8} = 0, \qquad h^4 \beta_{8/8} = \lambda h \beta_{6/2}, \qquad h^5 \beta_{6/2} = \lambda P^2 \beta_3, \qquad h^3 P^2 \beta_3 = 0.$$

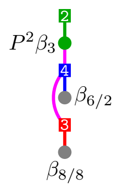

FIGURE 2. $E_2^{2,30}(\mathbb{S}_{\mathbb{F}_2})$

REFERENCES

[IKLRZ23] Daniel C. Isaksen, Hana Jia Kong, Guchuan Li, Yangyang Ruan, and Heyi Zhu. *The $\mathbb{C}$-motivic Adams-Novikov spectral sequence for topological modular forms.* 2023. arXiv: 2302.09123 [math.AT].