Gauge-Gravity Coupling in Manifold Quantum Gravity

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Abstract

We show that gauge-gravity coupling in Manifold Quantum Gravity arises directly from the Hessian expansion of the coherence functional. The Einstein-Yang-Mills sector emerges in the infrared with Newton's constant, gauge couplings and the cosmological constant determined internally by recursion geometry. Curvature-gauge mixing operators appear automatically with fixed coefficients, giving rise to testable signatures including curvature-dependent running of couplings, confinement scale modulation, birefringent lensing, gauge-wave energy exchange and the reinterpretation of the dark sector as modal potential. In addition, the ultraviolet sector is ghost-free and complete: the Hessian kernels generate entire nonlocal form factors, shown explicitly for a Gaussian regulator, and the results are independent of the chosen representation of the coherence functional. This establishes MQG as a variational framework in which both infrared and ultraviolet behaviour are coherently controlled, with no free counter-terms or hidden sectors.

Contents

1 Introduction

The unification of gauge and gravitational interactions remains a central problem in fundamental physics. While general relativity provides a geometric description of spacetime curvature [3],[4] and the Standard Model accounts for the gauge sector, their direct coupling is usually introduced via minimal substitution in the action. Beyond this, effective field theory arguments predict higher–order curvature gauge terms suppressed by the Planck scale [6],[5],[8], but these lack a derivation from first principles.

Manifold Quantum Gravity (MQG) offers such a derivation. Building on the recursive decoherence framework introduced in [1], and the emergence of the full $U(1) \times SU(2) \times SU(3)$ gauge sector demonstrated in [2], the present work develops the gauge–gravity interface. Specifically, we show that the second variation of the recursion functional induces curvature–dependent gauge terms of the schematic form

$$\Delta\mathcal{L} \;\sim\; \alpha_1 R\, F^2 + \alpha_2 R_{\mu\nu} F^{\mu\lambda} F^{\nu}{}_{\lambda} + \alpha_3 C_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}, \tag{1}$$

with calculable coefficients α_i set by the recursion Hessian.

The operators in (1) form a complete gauge– and diffeomorphism–invariant basis at dimension six for quadratic curvature–gauge couplings, up to total derivatives [6], [5].

These terms are not inserted ad hoc but emerge uniquely once gauge fields are projected from recursion geometry. Their presence implies specific phenomenology: curvature–dependent running of couplings, confinement scale modulation, polarisation–dependent lensing, energy exchange with gravitational waves and a structural reinterpretation of the dark sector in terms of modal potential.

The goal of this paper is twofold: (i) to present a rigorous derivation of gauge–gravity coupling within MQG, and (ii) to identify the resulting observational signatures, highlighting the predictive fingerprint that distinguishes MQG from both effective field theory extensions and quantum gravity candidates based on new fundamental constituents.

This work forms part of a sequence of studies developing Manifold Quantum Gravity (MQG). An initial formulation established the coherence functional and its variational role in unifying geometry and gauge dynamics [1], followed by a focused analysis of gauge sector emergence from the recursion Hessian [2]. The present paper extends that programme to include explicit gauge-gravity coupling. All necessary definitions are repeated here, so the results are self-contained, but the earlier works provide broader context on recursion geometry and the coherence functional framework.

2 Foundations

We begin by making the geometric sector of Manifold Quantum Gravity (MQG) dynamical, promoting the foliation layer from a fixed background to a fluctuating configuration. The fundamental fields are collected into the configuration

$$\Phi \;\equiv\; (g_{\mu\nu},\ \Psi,\ W^a_\mu), \qquad a \in \mathfrak{g} = \mathfrak{u}(1) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(3), \tag{2}$$

where $g_{\mu\nu}$ is the spacetime metric, Ψ the modal scalar field encoding recursive coherence and W^a_μ the gauge connections associated with the Standard Model gauge group. The corresponding curvature is

$$F^a_{\mu\nu} \;=\; \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + f^a{}_{bc} W^b_\mu W^c_\nu, \tag{3}$$

with $f^a{}_{bc}$ the structure constants of $\mathfrak{g}$.

2.1 Axioms and assumptions

The coherence functional $\mathcal{C}[\Phi]$ is the variational object defining MQG [1]. We impose three axioms which suffice to determine the infrared (IR) expansion:

(A1) $\mathcal{C}[\Phi]$ is invariant under spacetime diffeomorphisms and local G gauge transformations.

(A2) At a stationary background $\Phi_\star = (g_\star, \Psi_\star, W_\star = 0)$, the Hessian

$$\mathcal{H} \;\equiv\; \left.\frac{\delta^2 \mathcal{C}}{\delta\Phi\,\delta\Phi}\right|_{\Phi_\star} \tag{4}$$

is local to quadratic order in derivatives.

(A3) $\mathcal{H}$ is positive on the physical tangent space modulo diffeomorphism and gauge zero modes, ensuring healthy kinetic terms and absence of ghosts.

Locality at quadratic order. By (A2) we mean that the Hessian kernels admit a derivative expansion around $p^2 = 0$ with finitely many two-derivative terms:

$$\widehat{\mathcal{H}}(p) = A + B\,p^2 + \mathcal{O}(p^4),$$

in the background-field sense, so that the reconstructed action is a sum of local operators up to two derivatives.

Gauge fixing and ghosts. For practical inversions one may add background-field gauge-fixing terms and the corresponding Faddeev–Popov ghosts. Since our statements concern the gauge-invariant local action reconstructed from the physical blocks of $\mathcal{H}$, the gauge-fixed operators serve only as intermediate tools and do not alter the coefficients in (9)–(10).

These assumptions follow the standard logic of quantum field theory in curved spacetime [3], [4], [5] and effective action methods in gravity [6]. In particular, axiom (A1) ensures the uniqueness of the operator basis at quadratic order while (A2) and (A3) guarantee that the resulting local action is well defined and ghost free.

2.2 Projectors and inner product

To evaluate the Hessian blocks we employ the natural inner product on variations $\delta\Phi$ defined by MQG geometry [2]. Explicitly,

$$\langle \delta\Phi_1, \delta\Phi_2 \rangle \;=\; \int d^4x \sqrt{-g_\star}\left[\delta g^{(1)}_{\mu\nu}\, \mathcal{G}^{\mu\nu\rho\sigma}\, \delta g^{(2)}_{\rho\sigma} + \delta W^{a,(1)}_{\mu}\, \delta^{ab} g^{\mu\nu}\, \delta W^{b,(2)}_{\nu} + \delta\Psi^{(1)}\delta\Psi^{(2)}\right], \tag{5}$$

where $\mathcal{G}^{\mu\nu\rho\sigma}$ is the DeWitt supermetric on the space of metrics. The corresponding orthonormal projectors onto metric, gauge orbit and modal directions are denoted P_g, Π_a and P_Ψ respectively.

Let $\{\hat{e}^{(g)}_I\}$, $\{\hat{e}^{(a)}_A\}$ and $\{\hat{e}^{(\Psi)}_J\}$ denote orthonormal bases with respect to (5). Then the Hessian blocks project as

$$\kappa^{-2} \;\propto\; \langle \hat{e}^{(g)}_I,\, \mathcal{H}\, \hat{e}^{(g)}_I \rangle, \tag{6}$$

$$(g^{-2})_{ab} \;\propto\; \langle \hat{e}^{(a)}_A,\, \mathcal{H}\, \hat{e}^{(b)}_A \rangle, \tag{7}$$

$$c_i \;\propto\; \langle \hat{e}^{(g)}_I,\, \mathcal{H}\, \hat{e}^{(a)}_A \rangle, \tag{8}$$

where c_i are mixing coefficients associated with curvature–gauge operators such as $R\,\mathrm{tr}F^2$, $R_{\mu\nu}F^{\mu\rho}F^{\nu}{}_{\rho}$ and $C_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$ [7], [8].

2.3 Infrared action from the Hessian

Proposition 2.1 (Uniqueness of the quadratic IR basis)**.** *Under (A1)–(A3) the gauge- and diffeomorphism-invariant, local quadratic action up to two derivatives consists uniquely of R and $\mathrm{tr}F^2$, with no dimension-four mixing. The first curvature–gauge operators appear at dimension six as in (10), up to total derivatives and field redefinitions.*

Expanding $\mathcal{C}$ to quadratic order about $\Phi_\star$ yields a local IR action

$$\mathcal{S}_{\rm IR}[g,\Psi,W] \;=\; \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}R-\Lambda-\frac{1}{4}(g^{-2})_{ab}F^a_{\mu\nu}F^{b\,\mu\nu}+\mathcal{K}_\Psi(g,\Psi,W)-\mathcal{V}_\Psi(\Psi,g)+\mathcal{M}_{\rm mix}(g,W)\right], \tag{9}$$

where κ^{-2}, $(g^{-2})_{ab}$ and Λ are defined by the Hessian projections (6)–(8). The mixing sector has the explicit structure

$$\mathcal{M}_{\rm mix}(g,W) \;=\; c_1\,R\,\mathrm{tr}F^2+c_2\,R_{\mu\nu}\,\mathrm{tr}(F^{\mu\rho}F^{\nu}{}_{\rho})+c_3\,C_{\mu\nu\rho\sigma}\,\mathrm{tr}(F^{\mu\nu}F^{\rho\sigma})+\cdots, \tag{10}$$

with all coefficients fixed by Hessian projections.

Higher-order operators While (10) lists the unique curvature–gauge structures at dimension six, the Hessian expansion naturally continues to higher order. At dimension eight the independent, gauge- and diffeomorphism-invariant quadratic operators consist of cubic contractions of curvature with F^2 and of curvature-squared terms with F^2. A representative basis is

$$\mathcal{L}^{(8)}_{\rm mix} \;=\; d_1\,R^2\,\mathrm{tr}F^2+d_2\,R_{\mu\nu}R^{\mu\nu}\,\mathrm{tr}F^2+d_3\,R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\,\mathrm{tr}F^2+d_4\,R_{\mu\nu}\,\mathrm{tr}(F^{\mu\lambda}F^{\nu}{}_{\lambda})+d_5\,R_{\mu\nu\rho\sigma}\,\mathrm{tr}(F^{\mu\nu}F^{\rho\sigma})+\cdot \tag{11}$$

with coefficients d_i again determined by Hessian projections. Up to integrations by parts and field redefinitions, this list spans the complete dimension-eight basis of quadratic curvature–gauge couplings [6], [5].

Remark The operators in (11) are suppressed by two further powers of the Planck scale relative to (10). Although negligible in most infrared settings, they demonstrate that the MQG expansion provides a systematic, finite basis at each dimension, with all coefficients calculable from recursion geometry.

Equation (9) is the central structural result. The Einstein–Yang–Mills sector emerges automatically in the IR with Newton's constant, gauge couplings and the cosmological constant determined internally by MQG. No free constants are introduced by hand.

2.4 Ultraviolet behaviour and equivalence of representations

The infrared analysis above relied only on (A1)–(A3), which suffice to reconstruct the unique local quadratic basis and the dimension–six and –eight corrections. Away from the soft limit, the Hessian kernels acquire non-local momentum dependence that encodes recursion geometry. We now formalise two points: (i) the ultraviolet (UV) structure enforced by our axioms and (ii) the equivalence, at quadratic order, of several natural representations of the coherence functional.

Proposition 2.2 (UV completion under two admissible classes of form factors)**.** *Let $\widehat{\mathcal{H}}(p)$ be the momentum-space Hessian kernel on the physical subspace. If $\mathcal{C}[\Phi]$ obeys (A1)–(A3), then the quadratic form factors belong to one of two admissible classes ensuring ghost freedom:*

(Track A) Stieltjes/CBF class. *Suppose the form factors admit a Laplace-Stieltjes representation with a positive measure,*

$$\widehat{\mathcal{H}}(p_E^2) \;=\; \int_0^\infty d\mu(s)\, e^{-sp_E^2}, \qquad d\mu(s) \geq 0,$$

in Euclidean momenta $p_E^2 \geq 0$. Then $\widehat{\mathcal{H}}$ is entire, completely monotone, reflection positive and free of extra poles. The infrared expansion $\widehat{\mathcal{H}}(p^2) = A + Bp^2 + \mathcal{O}(p^4)$ reproduces the local basis [15], [14].

(Track B) Entire, zero-free class. *Suppose instead that $\widehat{\mathcal{H}}(p^2)$ is an entire function of p^2, real and positive for $p^2 \geq 0$, with no zeros on the real axis. Then the only pole of the propagator is at $p^2 = 0$, the IR limit again yields $A + Bp^2$, and tree-level ghost freedom and macrocausality are preserved.*

Thus MQG admits two rigorous UV completions: the Stieltjes/CBF class (Track A) with reflection positivity and exponential UV suppression, and the entire zero-free class (Track B) with ghost-free analytic continuation. Both reduce uniquely to the IR local action.

2.5 Equivalence of background realisations

We now show that the Hessian of $\mathcal{C}[\Phi]$ is unitarily equivalent across its three natural realisations: (i) the geometric Laplace-type operators on (S^3, g), (ii) the spectral triple $(\mathcal{A}, \mathcal{H}, D)$, and (iii) the Hilbert-recursion presentation $(\pi_\omega, \mathcal{H}_\omega, \mathcal{R})$.

Lemma 2.3 (Weitzenböck and Lichnerowicz identities)**.** *On a compact spin three–manifold (S^3, g):*

1. *For scalars, $\nabla^*\nabla = -\Delta_g$.*
2. *For spinors, $D^2 = \nabla^*\nabla + \frac{1}{4}\,\mathrm{Scal}(g)$.*

3. *For one–forms and TT tensors, the Hodge/Bochner Laplacians differ from D^2 by smooth curvature endomorphisms.*

Lemma 2.4 (Projector commutation)**.** *On S^3, the scalar, vector and transverse-traceless projectors are polynomial (hence Borel) functions of the relevant Laplacians. They therefore commute with $F(\mathcal{L}_g)$ for any bounded Borel function F.*

Lemma 2.5 (Spectral calculus under GNS recursion)**.** *Let $(\mathcal{A}, \mathcal{H}, D)$ be the commutative spectral triple of (S^3, g). Suppose the recursion generator $\mathcal{R}$ is positive, self-adjoint and has the same spectral measure as D^2 under the GNS representation $(\pi_\omega, \mathcal{H}_\omega)$. Then there exists a unitary $U : \mathcal{H} \to \mathcal{H}_\omega$ such that*

$$U\, F(D^2)\, U^{-1} \;=\; F(\mathcal{R}) \qquad \forall\, F \text{ bounded Borel.}$$

Theorem 2.6 (Equivalence of background realisations)**.** *Let $\mathbb{H}_g$, $\mathbb{H}_D$ and $\mathbb{H}_\mathcal{R}$ denote the Hessians in the geometric, spectral and recursion realisations, defined sector-wise by $\mathbb{H} = F(\mathcal{L}) + V$ with F any admissible form factor and V a smooth endomorphism. Under assumptions (A1)–(A3) there exist unitaries*

$$U_{G\to S} : L^2(S^3) \to \mathcal{H}, \qquad U_{S\to R} : \mathcal{H} \to \mathcal{H}_\omega$$

such that

$$U_{G\to S}\, \mathbb{H}_g\, U_{G\to S}^{-1} = \mathbb{H}_D, \qquad U_{S\to R}\, \mathbb{H}_D\, U_{S\to R}^{-1} = \mathbb{H}_\mathcal{R}.$$

Hence all three Hessians are unitarily equivalent, with identical quadratic forms, spectra and sectoral projectors.

Corollary 2.7 (Background independence)**.** *The MQG Hessian, its eigenmodes and projectors are representation-independent. Results derived from $\mathcal{H}$ are therefore insensitive to whether one works in geometric, spectral or recursion language.*

Lemma 2.8 (Equivalence of representations at quadratic order)**.** *Consider the following representatives for $\mathcal{C}$: (i) an information-geometric functional on Ψ (Fisher block), (ii) a spectral functional $\operatorname{Tr} f(D[\Phi]/\Lambda)$ with D a Dirac/Laplace-type operator, and (iii) a discrete modal (graph/Hilbert) functional whose continuum limit yields the spectrum of D. If their inner products are identified with (5), their quadratic Hessians coincide on the physical subspace up to unitary equivalence of basis vectors. Consequently the infrared operator basis and the curvature-gauge coefficients obtained in Sections 2–4 are representation-independent.*

Proof sketch. For (i) the Hessian of relative entropy at a stationary point is the Fisher metric, giving the Ψ-block [16], [17]. For (ii) varying $\mathrm{Tr}\, f(D/\Lambda)$ twice and using spectral calculus yields entire nonlocal form factors whose low–momentum expansion reproduces the Seeley-DeWitt series [15], [14]. For (iii) the graph Laplacian spectrum converges to the Laplace-Beltrami spectrum under mild regularity; the GNS construction identifies the inner product with (5). Matching projectors $\{P_g, \Pi_a, P_\Psi\}$ gives equality of blocks. $\square$

A hybrid model for UV form factors. As a concrete realisation of Proposition 2.2, consider the spectral action with Gaussian regulator $f(x) = e^{-x}$:

$$\mathcal{C}[\Phi] = \alpha\, \mathrm{Tr}\, f\left(\frac{D[g, \Psi, W]}{\Lambda_{\mathrm{rec}}}\right) + \beta \int d^4x \sqrt{-g}\, \mathcal{I}_F[\Psi; g] + \gamma \int d^4x \sqrt{-g}\ \mathrm{tr}(F_{\mu\nu} F^{\mu\nu}), \tag{12}$$

with $f(x) = e^{-x}$ smooth and positive. The corresponding Hessian kernels take the form

$$\widehat{\mathcal{H}}_{gg}(p) = \kappa_0^{-2}\, p^2\, F_g(p^2/\Lambda_{\mathrm{rec}}^2), \tag{13}$$
$$\widehat{\mathcal{H}}_{WW}(q) = Z\, q^2\, F_W(q^2/\Lambda_{\mathrm{rec}}^2), \tag{14}$$
$$\widehat{\mathcal{H}}_{gW}(p, q) = c\, p^2 q^2\, F_{\mathrm{mix}}(p^2/\Lambda_{\mathrm{rec}}^2, q^2/\Lambda_{\mathrm{rec}}^2). \tag{15}$$

Here $F_\bullet$ are entire completely monotone functions arising from the Laplace transform representation, placing the model squarely in Track A of Proposition 2.2. Alternative entire zero-free regulators, such as $\exp[-(p^2/\Lambda_{\mathrm{rec}}^2)^n]$ with $n \geq 1$, illustrate Track B. Thus MQG admits UV completions within both admissible classes.

Illustration: flat-background UV kernels. As an explicit instance of Track A, take $f(x) = e^{-x}$ in (12) and expand about flat space with vanishing gauge background. The quadratic kernels then read

$$\widehat{\mathcal{H}}_{gg}(p) = \kappa_0^{-2}\, p^2\, e^{-p^2/\Lambda_{\mathrm{rec}}^2}\, P^{\mathrm{TT}}(p) + \cdots, \qquad \widehat{\mathcal{H}}_{WW}(q) = Z\, q^2\, e^{-q^2/\Lambda_{\mathrm{rec}}^2}\, \Pi_T(q),$$

$$\widehat{\mathcal{H}}_{gW}(p, q) = c\, p^2 q^2\, e^{-(p^2+q^2)/\Lambda_{\mathrm{rec}}^2}\, (\text{projectors of Section 3.2}) + \cdots.$$

These kernels are entire and completely monotone, hence belong to the Stieltjes/CBF class. They exhibit exponential UV suppression, preserve reflection positivity and reduce smoothly to the infrared coefficients in (9).

A full covariant evaluation on curved backgrounds requires standard perturbation theory and is left to future work.

A rigorous operator–theoretic foundation for the positivity and spectral representation of the Hessian kernels (Laplace–Stieltjes form, complete monotonicity and absence of extra poles) is developed in Appendix A.

Lemma 2.9 (Gauge Ward identity with entire form factor)**.** *Let the quadratic gauge kernel in momentum space be*

$$\widehat{\mathcal{H}}^{\mu\nu}_{WW}(q) \;=\; Z\,q^2\,F_W\!\left(\frac{q^2}{\Lambda_{\rm rec}^2}\right)\,\Pi_T^{\mu\nu}(q),$$

where F_W is an entire function with $F_W(0)=1$ and $\Pi_T^{\mu\nu}(q)=\eta^{\mu\nu}-q^\mu q^\nu/q^2$ is the transverse projector. Then the Ward identity

$$q_\mu\,\widehat{\mathcal{H}}^{\mu\nu}_{WW}(q)=0$$

holds identically. Therefore, there are no spurious longitudinal poles and BRST transversality is preserved at quadratic order.

Proof. By definition $q_\mu\Pi_T^{\mu\nu}(q)=q_\mu\eta^{\mu\nu}-q_\mu q^\mu q^\nu/q^2=q^\nu-q^\nu=0$. Multiplying by the scalar factor $Z\,q^2\,F_W(q^2/\Lambda_{\rm rec}^2)$ does not alter transversality, therefore $q_\mu\,\widehat{\mathcal{H}}^{\mu\nu}_{WW}(q)=0$. Since F_W is entire and non-vanishing on the real axis, the only pole of the inverse kernel is at $q^2=0$ as in the infrared theory [15], [14]. □

Proposition 2.10 (FRW mixing form factor within Track A)**.** *With the hybrid functional* (12) *and Gaussian regulator $f(x)=e^{-x}$, the curvature–gauge cross kernel on slowly varying FRW backgrounds induces the nonlocal operator*

$$\mathcal{L}^{\rm FRW}_{\rm mix} \;=\; c_R\Big(-\frac{\Box}{\Lambda_{\rm rec}^2}\Big)\,R\,{\rm tr}F^2,$$

whose form factor belongs to Track A of Proposition 2.2. Explicitly,

$$c_R\Big(-\frac{\Box}{\Lambda_{\rm rec}^2}\Big) \;=\; \frac{1}{\Lambda_{\rm rec}^2}\;\int_0^1 d\xi\;\exp\Big[-\,\xi(1-\xi)\,\frac{-\Box}{\Lambda_{\rm rec}^2}\Big], \tag{16}$$

with small-momentum expansion

$$c_R\Big(-\frac{\Box}{\Lambda_{\rm rec}^2}\Big)=\frac{1}{\Lambda_{\rm rec}^2}\,\Big[\,1\;-\;\tfrac{1}{6}\,\frac{-\Box}{\Lambda_{\rm rec}^2}\;+\;\mathcal{O}(\Box^2/\Lambda_{\rm rec}^4)\Big].$$

Thus the leading IR coefficient is $c_R=\Lambda_{\rm rec}^{-2}$ as used in Section 4.

Proof sketch. Use the Schwinger representation with $f(x) = e^{-x}$. Expanding to first order in R and second order in F in covariant perturbation theory produces factorised integrals of the type

$$\mathcal{F}(-\Box) \;=\; \int_0^1 d\xi\, \exp\big[-\,\xi(1-\xi)\,(-\Box)/\Lambda_{\rm rec}^2\big],$$

multiplying the unique scalar contraction $R\,\mathrm{tr}F^2$. The IR limit follows by $\int_0^1 \xi(1-\xi)\,d\xi = 1/6$. Entirety and complete monotonicity place c_R squarely in Track A, ensuring ghost freedom and exponential UV suppression. $\Box$

Corollary 2.11 (IR limit and ghost freedom)**.** *Equation* (16) *reduces to* $c_R = \Lambda_{\rm rec}^{-2}$ *as* $-\Box/\Lambda_{\rm rec}^2 \to 0$*, while for finite momenta it defines an entire, ghost-free operator dressing that preserves IR locality and introduces only exponentially suppressed non-locality at high momentum.*

Lemma 2.12 (Closed form of the FRW mixing form factor within Track A)**.** *For the Gaussian spectral regulator* $f(x) = e^{-x}$ *the FRW scalar mixing form factor is an explicit element of the Stieltjes/CBF class and admits the closed expression*

$$c_R\Big(-\frac{\Box}{\Lambda_{\rm rec}^2}\Big) \;=\; \frac{1}{\Lambda_{\rm rec}^2}\,\exp\Big[-\,\frac{z}{4}\Big]\,\frac{\sqrt{\pi}}{\sqrt{z}}\,\mathrm{erfi}\Big(\frac{\sqrt{z}}{2}\Big), \qquad z \;\equiv\; -\frac{\Box}{\Lambda_{\rm rec}^2}, \tag{17}$$

with small-momentum expansion

$$c_R\Big(-\frac{\Box}{\Lambda_{\rm rec}^2}\Big) = \frac{1}{\Lambda_{\rm rec}^2}\left[1 - \frac{1}{6}\frac{-\Box}{\Lambda_{\rm rec}^2} + \frac{1}{60}\left(\frac{-\Box}{\Lambda_{\rm rec}^2}\right)^2 - \frac{1}{840}\left(\frac{-\Box}{\Lambda_{\rm rec}^2}\right)^3 + \cdots\right]. \tag{18}$$

Proof sketch. Starting from the Schwinger representation in Proposition 2.10, write

$$c_R\Big(-\frac{\Box}{\Lambda_{\rm rec}^2}\Big) = \frac{1}{\Lambda_{\rm rec}^2}\int_0^1 d\xi\, \exp\Big[-\,\xi(1-\xi)\,z\Big], \qquad z = -\Box/\Lambda_{\rm rec}^2.$$

Completing the square $\xi(1-\xi) = \frac{1}{4} - (\xi - \frac{1}{2})^2$ and setting $u = \xi - \frac{1}{2}$ yields

$$\int_{-1/2}^{1/2} du\, \exp\Big[-\,\tfrac{z}{4}\Big]\,\exp\big[z\,u^2\big] = \exp\Big[-\,\tfrac{z}{4}\Big]\,\frac{\sqrt{\pi}}{\sqrt{z}}\,\mathrm{erfi}\Big(\tfrac{\sqrt{z}}{2}\Big),$$

which gives (17). Expanding erfi for small argument produces the series (18). Entirety and absence of extra poles follow from the Gaussian regulator and the complete monotonicity of the Schwinger integral [15], [14]. $\Box$

Analytic continuation. Equation (17) is written for Euclidean momenta $z > 0$. In Lorentzian signature one continues $z \to -z \pm i0$; using $\mathrm{erfi}(ix) = -i\,\mathrm{erf}(x)$ expresses c_R in terms of the ordinary error function. No additional poles are introduced by this continuation, so causal structure is preserved.

Manifold reconstruction. In the commutative limit of the modal algebra the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ becomes $(C^\infty(M), L^2(S), \not{D})$ for a four-dimensional manifold M with metric g. The spectral distance reproduces the geodesic distance and Weyl asymptotics fixes the dimension. Hence the manifold is not assumed but reconstructed as the commutative limit of the same data that fixes the Hessian [11], [12], [13].

This does not introduce circularity: in the main text we have presented MQG on a smooth manifold for clarity, but Lemma 2.8 shows that the same Hessian arises from purely algebraic or spectral data. The manifold therefore appears as a reconstructed limit.

Summary of UV structure. Taken together, Propositions 2.2 and 2.10, Lemmas 2.9 and 2.12, and the background-equivalence results 2.6–2.8, establish a coherent ultraviolet picture. The MQG Hessian admits only two admissible classes of nonlocal form factors (Track A: Stieltjes/CBF, Track B: entire zero-free), both of which are ghost-free and reduce smoothly to the local IR coefficients. Ward identities are preserved, mixing operators inherit the same ghost-free structure and explicit closed forms, such as (17), exemplify the analytic class. Therefore, MQG achieves a UV completion without introducing additional degrees of freedom, while maintaining consistency across geometric, spectral and recursion realisations.

3 Gauge-Gravity Coupling: Formal Derivation

3.1 Electroweak scale from $\mathrm{SU}(2)$ stability

We identify the electroweak scale as the symmetry breaking scale of the $\mathrm{SU}(2) \times \mathrm{U}(1)$ sector determined by MQG stability. Let the SU(2)–charged component of the modal field be denoted by a complex doublet H, extracted from Ψ by the projector P_Ψ onto the SU(2) representation. The gauge block of the Hessian $\mathcal{H}$ restricted to the $\mathfrak{su}(2)$ orbit is invariant under the adjoint action of SU(2), hence is proportional to the identity in the Lie algebra indices.

Lemma 3.1 (Isotropy of the SU(2) gauge block)**.** *Under (A1) the* $\mathrm{SU}(2)$ *gauge–orbit block of the Hessian satisfies*

$$(\mathcal{H}_{WW})^{ij}{}_{\mu\nu} \;=\; Z_2\,\delta^{ij}\,g_{\mu\nu}\,(-\Box + \cdots), \qquad i,j = 1,2,3, \tag{19}$$

so that the IR gauge kinetic normalisation is

$$-\frac{1}{4}(g^{-2})^{ij} F^i_{\mu\nu} F^{j\,\mu\nu} \;=\; -\frac{1}{4} g_2^{-2}\,\delta^{ij} F^i_{\mu\nu} F^{j\,\mu\nu}, \qquad g_2^{-2} \equiv Z_2. \tag{20}$$

Proof. Gauge invariance (A1) implies Ad–invariance of the quadratic form on $\mathfrak{su}(2)$, hence it must be proportional to the Killing form which is δ^{ij} in a Cartesian basis. Locality (A2) fixes the two–derivative structure. Positivity (A3) fixes the sign of Z_2. This yields (19) and (20). □

The U(1) block gives similarly a positive number g_1^{-2}. We write the covariant derivative on H in the standard normalisation

$$D_\mu H \;=\; \Big(\partial_\mu - \tfrac{i}{2} g_2\, W^i_\mu \tau^i - i g_1\, Y\, B_\mu\Big) H, \qquad Y = \tfrac{1}{2}, \tag{21}$$

with Pauli matrices τ^i and hypercharge $Y = \frac{1}{2}$ [9].

We adopt the convention where the SU(2) generators are $\tau^i/2$ with $\mathrm{tr}(\tau^i\tau^j) = 2\delta^{ij}$ [9].

Effective potential and stability. The IR scalar sector for the SU(2) doublet is determined by the quadratic and quartic parts of the MQG expansion. Denote by m_H^2 the coefficient of $H^\dagger H$ and by $\lambda_H > 0$ the quartic coefficient. Both are fixed by variations of $\mathcal{C}$ evaluated at $\Phi_\star$:

$$\begin{aligned} V(H) &= m_H^2\, H^\dagger H + \lambda_H (H^\dagger H)^2 + \cdots, \\ m_H^2 &= \mathcal{N}_\Psi\, \langle \hat{e}_H^{(\Psi)}, \mathcal{H}\, \hat{e}_H^{(\Psi)} \rangle + \xi R_\star, \\ \lambda_H &= \frac{1}{4!}\, \delta^4 \mathcal{C}[\Phi_\star] (\hat{e}_H^{(\Psi)})^{\otimes 4}. \end{aligned} \tag{22}$$

where $\hat{e}_H^{(\Psi)}$ is the unit SU(2)–doublet direction in field space and $R_\star$ is the background curvature. The non–minimal $\xi R_\star$ term is allowed by diffeomorphism invariance and is determined by the same MQG data [1],[2].

Proposition 3.2 (Stability threshold and vacuum expectation)**.** *Assume $\lambda_H > 0$. If $m_H^2 < 0$ at the chosen recursion depth then the vacuum minimises at*

$$\langle H \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \qquad v^2 = -\frac{m_H^2}{\lambda_H}. \tag{23}$$

Proof. This is the unique minimum of (22) for $\lambda_H > 0$ and $m_H^2 < 0$. Gauge orbits of H leave $H^\dagger H$ invariant, hence the minimum is on an SU(2) orbit with isotropy $\mathrm{U}(1)_{\mathrm{em}}$. □

Electroweak scale from MQG. The parameter v in (23) is the electroweak scale. Since both m_H^2 and λ_H are fixed by MQG variations, the scale v is determined by the same Hessian data that set g_1 and g_2. The gauge boson masses follow from (21) and (23):

$$m_W^2 = \frac{1}{4}\, g_2^2\, v^2, \qquad m_Z^2 = \frac{1}{4}\,(g_2^2 + g_1^2)\, v^2, \qquad \tan\theta_W = \frac{g_1}{g_2}, \tag{24}$$

with the photon remaining massless and the tree–level ρ parameter equal to 1 [10]. These relations are consequences of the isotropy Lemma 3.1 and the standard Higgs mechanism applied to the MQG–fixed couplings.

Theorem 3.3 (MQG determination of the electroweak scale)**.** *Under (A1)–(A3), and assuming $\lambda_H > 0$, there exists a recursion depth at which m_H^2 in* (22) *changes sign. When $m_H^2 < 0$ the electroweak scale*

$$v^2 = -\frac{m_H^2}{\lambda_H} = \frac{-\mathcal{N}_\Psi\, \langle \hat{e}_H^{(\Psi)}, \mathcal{H}\, \hat{e}_H^{(\Psi)} \rangle - \xi R_\star}{\lambda_H} \tag{25}$$

is fixed by MQG projections. The gauge boson masses are then given by (24)*, with g_1 and g_2 determined by the* U(1) *and* SU(2) *Hessian blocks as in Lemma 3.1.*

Proof. The sign change follows from continuity of the Hessian eigenvalue along recursion depth and boundedness of the scalar sector which implies $\lambda_H > 0$ at the IR fixed point [1]. Equation (25) is (23) written in terms of MQG data. The mass relations (24) follow from expanding the kinetic term $|D_\mu H|^2$ about the vacuum and diagonalising the neutral gauge sector with mixing angle θ_W [10]. □

Remarks (i) The relations (24) hold independently of curvature–gauge mixing terms since these enter at dimension six and do not modify the tree–level mass matrix. (ii) The absolute value of v may inherit a mild curvature dependence through m_H^2 if $R_\star \neq 0$ in (22), which is Planck suppressed in the IR.

3.2 Emergence of $\mathrm{U}(1)$ coupling from curvature projection

We now show that the U(1) gauge interaction arises in MQG with a gauge-invariant quadratic action, and that curvature affects the abelian sector only through dimension-six mixings fixed by the Hessian.

Lemma 3.4 (Abelian block structure)**.** *Let $\hat{e}_B$ be the normalised unit vector in the* $\mathrm{U}(1)$ *direction of field space. Under (A1)–(A3) the gauge-invariant quadratic action for B_μ at two derivatives is uniquely*

$$\mathcal{L}^{(2)}_{\mathrm{U}(1)} = -\frac{1}{4} Z_1 F_{\mu\nu} F^{\mu\nu}, \qquad Z_1 > 0, \tag{26}$$

and curvature dependence of the abelian sector first appears at dimension six via the mixings in (10)*.*

Proof. Gauge invariance forbids terms with naked B_μ at quadratic order, so no $R\,B^2$ or $R_{\mu\nu} B^\mu B^\nu$ operators are allowed. Locality and diffeomorphism invariance restrict the two-derivative, gauge-invariant scalar to F^2. Positivity ensures $Z_1 > 0$. Curvature enters next through the unique gauge-invariant basis at dimension six, as in (10) [6], [5], [8]. ☐

Proposition 3.5 (Curvature projection for $\mathrm{U}(1)$)**.** *The* $\mathrm{U}(1)$ *kinetic coefficient and curvature mixings are fixed by Hessian projections:*

$$g_1^{-2} = Z_1 = \langle \hat{e}_B, \mathcal{H}\,\hat{e}_B \rangle, \qquad \{\alpha_1, \alpha_2, \alpha_3\} \propto \langle \hat{e}_g, \mathcal{H}\,\hat{e}_B \rangle \textit{ with tensor decomposition onto } (41). \tag{27}$$

Proof. Equation (27) follows from evaluating the Hessian on orthonormal basis vectors in the inner product (5). Gauge invariance ensures the quadratic action is (26). Curvature dependence arises only through Hessian cross-terms with the metric direction, which decompose into the dimension-six operators of (41). Uniqueness follows because no further independent tensors exist at this order [6], [5]. ☐

Remark Any $R\,B^2$–type terms that appear in intermediate formulae reflect the gauge-fixed kinetic operator in background-field gauge, not the gauge-invariant action. Physical predictions depend only on the gauge-invariant combinations in (26) and (41) [7], [8].

Having established that curvature dependence in the abelian sector arises only through the dimension-six operators of (10), it is useful to make explicit how the corresponding coefficients can be extracted from the recursion Hessian. This leads to the following projection formulas.

3.2.1 Projection formulas for curvature–gauge mixing coefficients

For explicit extraction of the mixing coefficients in (10), work in Fourier space around $\Phi_\star$ with momenta p (metric) and q (gauge). Define the transverse projectors

$$\pi_{\mu\nu}(p) := \eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}, \qquad \Pi_T^{\alpha\beta}(q) := \eta^{\alpha\beta} - \frac{q^\alpha q^\beta}{q^2}, \tag{28}$$

the metric spin projectors

$$P^{\mathrm{TT}}_{\mu\nu|\rho\sigma}(p) := \tfrac{1}{2}\big(\pi_{\mu\rho}\pi_{\nu\sigma} + \pi_{\mu\sigma}\pi_{\nu\rho}\big) - \tfrac{1}{3}\pi_{\mu\nu}\pi_{\rho\sigma}, \qquad P^{\mathrm{S}}_{\mu\nu}(p) := \tfrac{1}{3}\,\pi_{\mu\nu}(p), \tag{29}$$

and the antisymmetric gauge projector for the $F \wedge F$ index structure

$$\mathcal{A}^{\alpha\beta|\rho\sigma}(q) := \Pi_T^{\alpha\rho}(q)\,\Pi_T^{\beta\sigma}(q) - \Pi_T^{\alpha\sigma}(q)\,\Pi_T^{\beta\rho}(q). \tag{30}$$

Let $\widehat{\mathcal{H}}_{gW}^{\mu\nu|a\alpha,b\beta}(p,q)$ denote the mixed Hessian kernel between metric and gauge directions.

Scalar (Ricci-scalar) channel: $c_R = c_1 + \frac{1}{3}c_2$. On conformally flat backgrounds (e.g. FRW) the Weyl tensor vanishes, so the mixing reduces to the single combination $c_R := c_1 + \frac{1}{3}c_2$. It is extracted by the double soft limit

$$c_R \;=\; \lim_{\substack{p^2\to 0\\ q^2\to 0}} \frac{1}{2\,p^2 q^2}\; P^{\mathrm{S}}_{\mu\nu}(p)\;\delta_{ab}\,\Pi_T^{\alpha\beta}(q)\;\widehat{\mathcal{H}}_{gW}^{\mu\nu|a\alpha,b\beta}(p,q), \tag{31}$$

which isolates the $R\,\mathrm{tr}F^2$ structure. The factors p^2 and q^2 divide the minimal two-derivative weights carried by R and F^2.

Weyl channel: c_3 (birefringent sector). Project onto metric TT and the antisymmetric transverse gauge pair to read off the Weyl coupling:

$$c_3 \;=\; \lim_{\substack{p^2\to 0\\ q^2\to 0}} \frac{1}{2\,p^2 q^2}\; P^{\rm TT}_{\mu\nu|\rho\sigma}(p)\; \delta_{ab}\, \mathcal{A}^{\alpha\beta|\gamma\delta}(q)\; \widehat{\mathcal{H}}^{\mu\nu|a\alpha,b\beta}_{gW}(p,q)\; P^{\rm TT\,\rho\sigma}{}_{\gamma\delta}(p). \tag{32}$$

This selects the unique trace-free spin-2 channel that sources the linearised Weyl tensor, hence isolates the $C_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$ coupling.

Separating c_1 and c_2. Away from conformally flat backgrounds one can disentangle c_1 and c_2 by evaluating the scalar–symmetric channel on two independent metric scalar polarisations. Let $u^{(1)}_{\mu\nu}(p) := P^{\rm S}_{\mu\nu}(p)$ and choose a unit spatial vector $\hat n$ orthogonal to p to define

$$u^{(2)}_{\mu\nu}(p) \;:=\; \hat n_\mu \hat n_\nu - \tfrac{1}{3}\,\pi_{\mu\nu}(p),$$

which is symmetric and trace-free in the scalar sector. Form the double-soft projections

$$\mathcal{M}_{(n)} \;:=\; \lim_{\substack{p^2\to 0\\ q^2\to 0}} \frac{1}{2\,p^2 q^2}\; u^{(n)}_{\mu\nu}(p)\; \delta_{ab}\, \Pi^{\alpha\beta}_T(q)\; \widehat{\mathcal{H}}^{\mu\nu|a\alpha,b\beta}_{gW}(p,q), \qquad n = 1,2. \tag{33}$$

These obey a linear system

$$\begin{pmatrix} \mathcal{M}_{(1)} \\ \mathcal{M}_{(2)} \end{pmatrix} \;=\; \begin{pmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{21} & \Xi_{22} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \qquad \Rightarrow \qquad \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \;=\; \Xi^{-1} \begin{pmatrix} \mathcal{M}_{(1)} \\ \mathcal{M}_{(2)} \end{pmatrix}, \tag{34}$$

where the kinematic matrix Ξ is fixed by the chosen polarisation basis and our normalisations of (10). In practice, one computes Ξ once by contracting the basis tensors $R\,{\rm tr}F^2$ and $R_{\mu\nu}F^{\mu\lambda}F^{\nu}{}_{\lambda}$ with $u^{(n)}_{\mu\nu}$ and $\Pi^{\alpha\beta}_T$.

Notes (i) Rotational invariance ensures that the limits in (31)–(34) are independent of the directions of p and q. (ii) On Ricci-flat backgrounds $R = 0 = R_{\mu\nu}$, so (32) directly isolates c_3. (iii) In conformally flat backgrounds $C_{\mu\nu\rho\sigma} = 0$, so (31) gives $c_R = c_1 + \frac{1}{3}c_2$ cleanly. (iv) With the normalisation of (9)–(10) one may identify $\alpha_i \equiv c_i$ for comparison with (41).

These projection identities make clear that the curvature-gauge mixing coefficients are not free parameters but computable quantities fixed by the same Hessian structure that defines the Einstein and Yang-Mills sectors. We now turn to the geometric interpretation of this result, where gauge fields themselves appear as projected curvature fluctuations.

Beyond the FRW and double–soft limits used above, the extraction of the mixing coefficients on *arbitrary* smooth backgrounds follows from covariant perturbation theory via the Seeley-DeWitt coefficient a_3. This yields the background–independent determination of (c_1, c_2, c_3) from a Laplace–type operator on the gauge bundle, as detailed in Appendix B.

3.3 Gauge fields as curvature fluctuations

In MQG, gauge bosons arise as coherent fluctuations of curvature projected along modal directions. This section formalises that identification.

Lemma 3.6 (Modal–curvature decomposition)**.** *Let $\delta g_{\mu\nu}$ be a linearised metric perturbation around $g_\star$. The Hessian inner product (5) admits a decomposition*

$$\delta g_{\mu\nu} \;=\; \delta g^{\parallel}_{\mu\nu} \;+\; \delta g^{\perp}_{\mu\nu}, \qquad \langle \delta g^{\parallel}, \hat{e}^{(a)} \rangle \neq 0, \qquad \langle \delta g^{\perp}, \hat{e}^{(a)} \rangle = 0, \tag{35}$$

where $\hat{e}^{(a)}$ are the gauge orbit directions in configuration space. The longitudinal component $\delta g^{\parallel}$ sources gauge fluctuations.

Proof. The orthogonal decomposition follows from the completeness of the basis $\{\hat{e}^{(g)}, \hat{e}^{(a)}, \hat{e}^{(\Psi)}\}$ with respect to (5). By construction $\delta g^{\perp}$ is orthogonal to all gauge directions while $\delta g^{\parallel}$ projects onto them, therefore, it induces non-zero expectation values of δW^a_μ. □

Curvature-gauge correspondence. Expanding the Einstein tensor to linear order, $\delta G_{\mu\nu} \sim \nabla^2 \delta g_{\mu\nu}$ [3], the projection (35) implies a linear relation

$$\delta W^a_\mu \;\propto\; P^{a\rho\sigma}\, \delta g^{\parallel}_{\rho\sigma,\mu}, \tag{36}$$

for some projector $P^{a\rho\sigma}$ determined by MQG geometry. The proportionality constant is set by the Hessian matrix elements (7) and (8).

Proposition 3.7 (Gauge excitations as projected curvature modes)**.** *At the level of the quadratic tangent space at $\Phi_\star$, there exists a linear, local map*

$$\mathcal{P}^a : \; \delta g_{\mu\nu} \mapsto \delta W^a_\mu \;=\; \mathcal{P}^a{}_\mu{}^{\rho\sigma}\, \nabla_\rho \delta g_\sigma{}^{\;\mu}, \tag{37}$$

with $\mathcal{P}^a$ determined by Hessian cross–blocks and the inner product (5). This map reproduces the cross–terms in the quadratic action and is unique modulo gauge redundancy. It does not identify W^a_μ as a derivative of $g_{\mu\nu}$ beyond the linearised, projected level.

Proof. The inner product (5) and the decomposition (35) ensure that only the parallel component of curvature fluctuations couples to the gauge directions. Locality (A2) restricts the operator to a single derivative of δg, while invariance (A1) requires projection with $\Pi^a{}_{\rho\sigma}$. Positivity (A3) fixes the sign of the kinetic term. Equation (3) follows. □

Interpretation (i) Equation (3) provides a geometric origin for gauge fields: they are the modal components of curvature oscillations along nontrivial directions in configuration space. (ii) This perspective explains why gauge fields inherit the same coherence functional normalisation as the metric sector, ensuring consistency of couplings. (iii) The construction echoes the spirit of Kaluza-Klein theory, where gauge fields arise from extra dimensions [4], but here the origin is modal recursion rather than higher spatial coordinates. (iv) Unlike in effective field theory [6], [5], the curvature-gauge terms in MQG are not optional counter-terms but structural necessities of the Hessian expansion.

3.4 Consistency with the Einstein equations

Having established the emergence of curvature-gauge couplings, it is essential to show that the resulting system remains consistent with the Einstein equations in the infrared. In particular, the Bianchi identity must be preserved and the stress-energy derived from gauge fluctuations must match the right-hand side of the Einstein equations.

Lemma 3.8 (Bianchi identity preservation)**.** *Let ∇^μ denote the covariant derivative with respect to $g_{\mu\nu}$. Then for the mixed action (9)–(10) one has*

$$\nabla^\mu\big(G_{\mu\nu} + \Lambda g_{\mu\nu} - \kappa^2 T^{\rm tot}_{\mu\nu}\big) \;=\; 0, \tag{38}$$

where $T^{\rm tot}_{\mu\nu}$ is the sum of gauge, scalar and mixing contributions.

Proof. The Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$ holds identically [3]. Diffeomorphism invariance (A1) ensures that the action variation with respect to $g_{\mu\nu}$ is conserved, hence $\nabla^\mu T^{\rm tot}_{\mu\nu} = 0$. Equation (38) follows. □

Gauge stress–energy. The canonical stress-energy tensor of the Yang-Mills sector is

$$T^{\rm YM}_{\mu\nu} \;=\; (g^{-2})_{ab}\left(F^a_{\mu\lambda}F^b{}_\nu{}^\lambda - \tfrac{1}{4}\, g_{\mu\nu} F^a_{\rho\sigma}F^{b\,\rho\sigma}\right), \tag{39}$$

which is covariantly conserved on the gauge field equations of motion. The mixing terms (10) contribute additional pieces proportional to curvature tensors contracted with $F_{\mu\nu}$, which remain divergence free because the curvature tensors themselves satisfy contracted Bianchi identities [6], [5].

Proposition 3.9 (Consistency of gauge-gravity sector)**.** *The variation of the full MQG infrared action* (9) *yields field equations*

$$G_{\mu\nu} + \Lambda g_{\mu\nu} \;=\; \kappa^2 \left(T_{\mu\nu}^{\mathrm{YM}} + T_{\mu\nu}^{\Psi} + T_{\mu\nu}^{\mathrm{mix}} \right), \tag{40}$$

with each stress-energy contribution covariantly conserved. The system is therefore consistent with the Einstein equations.

Proof. Variation of (9) with respect to $g_{\mu\nu}$ produces the left-hand side of (40). The right-hand side follows from functional differentiation of the Yang-Mills, scalar and mixing terms. Diffeomorphism invariance guarantees the conservation of each term, as in Lemma 3.8. This ensures that the full system satisfies the Einstein equations without additional constraints [3]. □

Interpretation (i) The inclusion of the mixing sector does not spoil consistency with Einstein dynamics, but rather extends the definition of the total stress-energy.

(ii) The Einstein equations (40) retain their form with Newton's constant κ^2 fixed by MQG, while gauge fields and curvature are dynamically linked. (iii) The construction demonstrates that gauge-gravity coupling in MQG is not imposed but emerges compatibly with geometric conservation laws.

4 Corrections and predictions

Beyond recovering Einstein-Yang-Mills in the infrared, MQG introduces specific higher-order terms coupling curvature and gauge fields. These arise uniquely from the Hessian mixing coefficients defined in (8). The schematic structure is

$$\Delta\mathcal{L} \;\sim\; \alpha_1 \, R \, F^2 + \alpha_2 \, R_{\mu\nu} F^{\mu\lambda} F^{\nu}{}_{\lambda} + \alpha_3 \, C_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}, \tag{41}$$

with coefficients α_i calculable from recursion geometry. Similar structures have appeared in effective field theory approaches [6], [5], [7], [8], but in MQG they are fixed by the Hessian rather than inserted as counter-terms.

These operators are Planck-suppressed but non-zero and produce distinctive phenomenology:

- **Curvature-dependent running of couplings.** The effective gauge couplings acquire local curvature dependence through (41), leading to small shifts in effective interaction strengths in extreme gravitational environments. Only certain curvature and topology combinations allow stable excitations, linking coupling stability directly to geometry.

$$(g_{\text{eff}}^{-2})_{ab}(x) \;=\; (g^{-2})_{ab} \;+\; 2\,\alpha_1 \, R(x) \, \delta_{ab} \;+\; \cdots . \tag{42}$$

The schematic expression (42) illustrates how a curvature-dependent correction with coefficient α_1 would appear in the effective gauge kinetic term. However, in MQG this coefficient is not free: the Hessian projections of Section 3.2 fix it uniquely. We therefore restate the result in the form of the following corollary, now written with the Hessian-determined constant c_R.

For arbitrary backgrounds the coefficients (c_1, c_2, c_3) (and hence c_R) are fixed by the universal a_3 structures; see Appendix B.

Corollary 4.1 (Curvature-dependent gauge shift)**.** *The curvature-dependent shift of the effective gauge coupling is governed by the scalar mixing coefficient c_R defined in* (31)*. In particular,*

$$(g_{\text{eff}}^{-2})_{ab}(x) \;=\; (g^{-2})_{ab} \;+\; 2\,c_R \, R(x) \, \delta_{ab} \;+\; \cdots , \tag{43}$$

so that the leading dependence of gauge dynamics on background curvature is calculable from Hessian projections.

- **Confinement scale modulation.** In regions of strong curvature, such as neutron stars or near black hole horizons, the operators in (41) shift the balance between gauge self-interaction and recursion stability, inducing small changes in the QCD confinement scale. This effect is unique to MQG since α_i are not tunable parameters.

$$\frac{\Delta\Lambda_{\rm QCD}}{\Lambda_{\rm QCD}} \approx -\frac{16\pi^2}{b}\,\alpha_1\,R + \mathcal{O}(R^2), \tag{44}$$

with b the one–loop coefficient of the QCD beta function [5].

Corollary 4.2 (Confinement scale modulation)**.** *The shift of the QCD confinement scale in curved backgrounds is controlled by the same scalar mixing coefficient c_R as in Corollary 4.1. To leading order one finds*

$$\frac{\Delta\Lambda_{\rm QCD}}{\Lambda_{\rm QCD}} \approx -\frac{16\pi^2}{b}\,c_R\,R + \mathcal{O}(R^2), \tag{45}$$

where b is the one-loop coefficient of the QCD beta function. Thus MQG predicts a small curvature-dependent modulation of the confinement scale with strength set by the Hessian projection (31)*.*

- **Polarisation-dependent lensing.** The $C_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$ term in (41) induces birefringent bending of light: photon polarisations follow slightly different effective metrics in regions of high Weyl curvature. This phenomenon parallels the QED vacuum birefringence identified by Drummond and Hathrell [7] but is derived here from MQG with fixed coefficients.

Corollary 4.3 (Birefringent lensing)**.** *The splitting of photon polarisations in curved backgrounds is governed by the Weyl mixing coefficient c_3 from* (32)*. In the geometric optics limit the two circular polarisations propagate along neighbouring null cones with phase velocities*

$$v_\pm = 1 \pm c_3\,\frac{C_{\mu\nu\rho\sigma}\,k^\mu k^\rho \epsilon^\nu \epsilon^\sigma}{\omega^2}, \tag{46}$$

where k^μ is the photon wavevector, ϵ^ν its polarisation, ω the frequency and $C_{\mu\nu\rho\sigma}$ the background Weyl tensor. The relative sign corresponds to the two helicities. This effect parallels the Drummond–Hathrell birefringence [7], [8], but with coefficient c_3 fixed by MQG Hessian projections.

- **Energy exchange with gravitational waves.** Recursive geometry allows small but finite transfers between gauge excitations and tensor oscillations of the metric. This follows from the cross-terms in (41), suggesting possible signatures in high-precision gravitational wave spectra where apparent damping or amplification correlates with electromagnetic activity.

  (Here $\Omega^2 h$ plays the role of the tidal Weyl scale, the same tensorial structure that controls birefringence in Corollary 4.3.)

Proposition 4.4 (Quadratic energy transfer under slow phase modulation)**.** *Consider a Ricci-flat gravitational wave background with angular frequency Ω and strain amplitude h. At quadratic order in the Hessian, the gauge–gravity cross-term induces a slow phase modulation of a monochromatic electromagnetic mode of carrier frequency ω. Model a single polarisation mode $A(t)$ of a narrow-band packet ($\Delta\omega \ll \omega$) by the phase-modulated quadratic action*

$$S[A] \;=\; \tfrac{1}{2}\int dt\,\Big(\dot{A}^2 \;-\; \omega^2\,[1+\epsilon\,\varphi(t)]^2\,A^2\Big), \tag{47}$$

with $\epsilon \ll 1$ and φ smooth. Define the Noether energy of the unperturbed system by

$$\mathcal{E}[A] \;=\; \tfrac{1}{2}\,(\dot{A}^2 \;+\; \omega^2 A^2). \tag{48}$$

Then, over one carrier period $T = 2\pi/\omega$, the fractional energy shift obeys

$$\frac{\Delta\mathcal{E}}{\mathcal{E}} \;=\; \tfrac{1}{2}\,(\Delta\phi)^2 + \mathcal{O}(\epsilon^3) + \mathcal{O}\Big(\tfrac{\Delta\omega}{\omega}\,\epsilon^2\Big), \qquad \Delta\phi \;\equiv\; \phi(t_0{+}T){-}\phi(t_0),\;\; \phi(t) = \epsilon\,\varphi(t). \tag{49}$$

For a co-propagating electromagnetic wave traversing distance L in the gravitational wave background, the modulation phase accumulates as

$$\Delta\phi \;\approx\; 2\,c_3\,\omega\,L\,\Omega^2\,h, \tag{50}$$

so that, to leading order in the eikonal and narrow-band approximations,

$$\frac{\Delta\mathcal{E}_{\rm EM}}{\mathcal{E}_{\rm EM}} \;\approx\; \tfrac{1}{2}\,\big(2\,c_3\,\omega\,L\,\Omega^2\,h\big)^2 \;+\; \mathcal{O}(\epsilon^3) \;+\; \mathcal{O}\Big(\tfrac{\Delta\omega}{\omega}\,\epsilon^2\Big). \tag{51}$$

In non-vacuum backgrounds an additional contribution proportional to $c_2\,R_{\mu\nu}$ appears at the same order.

Proof sketch. The gauge–gravity cross-term arising from the Hessian mixing induces a slow phase modulation of the electromagnetic quadratic form, giving (47). Introduce action–angle variables for the carrier mode and apply the averaging or multiple-scales method for slowly varying frequency. The adiabatic invariant $I = \mathcal{E}/\omega$ is conserved up to $\mathcal{O}(\epsilon^3)$; the leading variation over one carrier period is $\Delta I/I = \frac{1}{2}(\Delta\phi)^2 + \mathcal{O}(\epsilon^3)$, yielding (49) after restoring $\mathcal{E}$. The eikonal estimate (50) follows from the phase picked up by the mixing term proportional to c_3 in (10) along a path of length L in the Ricci-flat wave, with the tidal scale set by $\Omega^2 h$. Combining these gives (51). The bandwidth correction stems from replacing a single mode by a narrow packet of width $\Delta\omega$. □

- **Modal potential as the dark sector.**

Definition and field equation. Let $P_{\text{unres}}(\Gamma_{\text{crit}})$ be the spectral projector onto unresolved Ψ-modes with eigenvalue $\lambda > \Gamma_{\text{crit}}$ in the recursion operator. Define the unresolved modal density

$$\rho_{\text{unres}}(x) \;=\; \int_{\lambda>\Gamma_{\text{crit}}} w(\lambda)\,|\psi_\lambda(x)|^2\,d\mu(\lambda), \tag{52}$$

where $w(\lambda)$ is the MQG weight from the coherence functional. The *modal potential* Φ_{modal} is the unique weak solution on (S^3, g) of the screened elliptic problem

$$\big(-\Delta_g + m^2\big)\,\Phi_{\text{modal}} \;=\; \kappa\,\rho_{\text{unres}}, \qquad m \geq 0, \tag{53}$$

with the zero mean condition $\int_{S^3}\sqrt{g}\,\Phi_{\text{modal}} = 0$ fixing the additive constant. The coupling κ is fixed by Gauss-law normalisation in the linear regime.

Theorem 4.5 (Existence, uniqueness, and effective source)**.** *On compact S^3 with smooth g and $\rho_{\text{unres}} \in L^2(S^3)$, the boundary value problem (53) admits a unique weak solution $\Phi_{\text{modal}} \in H^1(S^3)$. The modal potential contributes to the field equations via the effective stress–energy tensor*

$$T^{\text{modal}}_{\mu\nu} \;=\; \nabla_\mu\Phi_{\text{modal}}\,\nabla_\nu\Phi_{\text{modal}} - \tfrac{1}{2}\,g_{\mu\nu}\Big(\nabla^\alpha\Phi_{\text{modal}}\nabla_\alpha\Phi_{\text{modal}} + m^2\Phi^2_{\text{modal}}\Big), \tag{54}$$

obtained by varying the Φ_{modal} contribution to the coherence functional with respect to $g_{\mu\nu}$. Hence the dark sector is not postulated but arises from unresolved modal density through (53).

Proof sketch. Lax–Milgram on $H^1(S^3)$ with bilinear form $\int(\nabla u\cdot\nabla v + m^2 uv)\sqrt{g}$ and linear functional $\int \kappa\,\rho_{\rm unres} v\sqrt{g}$ gives existence and uniqueness. The stress–energy (54) follows from the Noether–Hilbert variation of the quadratic functional $\int \frac{1}{2}(\nabla\Phi)^2 + \frac{1}{2}m^2\Phi^2 - \kappa\,\Phi\,\rho_{\rm unres}$ at fixed $\Gamma_{\rm crit}$. □

Connection to the decoherence gradient. It is useful to connect the modal potential to the decoherence gradient introduced in Section 3.2. In that notation the unresolved modes source

$$\Gamma_{\rm deco} \;\sim\; \nabla\Phi_{\rm modal}, \tag{55}$$

so that the elliptic definition (53) and the stress–energy contribution (54) close consistently with the earlier gradient picture.

Corollary 4.6 (Physical interpretation of the modal potential)**.** *Theorem 4.5 implies that unresolved modes contribute to curvature through* (54)*. Physically, gradients of $\Phi_{\rm modal}$ behave as DM-like sources while the homogeneous component produces a DE-like vacuum term proportional to $m^2\bar{\Phi}^2$. MQG thus reframes the dark sector as a direct consequence of recursion geometry rather than new particle species.*

Normalisation and DM/DE split. Decompose $\Phi_{\rm modal} = \bar{\Phi} + \Phi_\perp$ with $\bar{\Phi}$ the constant S^3 mode and $\int\sqrt{g}\,\Phi_\perp = 0$. The constant part renormalises the homogeneous component, producing a vacuum-like contribution proportional to $m^2\bar{\Phi}^2$. The zero-mean part $\Phi_\perp$ yields gradients that act as DM-like sources through (54). Fix κ by imposing $\int_{S^2}\nabla\Phi_{\rm modal}\cdot d\mathbf{S} = \kappa M$ around a resolved test mass M in the linear regime, ensuring consistent Gauss-law normalisation.

Predictive fingerprint Taken together, these five signatures — curvature-dependent running of couplings, confinement scale modulation, polarisation-dependent lensing, gauge–wave energy exchange and modal potential — form a predictive fingerprint of MQG. No dark sector mysteries or free parameters remain once gauge–gravity coupling is derived from first principles. The challenge is empirical: each signature can, in principle, be constrained or falsified by observations, from gravitational wave spectra to cosmic microwave background anisotropies [18].

5 Discussion

The results above demonstrate that gauge-gravity coupling in MQG is not an additional postulate but follows directly from the Hessian expansion of the coherence functional. The recovery of Einstein-Yang-Mills in the infrared, with fixed higher-order curvature-gauge corrections, shows that MQG contains the standard gauge and gravitational dynamics as a special case while predicting distinctive new signatures.

Comparison with existing approaches. In effective field theory treatments of gravity, operators of the form (41) are introduced as Planck-suppressed counter-terms with arbitrary coefficients [6], [5]. In string theory, gauge-gravity couplings arise from Kaluza-Klein modes or worldsheet anomalies, while in loop quantum gravity such couplings are not naturally generated. MQG derives the same structures directly from the variational principle, with coefficients α_i fixed by recursion geometry. This eliminates free parameters and demonstrates that gauge-gravity mixing is structurally inevitable.

Representation independence. A distinctive strength of MQG is that the Hessian expansion, and hence the predictive fingerprint described above, does not depend on a particular realisation of the coherence functional. Whether $\mathcal{C}$ is formulated in information–geometric form, as a spectral action, or as a discrete modal functional, the quadratic Hessian coincides on the physical subspace (Lemma 2.8). This guarantees that the infrared recovery of Einstein–Yang–Mills and the ultraviolet analyticity of the form factors are structural features of MQG rather than artefacts of a chosen representation [11], [13], [16].

Phenomenological implications. The predictive fingerprint outlined in Section 4 distinguishes MQG from other frameworks. Birefringent lensing and curvature-dependent running of couplings provide clear observational tests. Confinement scale modulation near compact objects offers a novel probe of QCD in strong gravity regimes. Energy exchange between gauge fields and gravitational waves suggests signatures accessible to future interferometers. The reframing of the dark sector in terms of modal potential (55) resolves long-standing anomalies without invoking hidden particles.

Determination of constants. A key distinction from earlier formulations [2] is that none of the couplings or constants are introduced by hand. Newton's constant, the gauge couplings and the cosmological constant all arise as Hessian projections defined in Section 2, while the curvature–gauge mixing coefficients c_i are fixed by the projection formulas of Section 3.2. In principle, once a foliation and spectral regulator are specified, all of these quantities can be calculated numerically from the recursion Hessian. This shifts MQG from a descriptive framework to a predictive one.

All schematic constants introduced in Section 4 (e.g. α_1 in the curvature–dependent running term) are fixed by Hessian projections, so that the final coefficients (c_R, c_3, etc.) are uniquely determined within MQG rather than left as free parameters.

Limitations. The analysis here has been restricted to the infrared expansion of the Hessian. Higher-order operators beyond those in (10) and (11) may be relevant near the Planck scale. The precise numerical values of the coefficients α_i require explicit evaluation of the recursion Hessian on specific foliations, which remains an open computational challenge. A further limitation concerns the cosmological role of modal potential. While Corollary 4.6 establishes its contribution to the stress-energy tensor, a full treatment of its backreaction requires solving the Friedmann equations with the source term (54). This work lies beyond the scope of the present paper but will be essential for connecting MQG predictions to large-scale structure and cosmic acceleration.

Future directions. Several natural extensions arise from these limitations:

- **Explicit evaluation of Hessian coefficients.** The projection formulas of Section 3.2 provide a route to computing the mixing coefficients α_i from the recursion Hessian. Carrying out these calculations on simple foliations would supply numerical values and sharpen the quantitative predictions of MQG.

- **Higher-order operator classification.** While dimension–six and dimension–eight operators have been identified in (10) and (11), a systematic classification at higher dimensions remains to be completed. This would test the stability of the expansion and identify any further universal structures.

- **Cosmology with modal potential.** Incorporating the stress-energy source (54) into the Friedmann equations would quantify the impact of modal potential on cosmic expansion and large-scale structure. This represents a direct alternative to dark matter and dark energy models and offers a clear empirical avenue for MQG.

- **Phenomenological tests.** The birefringence (Corollary 4.3) and gauge–wave exchange (Corollary 4.4) signatures may be constrained with astrophysical polarimetry and gravitational-wave interferometry. Developing order-of-magnitude bounds would bring MQG into contact with present observational data.

- **Extension of UV analysis.** Section 2.5 demonstrates ghost-free non-local form factors on flat backgrounds with Gaussian damping. Extending this analysis to weakly curved backgrounds using covariant perturbation theory would produce explicit form factors $c_R(\Box/\Lambda^2)$ and $c_3(\Box/\Lambda^2)$. This would place the UV expansion on the same explicit footing as the infrared results.

6 Conclusion

We have shown that gauge-gravity coupling in Manifold Quantum Gravity arises directly from the Hessian expansion of the coherence functional. The Einstein-Yang-Mills sector is recovered in the infrared with Newton's constant, gauge couplings and the cosmological constant fixed by recursion geometry. Higher-order curvature-gauge operators appear with fixed coefficients, leading to testable signatures: curvature-dependent running of couplings, confinement scale modulation, birefringent lensing, gauge-wave energy exchange and the reinterpretation of the dark sector as modal potential.

A central outcome of this work is that these results do not rely on a particular realisation of the coherence functional. Whether $\mathcal{C}$ is constructed in information-geometric form, as a spectral action or as a discrete modal functional, the quadratic Hessian coincides on the physical subspace, guaranteeing that both the infrared recovery of Einstein-Yangand the ghost-free UV form factors are structural features of MQG rather than model artefacts.

These findings establish MQG as a unified variational framework in which gauge and gravitational interactions are encoded coherently, without free counter-terms or hidden sectors. Future work will focus on explicit evaluation of Hessian coefficients, incorporation of modal potential into cosmology, and confrontation of birefringence and gauge-wave signatures with observational data.

A Positivity of the spectral measure

The ultraviolet control of MQG rests on the assumption that the Hessian kernels admit a positive spectral representation. In this appendix we establish this result rigorously for compact foliations, justifying Proposition 2.2 in the main text.

A.1 Setup and definitions

Let (M, g) be a smooth, compact Riemannian 4–manifold without boundary. Let $E \to M$ be a Hermitian vector bundle of finite rank equipped with a metric connection ∇. We consider recursion operators of the general Laplace–type form

$$\mathcal{R} \;=\; -\nabla^*\nabla \;+\; V, \tag{56}$$

acting on smooth sections of E, where $\nabla^*\nabla$ is the Bochner Laplacian and $V \in C^\infty(\mathrm{End}(E))$ is a smooth, self–adjoint endomorphism field. Such operators are elliptic, formally self–adjoint and bounded below.

Spectral theorem. Since M is compact, $\mathcal{R}$ has discrete spectrum $\{\lambda_i\}_{i=0}^\infty$ with finite multiplicities, $\lambda_i \to \infty$, and admits a complete orthonormal basis of eigenvectors $\{e_i\}$ in $L^2(E)$. Thus

$$\mathcal{R}e_i \;=\; \lambda_i e_i, \qquad \langle e_i, e_j\rangle = \delta_{ij}, \qquad \lambda_i \geq \lambda_0 > -\infty. \tag{57}$$

By shifting V if necessary one may assume $\lambda_0 \geq 0$.

Heat kernel. The associated heat semigroup is defined by

$$K(s) \;=\; e^{-s\mathcal{R}}, \qquad s > 0, \tag{58}$$

with integral kernel $K(s; x, y)$ smooth on $(0, \infty) \times M \times M$. $K(s)$ is positivity preserving and contractive on $L^2(E)$.

A.2 Main theorem

Theorem A.1 (Spectral positivity and ghost freedom)**.** *Let $\mathcal{R}$ be a Laplace-type recursion operator of the form (56) on a compact Riemannian manifold (M, g). Then $\mathcal{R}$ is positive self-adjoint, and the corresponding Hessian kernel $\widehat{\mathcal{H}}(p^2)$ admits a representation*

$$\widehat{\mathcal{H}}(p^2) \;=\; \int_0^\infty e^{-sp^2}\, d\mu(s), \qquad p^2 \geq 0, \tag{59}$$

where $d\mu(s)$ is a positive measure determined by the spectral resolution of $\mathcal{R}$. Consequently $\widehat{\mathcal{H}}(p^2)$ is an entire, completely monotone function of p^2, and no additional poles or ghost degrees of freedom appear in the ultraviolet sector of MQG.

A.3 Proof of Theorem A.1

We break the proof into three steps: operator theoretic properties of $\mathcal{R}$, the heat semigroup and spectral calculus, and the Laplace-Stieltjes representation with positivity and monotonicity.

Lemma A.2 (Self-adjointness and lower boundedness)**.** *Let $\mathcal{R} = -\nabla^*\nabla + V$ with $V = V^* \in C^\infty(\mathrm{End}(E))$. Then the Friedrichs extension of $\mathcal{R}$ on $C^\infty(E)$ is self-adjoint on $L^2(E)$ with domain $H^2(E)$, elliptic and bounded below. After adding a constant multiple of the identity, if necessary, one may assume $\mathcal{R} \geq 0$.*

Proof. Elliptic regularity on compact M implies essential self-adjointness on $C^\infty(E)$ with domain $H^2(E)$. Since V is bounded and self-adjoint, $-\nabla^*\nabla$ is non-negative and the Kato-Rellich theorem gives that $\mathcal{R}$ is self-adjoint and bounded below. Shifting by a constant does not affect the conclusions of the theorem. □

Lemma A.3 (Heat semi-group and spectral resolution)**.** *For $s > 0$ the operator $K(s) = e^{-s\mathcal{R}}$ is a strongly continuous contraction semi-group on $L^2(E)$ with smooth kernel $K(s; x, y)$. There exists a projection-valued measure E_λ on $[0, \infty)$ such that*

$$\mathcal{R} \;=\; \int_{[0,\infty)} \lambda \, dE_\lambda, \qquad e^{-s\mathcal{R}} \;=\; \int_{[0,\infty)} e^{-s\lambda} \, dE_\lambda. \tag{60}$$

Proof. By Lemma A.2, $\mathcal{R}$ is self-adjoint and non-negative and, therefore, generates a strongly continuous contraction semi-group with smooth kernel on a compact manifold. The spectral theorem yields (60). □

Lemma A.4 (Admissible regulators and second variation)**.** *Let $f : [0, \infty) \to \mathbb{R}$ be an admissible regulator with a positive Laplace transform weight $\widehat{f} \in L^1_{\mathrm{loc}}([0, \infty))$, namely*

$$f(x) \;=\; \int_0^\infty \widehat{f}(s) \, e^{-sx} \, ds, \qquad \widehat{f}(s) \geq 0 \ \textit{a.e.}\ . \tag{61}$$

Consider the spectral functional $\mathcal{A}(\mathcal{R}) = \operatorname{Tr} f(\mathcal{R})$. For a smooth one-parameter family $\mathcal{R}_\tau$ with $\mathcal{R}_0 = \mathcal{R}$ and $\dot{\mathcal{R}} = \frac{d}{d\tau}\mathcal{R}_\tau|_{\tau=0}$ self-adjoint, the second variation reads

$$\frac{d^2}{d\tau^2}\Big|_{\tau=0} \operatorname{Tr} f(\mathcal{R}_\tau) \;=\; \int_0^\infty \widehat{f}(s) \int_0^s \operatorname{Tr}\Big[e^{-(s-u)\mathcal{R}}\,\dot{\mathcal{R}}\,e^{-u\mathcal{R}}\,\dot{\mathcal{R}}\Big]\,du\,ds. \tag{62}$$

Proof. Use (61) and the Duhamel formula $\frac{d}{d\tau}e^{-s\mathcal{R}_\tau} = -\int_0^s e^{-(s-u)\mathcal{R}_\tau}\dot{\mathcal{R}}_\tau e^{-u\mathcal{R}_\tau}du$. Twice differentiating under the trace and evaluating at $\tau = 0$ yields (62). Fubini is justified by positivity and local integrability of $\widehat{f}$ together with trace-class of the heat kernel on compact M for each $s > 0$. □

Lemma A.5 (Positivity of the quadratic form density)**.** *For $\dot{\mathcal{R}} = \dot{\mathcal{R}}^*$ and $0 \le u \le s$,*

$$\operatorname{Tr}\Big[e^{-(s-u)\mathcal{R}}\,\dot{\mathcal{R}}\,e^{-u\mathcal{R}}\,\dot{\mathcal{R}}\Big] \;=\; \operatorname{Tr}[X^*X] \;\ge\; 0, \qquad X \;\equiv\; e^{-u\mathcal{R}/2}\,\dot{\mathcal{R}}\,e^{-(s-u)\mathcal{R}/2}. \tag{63}$$

Proof. Since $\mathcal{R}$ is self-adjoint and non-negative, the semi-group factors as indicated. Then $X^* = e^{-(s-u)\mathcal{R}/2}\,\dot{\mathcal{R}}\,e^{-u\mathcal{R}/2}$ and X^*X is positive – therefore the trace is non-negative. □

Lemma A.6 (Laplace-Stieltjes representation of the form factor)**.** *Let $\widehat{\mathcal{H}}(p^2)$ denote the momentum-space Hessian kernel on the physical subspace associated with the second variation (62). Then there exists a positive Radon measure $d\mu(s)$ on $[0,\infty)$ such that*

$$\widehat{\mathcal{H}}(p^2) \;=\; \int_0^\infty e^{-sp^2}\,d\mu(s), \qquad p^2 \ge 0. \tag{64}$$

Proof. For Laplace-type $\mathcal{R}$, the quadratic kernel factors through the heat semi-group as in (62). Upon projecting to a momentum eigenmode with eigenvalue p^2 of the scalar Laplacian on the physical subspace, each heat operator contributes a factor e^{-up^2} or $e^{-(s-u)p^2}$ – therefore, the dependence on p^2 enters only through e^{-sp^2}. Define

$$d\mu(s) \;\equiv\; \widehat{f}(s)\,\left(\int_0^s \operatorname{Tr}\Big[e^{-(s-u)\mathcal{R}}\,\dot{\mathcal{R}}\,e^{-u\mathcal{R}}\,\dot{\mathcal{R}}\Big]\,du\right)ds.$$

By Lemma A.5 and $\widehat{f}(s) \ge 0$, $d\mu$ is a positive measure. Substituting this into (62) and reading off the p^2 dependence yields (64). □

Lemma A.7 (Complete monotonicity and holomorphy)**.** *The function* $\widehat{\mathcal{H}}(p^2)$ *in* (64) *is completely monotone on* $[0, \infty)$*:*

$$(-1)^n \frac{d^n}{d(p^2)^n} \widehat{\mathcal{H}}(p^2) \; \geq \; 0, \qquad n = 0, 1, 2, \ldots$$

and holomorphic for $\operatorname{Re} p^2 > 0$*. If, in addition,* $d\mu$ *has finite exponential moments of all orders (for example when* $\widehat{f}$ *decays faster than any exponential), then* $\widehat{\mathcal{H}}$ *extends to an entire function of* p^2*.*

Proof. Bernstein's theorem implies that Laplace-Stieltjes transforms of positive measures are completely monotone. Holomorphy for $\operatorname{Re} p^2 > 0$ follows from dominated convergence since $|e^{-sp^2}| \leq e^{-s \operatorname{Re} p^2}$ and μ is locally finite. If $\int_0^\infty e^{\alpha s} \, d\mu(s) < \infty$ for all $\alpha > 0$, the integral converges for all $p^2 \in \mathbb{C}$, yielding an entire extension. □

Lemma A.8 (Ghost freedom)**.** *Assume* $\widehat{\mathcal{H}}(p^2) > 0$ *for* $p^2 \geq 0$*; this holds for nontrivial* $\widehat{f}$ *and* $\dot{\mathcal{R}} \neq 0$ *by* (64)*. Then the quadratic propagator on the physical subspace has at most the massless pole at* $p^2 = 0$ *and no additional real poles.*

Proof. On the physical subspace, the quadratic operator takes the form $p^2 \, \widehat{\mathcal{H}}(p^2)$ up to positive normalisations and projectors. Since $\widehat{\mathcal{H}}(p^2) > 0$ for $p^2 \geq 0$, the only zero of $p^2 \, \widehat{\mathcal{H}}(p^2)$ on the real axis is at $p^2 = 0$. Complete monotonicity also forbids oscillatory sign changes that could generate additional real poles upon inversion. □

Proof of Theorem A.1. Combine Lemmas A.2 and A.3 for the operator framework, then Lemmas A.4–A.6 to obtain the Laplace-Stieltjes representation (59) with a positive measure $d\mu$. Lemma A.7 yields complete monotonicity and holomorphy for $\operatorname{Re} p^2 > 0$, with entire extension under mild decay of $\widehat{f}$. Finally Lemma A.8 gives the absence of extra real poles and thus ghost freedom at quadratic order. □

Remarks on scope

- The compactness of (M, g) ensures trace-class heat kernels and discrete spectrum. Extensions to non-compact M require control of essential spectrum and heat kernel bounds; the Laplace-Stieltjes conclusion remains valid under standard Gaussian off-diagonal estimates.

- For the Gaussian regulator $f(x) = e^{-x}$ used in the main text, $\widehat{f}(s) = \delta(s-1)$ and $d\mu$ is supported at a single s, so $\widehat{\mathcal{H}}(p^2)$ is entire and equals e^{-p^2} times a positive normalisation, which matches the Track A example.

B Mixing coefficients on curved manifolds

This appendix establishes, in a background–independent and covariant way, that the dimension–six curvature–gauge operators and their coefficients (c_1, c_2, c_3) are fixed by the Seeley–DeWitt coefficient $a_3(x)$ of a Laplace–type operator on the gauge bundle. This upgrades the FRW example of Section 2 to arbitrary smooth backgrounds (M, g).

B.1 Setup and operator class

Let (M, g) be a smooth compact 4–manifold without boundary. Let $E \to M$ be a Hermitian vector bundle carrying a unitary representation of the gauge group, with connection A_μ and curvature $\Omega_{\mu\nu} = [\nabla_\mu, \nabla_\nu] = F_{\mu\nu}$. Consider a Laplace–type operator on sections of E,

$$\Delta_A \;=\; -\,g^{\mu\nu}\nabla_\mu\nabla_\nu \;+\; \mathcal{E}, \qquad \mathcal{E} \in C^\infty(\mathrm{End}(E))^*, \tag{65}$$

where ∇ is the total covariant derivative (Levi–Civita plus A_μ) and $\mathcal{E}$ is a smooth self–adjoint endomorphism (bundle mass/curvature term). Let f be an admissible regulator with Laplace transform density $\widehat{f}(s) \geq 0$:

$$f(x) \;=\; \int_0^\infty \widehat{f}(s)\, e^{-sx}\, ds. \tag{66}$$

Define the spectral functional

$$\mathcal{C}[g, A] \;=\; \mathrm{Tr}\, f\!\left(\frac{\Delta_A}{\Lambda^2}\right). \tag{67}$$

B.2 Heat kernel and curvature expansion

By the spectral theorem and (66),

$$\mathcal{C}[g, A] \;=\; \int_0^\infty \widehat{f}(s)\, \mathrm{Tr}\, e^{-s\Delta_A/\Lambda^2}\, ds. \tag{68}$$

For Laplace–type Δ_A on a compact manifold, the heat trace admits the local asymptotic expansion (Seeley–DeWitt)

$$\mathrm{Tr}\, e^{-t\Delta_A} \;\sim\; \sum_{n=0}^{\infty} t^{(n-4)/2} \int_M d^4x\, \sqrt{g}\; a_n(x; \Delta_A), \qquad t \downarrow 0, \tag{69}$$

where $a_n(x)$ are universal polynomials in the Riemann tensor, $\Omega_{\mu\nu}$, $\mathcal{E}$, and their covariant derivatives [14], [15].

In $d = 4$, the **dimension–six** part (our target) comes from $n = 6$:

$$\Big[\mathcal{C}[g, A]\Big]_{d=6} \;=\; \Lambda^{-2} \left(\int_0^\infty \widehat{f}(s)\, s\, ds\right) \int_M d^4x\, \sqrt{g}\; a_3(x; \Delta_A). \tag{70}$$

B.3 Tensor basis and projection

For Laplace–type operators on bundles, $a_3(x)$ contains a linear combination of the independent curvature–gauge scalars at mass dimension six:

$$a_3(x;\Delta_A) \supset \alpha_1 R\,\mathrm{tr}\,(\Omega_{\mu\nu}\Omega^{\mu\nu})+\alpha_2 R_{\mu\nu}\,\mathrm{tr}\,(\Omega^{\mu\rho}\Omega^{\nu}{}_{\rho})+\alpha_3 R_{\mu\nu\rho\sigma}\,\mathrm{tr}\,(\Omega^{\mu\nu}\Omega^{\rho\sigma})+\cdots, \tag{71}$$

where "$\cdots$" denotes total derivatives and terms with derivatives of curvature (which can be removed or reshuffled by field redefinitions when matching to the quadratic, on–shell Hessian sector). Identifying $\Omega_{\mu\nu}$ with $F_{\mu\nu}$, the three displayed tensors span the basis used in the main text.

Theorem B.1 (Covariant determination of c_1, c_2, c_3)**.** *Under the assumptions above, the dimension–six curvature–gauge sector of $\mathcal{C}[g,A]$ is*

$$\int_M d^4x\,\sqrt{g}\,\big(c_1\,R\,\mathrm{tr}F^2 + c_2\,R_{\mu\nu}\,\mathrm{tr}(F^{\mu\rho}F^{\nu}{}_{\rho}) + c_3\,R_{\mu\nu\rho\sigma}\,\mathrm{tr}(F^{\mu\nu}F^{\rho\sigma})\big), \tag{72}$$

with

$$c_i \;=\; \frac{1}{(4\pi)^2}\,\beta_i\Big(\int_0^\infty \widehat{f}(s)\,s\,ds\Big)\,\frac{1}{\Lambda^2}, \qquad i=1,2,3, \tag{73}$$

where the β_i are universal numerical constants determined solely by the Gilkey–DeWitt coefficient a_3 for the operator class (65) *(and the chosen normalisation of* tr*). Thus (c_1, c_2, c_3) are background–independent functionals of f and the bundle data.*

Proof sketch. Insert the expansion (69) into (68) and extract the $n = 6$ term to obtain (70). The local invariant content of a_3 for a general Laplace–type operator is standard; see [14], [15]. Project a_3 onto the basis (71); the coefficients of each tensor structure are universal numbers $\tilde{\beta}_i$ that depend only on the operator class. After converting $\Omega_{\mu\nu}$ to $F_{\mu\nu}$ and matching normalisations of traces, one obtains (72) with c_i proportional to the spectral moment $\int \widehat{f}(s)s\,ds$ divided by Λ^2. The normalisation factor $(4\pi)^{-2}$ is the standard 4D heat kernel prefactor. This yields (73) with β_i the operator–class–dependent numbers. □

Remarks

- The result is **background–independent**: different spacetimes change only the values of the local curvature tensors in (72), not the constants c_i.
- For the Gaussian regulator $f(x) = e^{-x}$ one has $\widehat{f}(s) = \delta(s-1)$ and $\int \widehat{f}(s)s\,ds = 1$, so $c_i = \beta_i/[(4\pi)^2\Lambda^2]$.

- The precise β_i for a given operator class can be read off from the tabulated a_3 coefficients in [14], [15]. They depend on the representation normalisation through $\mathrm{tr}(T^a T^b) = \kappa\,\delta^{ab}$.

B.4 Universal a_3 tensor structures and projection

For a Laplace–type operator $\Delta_A = -g^{\mu\nu}\nabla_\mu\nabla_\nu + \mathcal{E}$ acting on a vector bundle with unitary connection, the local coefficient $a_3(x;\Delta_A)$ has a universal decomposition into curvature invariants [14], [15]. Restricting to the part bilinear in the gauge curvature and linear in the spacetime curvature, one can write

$$a_3(x;\Delta_A)\Big|_{RF^2} \;=\; \tau_1\, R\,\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) + \tau_2\, R_{\mu\nu}\,\mathrm{tr}(F^{\mu\rho}F^{\nu}{}_{\rho}) + \tau_3\, R_{\mu\nu\rho\sigma}\,\mathrm{tr}(F^{\mu\nu}F^{\rho\sigma}) + \nabla\cdot(\cdots), \tag{74}$$

where τ_i are operator–class–dependent universal constants and $\nabla\cdot(\cdots)$ denotes total derivatives and terms with derivatives of curvatures (which do not contribute to the quadratic, on–shell Hessian sector after field redefinitions; see Section B.7 below). The overall normalisation of tr is fixed by $\mathrm{tr}(T^aT^b) = \kappa\,\delta^{ab}$ in the chosen representation.

Comparing (74) with (72) and using (73) gives the identification

$$\beta_i \;=\; (4\pi)^2\,\tau_i\,, \qquad c_i \;=\; \tau_i\,\Big(\int_0^\infty \widehat{f}(s)\,s\,ds\Big)\,\frac{1}{\Lambda^2}, \qquad i = 1,2,3. \tag{75}$$

Thus it suffices to compute the τ_i once for the operator class.

B.5 Background triad for algebraic extraction of τ_i

Although the τ_i can be read directly from the tabulated a_3 coefficient for Δ_A, it is sometimes convenient to extract them by evaluating (74) on three diagnostic background classes that project onto linearly independent combinations:

(I) Ricci–flat backgrounds. Take $R_{\mu\nu} = 0$ but $R_{\mu\nu\rho\sigma} \neq 0$ (e.g. a vacuum gravitational wave or Schwarzschild exterior). Then (74) reduces to

$$a_3\Big|_{RF^2}^{(\mathrm{I})} \;=\; \tau_3\, R_{\mu\nu\rho\sigma}\,\mathrm{tr}(F^{\mu\nu}F^{\rho\sigma}),$$

so a single nonzero evaluation fixes τ_3.

(II) Einstein spaces. Take $R_{\mu\nu} = \frac{R}{4}\, g_{\mu\nu}$ with $R \neq 0$ and arbitrary Weyl curvature. Using $R_{\mu\nu}\, \mathrm{tr}(F^{\mu\rho} F^{\nu}{}_{\rho}) = \frac{R}{4}\, \mathrm{tr}(F_{\mu\nu} F^{\mu\nu})$, one obtains

$$a_3\Big|_{RF^2}^{(\mathrm{II})} \;=\; \Big(\tau_1 + \tfrac{1}{4}\tau_2\Big)\, R\, \mathrm{tr}(F_{\mu\nu} F^{\mu\nu}) \;+\; \tau_3\, C_{\mu\nu\rho\sigma}\, \mathrm{tr}(F^{\mu\nu} F^{\rho\sigma}),$$

where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor. Combining with the value of τ_3 from (I) isolates $\tau_1 + \frac{1}{4}\tau_2$.

(III) Conformally flat, non–Einstein backgrounds. Take $C_{\mu\nu\rho\sigma} = 0$ but $R_{\mu\nu} \neq \frac{R}{4} g_{\mu\nu}$ (e.g. inhomogeneous FRW). Then

$$a_3\Big|_{RF^2}^{(\mathrm{III})} \;=\; \tau_1\, R\, \mathrm{tr}(F_{\mu\nu} F^{\mu\nu}) + \tau_2\, R_{\mu\nu}\, \mathrm{tr}(F^{\mu\rho} F^{\nu}{}_{\rho}),$$

which, together with (II), separates τ_1 and τ_2.

In practice one can use these three projections as a check on the direct readout from the a_3 tables. Since a_3 is local and universal, the τ_i obtained in this way do not depend on the particular representative of each class.

B.6 Example: minimal Laplace–type operator on the adjoint bundle

Consider the minimal Laplace–type operator on the adjoint bundle,

$$\Delta_A^{\mathrm{adj}} \;=\; -g^{\mu\nu} \mathcal{D}_\mu \mathcal{D}_\nu, \qquad \mathcal{D}_\mu = \nabla_\mu + [A_\mu, \cdot], \tag{76}$$

with $\mathcal{E} = 0$. For this class, the a_3 coefficient bilinear in $F_{\mu\nu}$ and linear in spacetime curvature has the form (74) with universal numbers τ_i^{adj} that depend only on the dimension and on the chosen trace normalisation. Matching to the standard Gilkey basis one finds

$$\tau_1^{\mathrm{adj}} \;=\; \alpha_1\, \kappa, \qquad \tau_2^{\mathrm{adj}} \;=\; \alpha_2\, \kappa, \qquad \tau_3^{\mathrm{adj}} \;=\; \alpha_3\, \kappa, \tag{77}$$

where κ is defined by $\mathrm{tr}(T^a T^b) \;=\; \kappa\, \delta^{ab}$ and α_i are the universal scalar coefficients quoted in the heat kernel literature for Laplace–type operators on vector bundles. Substituting (77) into (75) yields

$$c_i^{\mathrm{adj}} \;=\; \alpha_i\, \kappa \left(\int_0^\infty \widehat{f}(s)\, s\, ds \right) \frac{1}{\Lambda^2}, \qquad i = 1, 2, 3.$$

For the Gaussian regulator $f(x) = e^{-x}$ one has $\int \widehat{f}(s) s\, ds = 1$, so

$$c_i^{\mathrm{adj}} \;=\; \alpha_i\, \kappa\, \Lambda^{-2}.$$

This shows explicitly how the FRW result $c_R = \Lambda^{-2}$ in the main text corresponds to $\alpha_1\, \kappa = 1$ in that normalisation, while α_2 and α_3 fix the Ricci and Weyl mixing strengths.

Comment on nonminimal operators. If $\mathcal{E} \neq 0$ includes curvature endomorphisms (for example $\mathcal{E} \propto R\,\mathbf{1}$ in a nonminimal gauge choice), the same procedure applies. The τ_i then depend on the decomposition of $\mathcal{E}$ into the Gilkey basis; they are still universal numbers for the given operator class and trace convention.

B.7 Field redefinitions and removal of derivative terms

The coefficient a_3 contains, in addition to (74), total derivatives and terms with covariant derivatives of curvature, e.g.

$$\nabla_\alpha R\,\mathrm{tr}(F^{\alpha\beta}A_\beta), \qquad \nabla_\alpha \mathrm{tr}(F^{\alpha\beta}\nabla^\mu F_{\mu\beta}),$$

and similar. At quadratic order in fluctuations and for on–shell backgrounds, these terms can be removed from the quadratic effective action by local field redefinitions and integrations by parts, without affecting the physical Hessian coefficients. More precisely:

Lemma B.2 (Quadratic equivalence under local redefinitions)**.** *Let $S[A,g]$ be a local functional whose quadratic part contains terms of the form $\int \sqrt{g}\,\nabla\cdot J$ or $\int \sqrt{g}\,\mathcal{O}^{\mu\nu}(g)\,\nabla_\mu X_\nu$, with $\mathcal{O}^{\mu\nu}$ a local tensor made of curvature. Then there exists a local field redefinition $A_\mu \mapsto A_\mu + \delta A_\mu$ with $\delta A_\mu = \mathcal{O}(\nabla X)_\mu$ and integrations by parts such that these terms do not contribute to the quadratic Hessian on the physical transverse subspace.*

Proof sketch. Work in background field gauge and decompose $A_\mu = A_\mu^\perp + \nabla_\mu\phi$. Projecting to the transverse sector eliminates pure divergence structures. The remaining derivative couplings can be cancelled by choosing δA_μ to complete squares at quadratic order. Since we only modify higher order terms or pure gauges, the physical quadratic kernel is unchanged. □

By Lemma B.2, only the three scalars in (74) are relevant for the quadratic Hessian coefficients. All derivative and boundary terms can be ignored for the purpose of extracting (c_1, c_2, c_3).

B.8 Summary of Appendix B

The main conclusions are:

- The dimension–six curvature–gauge mixing in MQG is fixed by the universal a_3 coefficient of a Laplace–type operator on the gauge bundle. No background–specific parameter enters.

- The coefficients are

$$c_i \;=\; \tau_i \left(\int_0^\infty \widehat{f}(s)\, s\, ds\right) \Lambda^{-2}, \qquad i = 1, 2, 3,$$

  with τ_i read from the Gilkey–DeWitt tables or extracted by the background triad of Section B.5.

- For the Gaussian regulator, $c_i = \tau_i\, \Lambda^{-2}$. The FRW closed form in the main text realises this with $i = 1$ and provides a cross–check for τ_1.

References

1 J. D. Coulson, “Manifold Quantum Gravity: From First Principles to Cosmic Phenomena,” Zenodo Preprint (2025). DOI: 10.5281/zenodo.16214915.

2 J. D. Coulson, “Emergence of the Gauge Sector in Manifold Quantum Gravity,” Zenodo Preprint (2025). DOI: 10.5281/zenodo.16374596.

3 R. M. Wald, *General Relativity*, University of Chicago Press (1984).

4 S. M. Carroll, *Spacetime and Geometry: An Introduction to General Relativity*, Addison–Wesley (2004).

5 L. E. Parker and D. J. Toms, *Quantum Field Theory in Curved Spacetime*, Cambridge University Press (2009).

6 I. L. Buchbinder, S. D. Odintsov and I. L. Shapiro, *Effective Action in Quantum Gravity*, IOP Publishing (1992).

7 I. T. Drummond and S. J. Hathrell, “QED vacuum polarization in a background gravitational field and its effect on the velocity of photons,” *Phys. Rev. D* **22** (1980) 343–355. DOI: 10.1103/PhysRevD.22.343.

8 G. M. Shore, “Quantum gravitational optics,” *Contemp. Phys.* **44** (2003) 503–521; arXiv:gr-qc/0304059.

9 M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory*, Westview Press (1995).

10 S. Weinberg, “A Model of Leptons,” *Phys. Rev. Lett.* **19** (1967) 1264–1266.

11 A. Connes, *Noncommutative Geometry*, Academic Press (1994).

12 A. Connes and J. Lott, “Particle models and noncommutative geometry,” *Nucl. Phys. Proc. Suppl.* **18B** (1991) 29–47; A. Connes, “Noncommutative geometry and reality,” *J. Math. Phys.* **36** (1995) 6194–6231.

13 A. H. Chamseddine, A. Connes and M. Marcolli, “Gravity and the standard model with neutrino mixing,” *Adv. Theor. Math. Phys.* **11** (2007) 991–1089; arXiv:hep-th/0610241.

$\boxed{\mathbf{14}}$ P. B. Gilkey, *Asymptotic Formulae in Spectral Geometry*, Chapman & Hall/CRC (2004).

$\boxed{\mathbf{15}}$ D. V. Vassilevich, “Heat kernel expansion: user’s manual,” *Phys. Rept.* **388** (2003) 279–360; arXiv:hep-th/0306138.

$\boxed{\mathbf{16}}$ S. Amari and H. Nagaoka, *Methods of Information Geometry*, Translations of Mathematical Monographs vol. 191, American Mathematical Society (2000).

$\boxed{\mathbf{17}}$ C. Caticha, “Entropic dynamics: an inference approach to quantum theory, time and measurement,” *Entropy* **22** (2020) 908; arXiv:1908.04693.

$\boxed{\mathbf{18}}$ Planck Collaboration: N. Aghanim *et al.*, “Planck 2018 results. VI. Cosmological parameters,” *Astron. Astrophys.* **641** (2020) A6 [arXiv:1807.06209].