Fixing the Measure: Deriving $|\Psi|^2$ from Symmetry in Deterministic Geometry

A Preprint

Zayn Blore
zblore@gmail.com

August 5, 2025

Abstract

This paper derives the Born rule from first principles by identifying the unique measure over complex Hilbert space that is invariant under two physically motivated symmetries: complex-scaling homogeneity, and unitary covariance. Assuming only that quantum amplitudes Ψ inhabit a finite-dimensional Hilbert space $\mathbb{C}^n$, we show that the only measure consistent with deterministic, volume-preserving dynamics and these symmetries is proportional to $|\Psi|^2$. This result explains the empirical success of the Born rule as a geometric necessity, not a probabilistic axiom. When applied to systems with disjoint outcome regions, as in volume-based formulations of branching dynamics, this measure yields outcome frequencies that match quantum predictions exactly. The derivation introduces no probabilistic or postulated elements beyond geometry and symmetry. Outcome weights arise solely from the invariant structure of finite-dimensional amplitude space under deterministic, volume-preserving flow.

Keywords First keyword · Second keyword · More

1 Introduction

In standard quantum mechanics, the Born rule is introduced as a postulate: the probability of a measurement outcome is given by the squared modulus of the corresponding component of the wavefunction. This rule, though empirically successful, lacks a derivation from more primitive physical principles. Why should outcome frequencies match squared amplitudes? Why does this specific functional form arise across all quantum systems?

This paper offers a purely geometric answer. We consider a finite-dimensional complex Hilbert space $\mathcal{H} = \mathbb{C}^n$ and ask: under what conditions is a unique measure over the state space determined? We assume only that the system evolves deterministically, conserves a volume measure, and respects three physically natural symmetries:

- Complex-scaling symmetry
- Unitary covariance

These symmetries are not imposed arbitrarily. They reflect essential properties of real quantum systems, such as rotational invariance, global phase redundancy, and basis independence of outcome statistics.

We show that these symmetry constraints uniquely fix the measure over state space to be proportional to $|\Psi|^2$. This derivation does not appeal to stochasticity, hidden variables, or interpretive constructs. Instead, $|\Psi|^2$ arises as the only measure consistent with the structural symmetries of finite-dimensional quantum amplitudes under deterministic evolution.

Our aim is not to reinterpret the quantum formalism, but to explain one of its most fundamental features, its outcome frequencies, as a necessary consequence of geometric structure. The result complements recent work on volume-based derivations of empirical frequencies, providing a rigorous explanation of why those volumes match squared amplitudes.

Note: We work on finite $\mathbb{C}^n$ because every laboratory apparatus defines an effective finite basis cut-off. Appendix B sketches how the two-symmetry proof extends to any Hilbert space by taking inductive limits $\mathbb{C}P^n \to \mathbb{C}P^\infty$.

The paper is structured as follows: Section 2 introduces the symmetry assumptions and proves that only the $|\Psi|^2$ measure satisfies them. Section 3 shows how these weights correspond to observed frequencies under deterministic evolution, extending the typicality framework. Section 4 discusses limitations, scope, and possible generalisations. Section 5 outlines falsifiable predictions and potential experimental tests.

1.1 Framework Overview

This paper investigates how the Born rule can emerge from deterministic geometry, without invoking probabilities or interpretive assumptions. We consider quantum systems represented by state vectors Ψ in a finite-dimensional complex Hilbert space $\mathbb{C}^n$. The structure of $\mathbb{C}^n$ is taken as operationally valid for systems with a finite number of resolvable outcomes. Our aim is to identify the natural measure over this state space that governs long-run outcome frequencies under deterministic dynamics.

We assume the system evolves according to a smooth, measure-preserving map φ_t on the unit sphere in $\mathbb{C}^n$. At key junctures, such as measurements or interactions with coarse-grained observables, state space is partitioned into disjoint regions $\{\Omega_i\}$, each corresponding to a macroscopically distinct outcome. This defines a branching structure: although the dynamics are deterministic, different initial states evolve into different outcome regions. These regions are taken to be measurable and invariant under the system's natural symmetries.

The core principle used throughout this paper is volume-typicality: the empirical frequency with which a given outcome is observed corresponds to the fraction of state space volume its region occupies under an invariant measure. In the absence of additional structure or hidden variables, all microstates consistent with the dynamics are treated equally. This is analogous to classical statistical mechanics, where macro-observables are derived from uniform measures over phase space constrained by conservation laws.

The central question is therefore: which measure over the unit sphere in $\mathbb{C}^n$ satisfies the physical symmetries of the system and leads to consistent outcome frequencies? We impose two natural constraints:

1. **Complex-scaling symmetry:** $\Psi \to \lambda\Psi$ ($\lambda \in \mathbb{C}^*$) does not affect predictions.
2. **Unitary covariance:** transformations $\Psi \to U\Psi$ ($U \in SU(n)$) preserve outcome relations.

These constraints determine the structure of permissible outcome regions and the allowed form of the measure. We will show that the only volume measure consistent with these assumptions is proportional to $|\Psi|^2$, matching the Born rule, but derived from geometric structure alone.

1.2 Postulates addressed

Standard postulate	Status	Location
1. State = Hilbert vector	Deferred: Requires justification of Ψ-space	Paper C
2. Observables = Hermitian operators	Deferred: Requires operator emergence	Paper C
3. Measurement outcomes are stochastic	Replaced: Outcomes arise from deterministic branching structure	Blore [2025], this work
4. Born rule: $P(i) = |\Psi_i|^2$	Derived: Unique unitarily invariant (Haar) measure under symmetry constraints	this work
5. Superposition principle	Deferred: Requires linear structure derivation	Paper C
6. Unitary evolution	Deferred: Requires conservation from underlying dynamics	Paper C
7. Probability (interpretive rule)	Replaced: Frequencies emerge from volume ratios in deterministic geometry	Blore [2025]

2 Symmetries and Measure on Complex State Space

The goal of this section is to identify the unique volume measure on complex amplitude space that is preserved under minimal, physically motivated symmetries. This measure will serve as the deterministic analogue of quantum probability. The derivation assumes only that complex flow coordinates Ψ reside in a smooth, finite-dimensional complex vector space S, and that time evolution is invertible and measure-preserving.

Following the construction in Blore [2025], we define the state space S as a smooth, compact manifold equipped with a volume-preserving flow $\varphi_t : S \to S$ and measure μ.

Note: We embed the finite complex manifold S in $\mathbb{C}^N$; after quotienting by global phase, S is isomorphic to the projective Hilbert space $\mathbb{C}P^{N-1}$. The $SU(N)$ symmetry constraints derived below pick out the unitarily invariant (Haar) measure density $|\Psi|^2 \mathrm{d}V$. This density coincides with the conventional Born weighting, yet it follows solely from geometric invariance, with no Hilbert-space postulates assumed. The framework is therefore geometrically self-contained while remaining fully compatible with standard Hilbert-space calculations.

We impose two global symmetries on this space:

1. **Complex-scaling symmetry:** The volume measure is invariant under the transformation $\Psi \to \alpha\Psi$ for all $\alpha \in \mathbb{C}^* \setminus \{0\}$, ensuring that global rescalings do not affect outcome ratios.
2. **Unitary covariance:** The measure is invariant under all unitary transformations $U \to U(n)$, ensuring symmetry under change of basis and compatibility with structure, preserving evolution in Ψ coordinates.

These symmetries are not interpretive or axiomatic, they reflect well-established structural features of amplitude space in quantum theory.

We do not assume any probabilistic interpretation of Ψ, nor do we postulate $|\Psi|^2$. Instead, we ask: what measure μ over Ψ-space is invariant under these transformations and respects unitarily invariant measure?

Note on the Liouville measure: It describes the invariant volume in real phase space. Here, the analogous structure in the complex setting is the unitarily invariant (Haar) measure on the unit sphere, which descends to the Fubini–Study measure on projective space $\mathbb{C}P^{n-1}$.

Note on Compactness and Scaling: While the transformation $\Psi \to \lambda\Psi$ with arbitrary $\lambda \in \mathbb{C}^*$ formally ranges over an unbounded set, the physically relevant state space is the projective space $\mathbb{C}P^{n-1}$ obtained by quotienting out global rescaling (including phase). Thus, although we use scaling symmetry to constrain the measure's homogeneity exponent k, the measure itself ultimately lives on the compact manifold $\mathbb{C}P^{n-1}$, ensuring normalisability and consistency with standard finite-dimensional volume forms (Fubini–Study measure).

In Section 2.5, we show that these two symmetries uniquely select the squared-norm measure $\mathrm{d}\mu = |\Psi|^2 \mathrm{d}V_H$, where $\mathrm{d}V_H$ is the standard volume form on S.

This result does not rely on Hilbert space axioms, probabilistic postulates, or interpretive frameworks. It follows directly from differential geometry and symmetry constraints.

Section 2.1 introduces the finite-dimensional Hilbert space $\mathbb{C}^N$ and its volume structure. Sections 2.2 through 2.4 define and apply each of the two symmetry constraints in turn, examining their consequences for any invariant measure. Section 2.5 then combines these constraints to prove that the squared-norm measure $\mathrm{d}\mu \propto |\Psi|^2 \mathrm{d}V_H$ is the unique solution. Section 2.6 summarizes the result in boxed form and sets the stage for its integration into the deterministic branching framework developed in Blore [2025].

2.1 State Space, Evolution, and Outcome Regions

We consider a dynamical system with the following properties:

- The complete state of the system at a given time is represented by a point $x \in S$, where S is a smooth, finite-dimensional, compact, Hausdorff manifold equipped with a normalised volume measure μ.
- The system evolves according to a deterministic, continuous flow $\varphi_t : S \to S$, such that for all measurable $A \subseteq S$, the measure is preserved:
$$\mu(\varphi_t(A)) = \mu(A).$$
- The flow is invertible and autonomous; that is, the evolution does not depend explicitly on time.

An observable is defined as a coarse-graining function $f : S \to O$, mapping each microstate to a macroscopic outcome label. The range $O = \{o_1, o_2, \ldots, o_N\}$ is finite and corresponds to distinguishable outcomes in a measurement.

For a given observable f, we define the outcome regions $\Omega_i \supset S$ as the preimages:

$$\Omega_i = f^{-1}(o_i),$$

such that the collection $\{\Omega_i\}$ forms a measurable partition of S up to null sets:

$$S = \bigsqcup_{i=1}^{N} \Omega_i, \quad \mu(\Omega_i \cap \Omega_j) = 0 \text{ for } i \neq j.$$

We interpret Ω_i as the subset of microstates that deterministically evolve, under the flow φ_t, toward macroscopic outcome o_i. Each region Ω_i is assumed to be measurable with respect to the sigma-algebra Σ on S, and the volume $\mu(\Omega_i)$ represents the proportion of initial states that result in outcome o_i when subjected to the specified observable.

This motivates the definition of the outcome weight:

$$w_i = \frac{\mu(\Omega_i)}{\mu(\Omega_0)}.$$

Given that $\mu(\Omega_i)$ is finite and invariant under ϕ_t, these weights satisfy:

$$\sum_{i=1}^{N} w_i = 1, \quad (w_i \geq 0).$$

These weights are not introduced as postulates but are derived from the geometric properties of the outcome partition. They serve as the foundational quantities from which we will reconstruct standard quantum probability assignments under symmetry constraints in later sections.

In this geometric framework, measurement observables are defined not by Hermitian operators, but by partitions of S into disjoint outcome regions that respect the system's underlying symmetry group. Each such observable corresponds to a map:

$$O : S \to \{o_i\},$$

where $\{o_i\}$ is a finite set of outcomes and $O^{-1}(i) = \Omega_i \subset S$ defines the measurable region associated with outcome i. For the measure μ to be meaningful across different observables, the regions Ω_i must be invariant under subgroups of $U(d)$. This ensures that the derived outcome weights are well-defined under allowed symmetry-preserving transformations.

For example, a spin-$1/2$ measurement along axis $\hat{n}$ may correspond to a partition of S into two hemispheres invariant under $U(1)$ rotations stabilizing $\hat{n}$. These geometric outcome regions reproduce Born rule weights when measured with respect to the derived measure $\mu \propto |\Psi|^2$.

This construction generalizes to higher dimensions without reference to operator eigenvalues: observables arise as symmetry-breaking partitions, with outcome probabilities set by invariant volume ratios over the manifold S.

The state space S should be understood not as a mathematical abstraction or tool of inference, but as the physically real configuration space of a deterministic system. The real-valued manifold S encodes all possible microstates of the universe or subsystem. Local complex coordinates $\Psi \in \mathbb{C}^n$ are constructed on the constraint surface $\Sigma \subset S$ for analytical convenience but are not themselves elements of S.

Observables and outcome frequencies emerge from the geometric and symmetry structure of this space, not from ignorance, collapse, or statistical randomness.

This structural perspective mirrors classical statistical mechanics, where macroscopic regularities emerge from conserved geometric measures and deterministic dynamics, without invoking probabilistic assumptions. Similarly, here, probabilities arise from invariant volume ratios over S, grounding quantum statistics in geometric typicality rather than stochastic evolution or many-worlds branching.

2.2 Volume, Symmetry, and Typicality

To proceed from the definition of outcome regions to a derivation of specific probability weights, we introduce a key physical constraint: the outcome weights must respect the underlying symmetries of the state space and observable. This requirement restricts both the measure μ and the construction of outcome partitions $\{\Omega_i\}$, ultimately determining the functional form of the weights w_i.

Invariance Under Symmetry Transformations

Let G be the group of physical symmetries under which the outcome measure is invariant. In systems where local Ψ-coordinates are defined, we are particularly interested in the group $G \cong SU(d)$, which acts as unitary transformations on Ψ-space and preserves the derived volume measure $\mathrm{d}\mu \propto |\Psi|^2 \mathrm{d}V$. These symmetries correspond to geometry-preserving transformations of the surface that respect local complex structure. We require that the measure μ on the state space S be invariant under the action of G:

$$\mu(g \cdot A) = \mu(A), \quad \forall g \in G, A \subseteq S.$$

This condition ensures that no subset of microstates is privileged under a transformation of the state or observable. In the case where $S \cong S^2$ (e.g. for qubits), the unique rotation-invariant measure is the standard surface area measure on the 2-sphere.

Geometry of Outcome Regions

Similarly, we require that outcome regions Ω_i be defined solely by their geometric relation to the state and not by arbitrary coordinate labels. That is, if the state and observable are rotated jointly under a group transformation $g \in G$, the outcome probabilities must remain unchanged.

Formally, if the observable is transformed so that outcome region Ω_i becomes $g \cdot \Omega_i$, and the system state is likewise transformed by g, then:

$$\mu(\Omega_i) = \mu(g \cdot \Omega_i).$$

This condition ensures that the outcome partition is frame-independent, it reflects intrinsic geometric structure rather than any specific labeling convention or coordinate choice.

Thus, outcome regions must be constructed in a way that is covariant under symmetry, typically as regions bounded by planes or surfaces of constant angular relation to the system state or observable axes. The only partitions that satisfy this symmetry constraint are those defined by geodesic separation, such as spherical caps when $S \cong S^2$.

Typicality and Relative Volume

We interpret the volume $\mu(\Omega_i)$ as expressing the typicality of outcome i under the deterministic flow φ_t, when initial conditions are drawn from $\Omega_0 \subset S$ according to the symmetry-invariant measure μ.

This notion of typicality is purely geometric: it reflects the fraction of microstates that evolve to outcome i, assuming no bias in the initial distribution. Under volume-preserving dynamics, these weights are stable across time, independent of the detailed microstate trajectories.

By combining:

- measure invariance under symmetry,
- volume preservation under flow,
- and indistinguishability of microstates given macroscopic preparation,

we obtain a uniquely defined set of outcome weights $\{w_i\}$, grounded entirely in geometry.

In the next section, we apply this framework to a one-qubit system and show that the weights w_i, derived purely from volume symmetry and deterministic flow, coincide exactly with the standard Born rule assignments for spin measurements.

2.3 Worked Example: One-Qubit Measurement

We now demonstrate how the framework of volume-preserving dynamics and symmetry-constrained partitions yields the standard Born rule in a concrete example. Specifically, we consider a two-level system whose effective state space admits a natural identification with the unit 2-sphere S^2. We show that the volume ratios associated with hemispherical outcome regions exactly match the conventional quantum weights given by squared spinor components, without assuming Hilbert space, measurement postulates, or stochasticity.

System Setup

Consider a two-level system described by a point on the unit 2-sphere S^2, with polar angle $\theta \in [0, \pi]$, measured from the north pole. We define the measurement axis $\hat{n}$ such that outcome "0" corresponds to the upper spherical cap centered at the north pole, and outcome "1" to the lower cap centered at the south pole. The angle θ then determines the system's position relative to the measurement axis, and the volume of each cap defines the corresponding outcome weight.

The microstate space S is taken to be the unit 2-sphere S^2, equipped with the standard rotation-invariant surface area measure μ, and the evolution ϕ_t is assumed to be measure-preserving.

A pure system configuration is represented by a point on the unit 2-sphere S^2, specified by the Bloch vector.

The state $|\Psi\rangle$ maps to the Bloch vector

$$v = (\sin\theta, 0, \cos\theta)$$

for some polar angle $\theta \in [0, \pi]$.

We define two outcome regions $\Omega_0, \Omega_1 \cong S^2$ as follows:

- Ω_0 is the set of microstates geometrically closer to $\vec{\mathbf{v}}_0 = (0, 0, 1)$ (the north pole) than to $\vec{\mathbf{v}}_1 = (0, 0, -1)$ (the south pole),
- $\Omega_1 = S^2 \setminus \Omega_0$ is the complementary region.

The boundary between these regions lies on the great circle orthogonal to $\vec{\mathbf{v}}$, passing through the equator. This partitions the sphere into two spherical caps, each defined solely by angular distance from the poles. The cap Ω_0, centred on the north pole, has an angular radius of θ.

This construction respects the symmetry constraints outlined in Section 2.2 and ensures that outcome weights are invariant under rotations stabilizing the measurement axis.

Volume Computation

The surface area of a spherical cap on S^2 of angular radius $\alpha \in [0, \pi]$ is given by:

$$A(\alpha) = 2\pi(1 - \cos\alpha).$$

Hence, the area of the cap corresponding to outcome "1" is:

$$\mu(\Omega_1) = 2\pi(1 - \cos\theta)$$

and the complementary region has:

$$\mu(\Omega_0) = 4\pi - \mu(\Omega_1) = 2\pi(1 + \cos\theta).$$

Normalising by the total surface area $\mu(S^2) = 4\pi$, we obtain the outcome weights:

$$w_0 = \frac{\mu(\Omega_0)}{4\pi} = \frac{1 + \cos\theta}{2} = \cos^2\left(\frac{\theta}{2}\right) \quad , \quad w_1 = \frac{\mu(\Omega_1)}{4\pi} = \frac{1 - \cos\theta}{2} = \sin^2\left(\frac{\theta}{2}\right).$$

Interpretation

These volume ratios exactly reproduce the standard quantum outcome weights prescribed by the Born rule, with:

$$w_0 = \cos^2\left(\frac{\theta}{2}\right) \quad , \quad w_1 = \sin^2\left(\frac{\theta}{2}\right).$$

although in our framework these expressions arise purely from geometric considerations, specifically, the ratio of rotationally invariant volumes on the sphere, rather than from inner products or postulated measurement rules.

In the next section, we generalise these results and clarify the structural properties of the volume-based weights for arbitrary finite outcome partitions.

2.4 General Properties of Volume-Based Outcome Weights

The preceding example illustrates how, in a highly symmetric setting, the combination of deterministic dynamics and volume-preserving flow can reproduce standard quantum probabilities as relative volumes of outcome regions. In this section, we abstract these observations and establish general properties of the outcome weights $w_i = \mu(\Omega_i)/\mu(\Omega_0)$ for arbitrary finite outcome partitions.

As shown in Blore [2025], volume-typicality ensures that these weights correspond to long-run frequencies under repeated macroscopic preparations.

Normalisation and Positivity

By construction, the regions $\Omega_1, \Omega_2, \ldots, \Omega_N \cong S$ form a measurable partition of the prepared region:

$$\Omega_0 = \bigsqcup_{i=1}^{N} \Omega_i,$$

with disjoint interiors and possibly overlapping boundaries of measure zero. Since μ is a finite measure on a compact manifold, and each Ω_i is measurable, it follows that:

$$\sum_{i=1}^{N} w_i = \sum_{i=1}^{N} \frac{\mu(\Omega_i)}{\mu(S)} = 1, \quad w_i \geq 0.$$

Thus, the weights $\{w_i\}$ satisfy the axioms of a probability distribution, yet arise entirely from deterministic geometry and symmetry-constrained volume flow.

2.5 Invariance Under Reparameterisation

The weights w_i are invariant under any smooth bijective reparameterisation of S that preserves the volume measure μ. That is, if $\Psi : S \to S$ is a diffeomorphism such that $\mu(\Psi(A)) = \mu(A)$ for all measurable $A \subseteq S$, then:

$$\mu(\Psi(\Omega_i)) = \mu(\Omega_i)$$

and the outcome weights remain unchanged. This guarantees that the results do not depend on arbitrary coordinate choices or surface labels, but only on the intrinsic geometry of the partition.

Dependence on Observable Partition

The mapping $f : S \to O$ defines the outcome regions $\Omega_i = f^{-1}(o_i)$. The weights w_i depend on the specific observable used, as different coarse-grainings will yield different partitions.

However, if two observables are related by a symmetry of the system (e.g. unitary rotation), and the state is transformed accordingly, the corresponding outcome weights remain invariant, provided that the measure μ is itself invariant under that symmetry.

This reflects the physical requirement that equivalent measurements under transformation groups yield the same statistical predictions.

Stability Under Evolution

Because the flow φ_t preserves volume and determinism, the outcome weights are invariant under forward or backward evolution of the initial ensemble. That is, for any time t,

$$\mu(\varphi_t(\Omega_i)) = \mu(\Omega_i)$$

so the set of microstates that evolve toward outcome i maintains the same volume regardless of when the observable is applied. This ensures that the outcome frequencies do not fluctuate in time, and that the limiting frequencies over repeated trials are stationary.

Non-Uniqueness of Partition Boundaries

The precise shape of the boundary between adjacent outcome regions may vary without affecting the weights w_i as long as the boundary has zero measure. In this sense, the volume-based weights are insensitive to arbitrary microscopic refinements or boundary adjustments and depend only on the gross partitioning of state space.

This is essential for ensuring physical robustness: small changes in the microstructure of the model should not alter empirical predictions.

2.6 Summary

The outcome weights w_i derived from volume ratios satisfy the same formal properties as conventional probability distributions but are determined entirely by the geometric structure of the state space, the symmetry-invariant measure μ, and the observable-defined partition Ω_i. No stochastic input is required.

This sets the stage for Section 3, where we analyse under what conditions these volume-based weights must match the squared modulus of amplitudes in conventional quantum mechanics.

While Blore [2025] established how outcome frequencies can be assigned without stochastic assumptions, it remained open whether these weights must match the squared amplitudes of conventional quantum theory. We now show that symmetry requires precisely this.

Table 1: Symmetries constraining the invariant measure $\mathrm{d}\mu$ on complex amplitude space

Symmetry	**Transformation**	**Invariance Requirement**
Complex-scaling symetry	$\Psi \to \alpha\Psi$	Measure must scale consistently under complex rescaling
Unitary covariance	$\Psi \to U\Psi, U \in U(n)$	Measure must be invariant under all unitary rotations

Together, these imply $\mathrm{d}\mu = |\Psi|^2 \mathrm{d}V_H$.

Table 2: Glossary Table

Symbol	**Meaning**
S	The compact, smooth state space manifold over which dynamics evolve
φ_t	Deterministic, volume-preserving evolution map: $\varphi_t : S \to S$
μ	Invariant volume measure on S, preserved under φ_t
O	Observable or coarse-graining function: $O : S \to \{1, 2, \ldots, N\}$
Ω_i	Outcome region: $\Omega_i = O^{-1}(i) \subset S$
w_i	Weight assigned to outcome i : $w_i = \mu(\Omega_i)/\mu(\Omega_0)$
$SU(d)$	Special unitary group preserving overlaps and volume in $\mathbb{C}^d$
$\mathbb{C}P^{d-1}$	Complex projective space of unit vectors modulo global phase

3 Deriving the Born Rule from Symmetry Constraints

The preceding sections established that deterministic, volume-preserving dynamics on a symmetric state space naturally define outcome weights as relative volumes $w_i = \mu(\Omega_i)/\mu(\Omega_0)$, where $\{\Omega_i\}$ is a coarse-grained partition corresponding to a given observable.

These weights are well-defined, stable, and physically meaningful. But this alone does not explain why, across all known quantum systems, the outcome weights w_i numerically match match the Born rule:

$$w_i = |\langle \phi_i | \Psi \rangle|^2.$$

This section addresses that puzzle.

We now ask: under what conditions does the volume of an outcome region necessarily match the squared amplitude of its projection?

The key lies in the constraint imposed by symmetry. In quantum mechanics, the relevant symmetries are given by unitary transformations, elements of the group $SU(d)$, which preserve inner products and hence probabilities in conventional formulations. A valid geometric assignment of outcome weights must be invariant under such transformations. This symmetry requirement turns out to be extremely restrictive.

We show that if:

- The state space admits a unitarily invariant measure,
- Outcome regions depend only on the angle between the system state and the measurement axis,
- And outcome weights are determined by the volume of these regions,

Under the assumptions of rotational symmetry, invariance under reparameterisation, and deterministic partitioning, the only volume-based weighting rule consistent with normalization, observable equivalence, and group invariance is:

$$w_i = \frac{\mu(\Omega_i)}{\mu(\Omega_0)}.$$

In systems where outcome regions correspond to standard quantum projectors, these weights numerically match the Born rule values $w_i = |\langle\phi_i|\Psi\rangle|^2$.

Any alternative assignment, such as weighting linearly with angle, using raw amplitude, or applying nonlinear transformations, fails to preserve one or more of the essential constraints: symmetry invariance, correct normalization, or frame independence.

In what follows, we first formalise the geometric action of $SU(d)$ on the state space and outcome partitions (Section 3.1). We then prove that only the squared-modulus weight assignment is consistent with these constraints (Section 3.2) and finally summarise the complete derivation logic (Section 3.3).

This result completes the structural derivation: deterministic geometry, conserved volume, and symmetry constraints uniquely fix the outcome weights to match $|\Psi|^2$, with no need for additional probabilistic postulates.

3.1 Symmetry Constraints on Outcome Geometry

To determine which volume assignments are physically valid, we must first characterise the symmetry properties of the state space and the measurement process. In standard quantum mechanics, these are governed by the action of the unitary group $SU(d)$ on the projective Hilbert space of pure states.

We assume the following:

- The state space S admits a measure μ that is invariant under the action of $SU(d)$.
- Each measurement outcome corresponds to a pure state $|\phi_i\rangle \in \mathbb{C}^d$, with the observable defined by the orthonormal set $\{|\phi_i\rangle\}$.
- The system's preparation state is $|\Psi\rangle \in \mathbb{C}^d$, and the outcome regions $\Omega_i \subset S$ are determined by the geometric relationship between $|\Psi\rangle$ and $|\phi_i\rangle$.

We now impose the following symmetry requirement:

Symmetry Principle: The outcome weights w_i must remain invariant under joint unitary transformations of both the preparation state $|\Psi\rangle$ and the measurement basis $\{|\phi_i\rangle\}$.

That is, for all $U \in SU(d)$, $w_i(U|\Psi\rangle, \{U|\phi_j\rangle\}) = w_i(|\Psi\rangle, \{|\phi_j\rangle\})$.

This requirement implies that the weights w_i cannot depend on any particular coordinate system or labeling of the outcome states. Instead, they must be expressible purely as functions of invariant quantities under $SU(d)$ action.

The only such invariant between two pure states $|\Psi\rangle$ and $|\phi_i\rangle$ is the squared modulus of their inner product:

$$|\langle\phi_i|\Psi\rangle|^2.$$

Thus, for any geometric partition of the state space to respect $SU(d)$ symmetry, the region Ω_i associated with outcome i must depend only on the value of $|\langle\phi_i|\Psi\rangle|^2$. In other words, for fixed $|\phi_i\rangle$, all states $|\Psi\rangle$ that yield the same $|\langle\phi_i|\Psi\rangle|^2$ must lie on the same boundary of Ω_i.

We conclude that:

- Outcome regions Ω_i must be defined solely in terms of squared inner products $|\langle\phi_i|\Psi\rangle|^2$.
- The volume measure μ must assign the same relative weight to outcomes across all coordinate frames.
- Any volume-based outcome weight assignment inconsistent with this geometry violates the $SU(d)$ symmetry of the underlying physics.

Example: Qubit Branching Geometry and Volume-Typicality

Consider a two-dimensional quantum system ($d = 2$), where the projective state space $\mathbb{C}P^1$ is equivalent to the surface of a sphere, the Bloch sphere. The system state is represented by a unit vector $|\Psi\rangle \in \mathbb{C}^2$, and measurement is performed in an orthonormal basis $\{|\phi_1\rangle, |\phi_2\rangle\}$.

Let the system be prepared in the state:

$$|\Psi\rangle = \cos\left(\frac{\theta}{2}\right)|\phi_1\rangle + \sin\left(\frac{\theta}{2}\right)|\phi_2\rangle,$$

for some $\theta \in [0, \pi]$. The squared overlap with each outcome basis vector is:

$$|\langle\phi_1|\Psi\rangle|^2 = \cos^2\left(\frac{\theta}{2}\right) \quad , \quad |\langle\phi_2|\Psi\rangle|^2 = \sin^2\left(\frac{\theta}{2}\right).$$

Now, interpret this setup geometrically:

- The Bloch sphere is a unit-radius 2-sphere S^2, equipped with the $SU(2)$-invariant measure (surface area).
- Each outcome region Ω_i corresponds to a spherical cap centered on $|\phi_i\rangle$, subtending an angle θ at the origin.

The surface area A of a spherical cap of angular radius θ on a unit sphere is:

$$A(\theta) = 2\pi(1 - \cos\theta),$$

but since the angular separation between $|\Psi\rangle$ and $|\phi_i\rangle$ is θ, the area of the cap assigned to ϕ_2 is:

$$A_2 = 4\pi\sin^2\left(\frac{\theta}{2}\right).$$

This area is proportional to the squared amplitude $|\langle\phi_2|\Psi\rangle|^2 = \sin^2(\theta/2)$, so:

$$\frac{\mu(\Omega_2)}{\mu(\Omega_0)} = \frac{A_2}{4\pi} = \sin^2\left(\frac{\theta}{2}\right).$$

Thus, the relative volume of each outcome region matches the Born rule weight:

$$P(i) = \frac{\mu(\Omega_i)}{\mu(\Omega_0)} = |\langle\phi_i|\Psi\rangle|^2.$$

This provides a concrete, deterministic realisation of volume-typicality: microstates on the Bloch sphere deterministically evolve into macroscopically distinct outcomes, and the ratio of volumes corresponding to each region directly yields the expected quantum frequencies.

In the next section, we show that the only volume function consistent with these constraints is the squared amplitude itself.

3.2 Uniqueness of the Squared-Amplitude Volume Rule

Theorem 1 (Born Measure Uniqueness)

Let Ψ inhabit a smooth, finite-dimensional complex vector space $\mathbb{C}^n$ equipped with a normalised, volume-preserving measure μ. Suppose the measure obeys two symmetry constraints:

1. Complex-scaling symmetry: $\mu(\lambda\Psi) = |\lambda|^k \mu(\Psi)$ for all $\lambda \in \mathbb{C}^*$ and some fixed k.
2. Unitary covariance: $\mu(U\Psi) = \mu(\Psi)$ for all $U \in U(n)$.

Then, normalising μ to unity over the unit sphere implies $k = 2$, yielding the squared-norm measure:

$$\mu(\Psi) = |\Psi|^2.$$

Proof

By complex-scaling symmetry, μ transforms homogeneously:

$$\mu(\lambda\Psi) = |\lambda|^k \mu(\Psi)$$

for some exponent k. This condition constrains how the measure responds to global amplitude rescaling, including phase (as phase shifts are $\lambda \in U(1)$, a subset of $\mathbb{C}^*$).

By unitary covariance, outcome regions defined via partitions in $\mathbb{C}^n$ must yield identical relative weights under any change of basis via $U \in U(n)$. Consequently, μ must depend only on the magnitude $|\Psi|$, not on Ψ's direction in $\mathbb{C}^n$.

Given these constraints, normalise μ on the unit sphere $S^{2n-1} \subset \mathbb{C}^n$. Under scaling $\Psi \to \lambda\Psi$, the condition $\mu(\lambda\Psi) = |\lambda|^k \mu(\Psi)$ implies that extending the integration region radially forces $k = 2$ to preserve the overall probability sum to 1. Any other k leads to divergence or vanishing integral in the limit of integration over the entire $\mathbb{C}^n$.

3.3 Summary of Derivation Logic

Having established in Section 3.2 that symmetry constraints uniquely fix the outcome weights to be $w_i = |\langle\phi_i|\Psi\rangle|^2$, we now show that these weights correspond to empirical frequencies under deterministic evolution

We now summarise the full logical structure that leads from symmetry-constrained deterministic geometry to the Born rule. Each step rests on structural principles alone, with no supplementary assumptions or interpretive insertions.

Step 1: Define a Deterministic State Space

We assume a system evolves on a smooth, compact, finite-dimensional manifold S, equipped with:

- A deterministic, measure-preserving flow $\varphi_t : S \to S$,
- A symmetry-invariant measure μ,
- A coarse-graining map $f : S \to O$ that assigns macroscopic outcomes.

From this, outcome regions $\{\Omega_i\}$ are defined as measurable subsets of S, and outcome weights are:

$$w_i = \frac{\mu(\Omega_i)}{\mu(\Omega_0)}.$$

Step 2: Require Symmetry under Observable Transformations

The mapping f must respect the physical symmetries of the system. In quantum systems, this is captured by unitary transformations $U \in SU(d)$, under which both the preparation state $|\Psi\rangle$ and observable basis $\{|\phi_i\rangle\}$ transform.

This symmetry requirement forces the geometry of Ω_i to depend solely on the overlap $|\langle\phi_i|\Psi\rangle|^2$. Any asymmetry in how regions are constructed would violate this invariance.

Step 3: Impose Additivity and Invariance

Given that:

- Microstates evolve deterministically,
- The measure is invariant under flow and symmetry,
- Outcome weights must sum to one for any orthonormal basis,

the only volume assignment that satisfies all constraints is:

$$w_i = |\langle\phi_i|\Psi\rangle|^2.$$

This emerges not from assumption, but from the structure of geometric typicality under symmetry. It is the only volume rule consistent with deterministic evolution, observable equivalence, and ensemble invariance.

The logic of volume-typicality presented here aligns with a growing body of recent work exploring deterministic foundations of statistical mechanics [Goldstein, 2011]. These approaches reject intrinsic stochasticity in favour of measure-theoretic typicality and symmetry-constrained dynamics. Our construction shares this spirit but applies it to outcome frequencies under $SU(d)$-invariant geometry.

3.4 Environmental Structure and Robustness of Volume-Based Outcomes

In practical settings, quantum systems are never fully isolated. To preserve the geometric determinism of this framework, we model environmental influence by extending the state space: the total system lives in an enlarged manifold $S_{\text{tot}} = S_{\text{sys}} \times S_{\text{env}}$, where both subspaces obey symmetry-preserving dynamics.

A key question is whether the volume-based outcome structure, central to our derivation of $\mu \propto |\Psi|^2$, remains intact when the environment induces coupling between outcome regions.

Let the outcome regions $\Omega_i \subset S_{\text{sys}}$ be defined within the state space of the system under study. In realistic scenarios, the system evolves in conjunction with additional degrees of freedom, such as measurement devices or auxiliary subsystems, forming a joint state space $S_{\text{tot}} = S_{\text{sys}} \times S_{\text{env}}$. When interactions between the system and its environment are weak and unitary, the joint evolution remains deterministic and volume-preserving. In this setting, outcome regions can be extended to product regions of the form:

$$\Omega_i = \{(\Psi_{\text{sys}}, \Psi_{\text{env}}) \in S_{\text{tot}} | \Psi_{\text{sys}} \in \Omega_i, \Psi_{\text{env}} \in E_i\}$$

where each E_i is a measurable subset of the environment's state space correlated with Ω_i. These environmental regions E_i may have partial overlap depending on the strength and structure of the coupling, but as long as the system regions Ω_i remain disjoint and macroscopically distinguishable, the total volume assigned to each branch remains well-defined. The geometric weights derived from Ω_i continue to determine the observed frequencies, even in the presence of mild entanglement or auxiliary dynamics.

Provided the overlap between E_i and E_j for $i \neq j$ is small, the effective volume ratio for outcome i remains stable:

$$\frac{\mu\left(\widetilde{\Omega}_i\right)}{\mu\left(\widetilde{\Omega}_0\right)} \approx \frac{\mu(\Omega_i)}{\mu(\Omega_0)}.$$

This demonstrates that decoherence-like effects, in which information disperses into inaccessible degrees of freedom, can be geometrically modeled within this framework, without altering the derived probabilities.

3.5 Justifying Typicality and Uniform Distribution

A central assumption in our framework is that microstates consistent with a macroscopic preparation are distributed uniformly with respect to the symmetry-invariant measure μ. This uniform distribution underpins the correspondence between volume fractions and empirical frequencies in repeated experiments.

Mathematically, typicality arguments [Goldstein, 2011] show that in deterministic systems preserving an invariant measure, the vast majority of initial conditions yield macroscopic frequencies matching the measure-theoretic volume fractions. Under measure-preserving dynamics, ensembles uniformly sampled from μ remain uniform across time, guaranteeing the stability of outcome frequencies.

Thus, the typicality assumption is not an additional probabilistic postulate but a structural consequence of deterministic volume-preserving flow on compact state space, consistent with standard results in statistical mechanics.

3.6 Connection to Probabilities in Standard Quantum Mechanics

In conventional quantum theory, the Born rule assigns probabilities to measurement outcomes according to the squared modulus of the projection of the state vector onto measurement basis states. Here, we reinterpret this probability assignment as a geometric volume ratio over the state space.

In our framework, the invariant measure μ on $\mathbb{C}P^{n-1}$ assigns a volume to each outcome region ω_i defined by coarse-graining the continuous state space according to the measurement observable. Under deterministic, volume-preserving flow, the relative volume $\mu(\Omega_i)/\mu(\Omega_0)$ corresponds to the long-run frequency with which microstates evolve into the macroscopic outcome associated with Ω_i.

Thus, what standard quantum theory calls "probability" is here reinterpreted as the geometric fraction of state space consistent with a given outcome under symmetry constraints and deterministic dynamics. This paper explains why the volume-derived weights numerically match the conventional Born rule probabilities while maintaining a strictly deterministic foundation.

In the next sections, we explore the implications of this result, its relationship to conventional quantum postulates, and potential avenues for empirical falsification.

4 Discussion

The derivation presented above shows that the Born rule follows necessarily from a deterministic, symmetry-constrained geometric framework: outcome weights emerge from volume ratios over a compact, invariant state space, with no requirement for probabilistic or interpretive inputs. In this section, we discuss the broader implications of this result.

First, we clarify how volume-typicality connects to observable outcome frequencies, and why no probability postulate is needed (Section 4.1). Next, we compare this approach with the conventional axioms of quantum mechanics, identifying which postulates are replaced or made redundant (Section 4.2). Finally, we outline the limitations of the current framework and identify key open questions for future work (Section 4.3).

Throughout, the focus remains on what this derivation requires and what it does not. The success of the volume-weighted approach does not depend on philosophical stance. It is a structural result grounded in symmetry, continuity, and the intrinsic geometry of the state space.

4.1 Volume-Typicality and Frequencies

The outcome weights $w_i = \mu(\Omega_i)/\mu(S)$ derived in Sections 2 and 3 have the form of a probability distribution but arise from purely geometric and deterministic considerations. No stochastic process, random collapse, or statistical sampling is involved in their definition. Instead, these weights describe the relative volume of each outcome region under a measure-preserving flow on a compact state space.

This supports the notion of volume-typicality introduced in Blore [2025], where observed outcome frequencies are understood as the long-run frequencies of macroscopic events generated by a deterministic system, given uncertainty about the precise microstate. If the system is prepared repeatedly under the same macroscopic conditions, and the set of corresponding microstates is uniformly distributed with respect to the symmetry-invariant measure μ, then the observed frequency of each outcome will, in the limit, approach w_i.

This approach to frequency is entirely deterministic:

- The system evolves predictably from each microstate via the flow φ_t.
- The partition $\{\Omega_i\}$ is defined geometrically by the observable.
- The outcome label i is a deterministic function of the initial microstate.
- The outcome weights arise solely from the relative measure of the outcome regions.

What appears probabilistic at the macroscopic level is thus the consequence of two factors: our ignorance of the specific microstate, and the fact that outcome labels arise from coarse-graining a continuous state space. The term "typicality" reflects this: most microstates consistent with a given preparation lie in outcome regions in proportion to their volume.

Importantly, this view differs sharply from conventional probabilistic interpretations of quantum mechanics. Here, frequencies emerge not from intrinsic indeterminism, but from the geometric structure of the dynamical system and its symmetries.

This closes the explanatory loop: the Born rule is recovered not by assumption or statistical postulate, but by identifying how deterministic systems produce stable ensemble frequencies under well-defined geometric constraints.

4.2 Comparison with Quantum Postulates

Standard formulations of quantum mechanics rely on several core postulates, including:

1. **State Postulate:** A system is fully described by a unit vector $|\Psi\rangle$ in a Hilbert space $\mathcal{H}$.
2. **Observable Postulate:** Physical observables correspond to Hermitian operators on $\mathcal{H}$, with eigenvalues representing possible outcomes.
3. **Evolution Postulate:** The state evolves unitarily via the Schrödinger equation.
4. **Measurement Postulate (Born Rule):** Upon measurement in basis $\{|\phi_i\rangle\}$, the probability of outcome i is given by $P(i) = |\langle\phi_i|\Psi\rangle|^2$.
5. **Collapse Postulate:** After measurement, the system collapses to the eigenstate corresponding to the observed outcome.

The framework presented in this paper recovers the outcome statistics implied by Postulate 4 but does so without invoking it. Instead, the result $w_i = |\langle\phi_i|\Psi\rangle|^2$ emerges from deterministic dynamics and symmetry-constrained geometry alone.

Specifically:

- The Born rule is not assumed; it is shown to be the unique outcome of volume-based typicality under $SU(d)$ symmetry.
- No probabilistic interpretation is invoked. Frequencies arise from volume ratios over coarse-grained outcome regions, not from stochastic collapse or epistemic randomness.

- The derivation is agnostic to ontological commitments. It does not rely on hidden variables, wavefunction branching, or subjective belief updating.

This geometric account thus provides an explanatory foundation for outcome frequencies that is logically prior to the standard postulates. It suggests that if the quantum formalism is an emergent statistical description of an underlying deterministic system, then the Born rule need not be fundamental, it can be derived.

Moreover, by grounding outcome statistics in measurable geometric volumes on a symmetric state space, the approach sidesteps long-standing controversies over collapse and interpretation. It offers a clear distinction between what is observed (frequencies) and what is explained (volume typicality), without needing to assert how or whether the wavefunction itself is ontic.

In this way, the approach complements the operational success of standard quantum theory while offering a deeper structural account of one of its most fundamental statistical rules.

4.3 Limitations and Open Questions

While this paper establishes that the Born rule follows uniquely from deterministic volume geometry under symmetry constraints, several important limitations remain. These concern the scope of applicability, the treatment of composite systems, and the possible extension to more general observables.

Finite-Dimensional Systems

The derivation presented here is restricted to systems represented by finite-dimensional state spaces. The symmetry constraints used in Section 3 rely on the compactness of the projective state space and the unitary invariance of the measure μ.

While our derivation has focused on finite-dimensional quantum systems with compact projective state space $S \cong \mathbb{C}P^{d-1}$, the principles outlined here may extend to continuous or infinite-dimensional systems by coarse-graining. In such cases, effective discretization methods, such as those used in lattice field theory [Wilson, 1974], allow the continuum to be approximated by a finite number of distinguishable degrees of freedom within a compact geometry. These discretized state spaces can preserve the same geometric symmetries (e.g. unitary invariance) and support a derived volume measure that converges to the continuum form in the limit of finer resolution. Thus, while the present analysis does not address infinite-dimensional Hilbert spaces directly, the volume-based symmetry argument may remain valid in physically relevant limits.

Beyond Projective Measurements

Only sharp measurements, i.e., those defined by orthonormal projective bases, are considered in this framework.

Empirical Signatures

Although the volume-based framework reproduces the Born rule under well-defined assumptions, it currently makes no novel empirical predictions. This limits its immediate falsifiability.

Summary

This paper establishes that deterministic evolution on a symmetry-constrained state space yields outcome frequencies matching the Born rule, with no need for stochastic postulates. What remains is to extend, test, and refine the model:

- Can the framework accommodate entanglement, decoherence, or information flow?
- Can or should it be generalised to infinite dimensions or continuous spectra?
- Are there regimes, cosmological, mesoscopic, or gravitational, where deviations might appear?

These questions define the natural next steps for a volume-based approach to quantum foundations.

5 Falsifiability and Empirical Tests

Any proposed explanation of the Born rule must be subject to empirical scrutiny. While the standard quantum formalism takes $P(i) = |\langle\phi_i|\Psi\rangle|^2$ as an axiom, the present framework derives it from deterministic geometry. This opens the door to testing the assumptions under which the derivation holds.

The key empirical claim is this:

If macroscopic preparation procedures select microstates uniformly (with respect to a symmetry-invariant, volume-preserving measure μ) over the state space S, and if observable outcomes correspond to well-defined partitions $\{\Omega_i\}$, then the frequency of each outcome should converge to $w_i = \mu(\Omega_i)/\mu(\Omega_0) = |\langle\phi_i|\Psi\rangle|^2$.

Any persistent deviation from this rule, under controlled, repeatable conditions, would falsify one or more assumptions of the volume-typicality framework.

For example, Interferometry as a Test of the Measure.

The geometric symmetry-based framework predicts outcome frequencies proportional to $|\Psi|^2$ for all quantum systems. Any consistent, reproducible deviation from this rule, for example, frequencies proportional to $|\Psi|^{2+\epsilon}$ for some small $\epsilon \neq 0$, would falsify the derived measure and rule out this framework.

One feasible test platform is quantum interferometry, such as a Mach–Zehnder interferometer with adjustable phase delay. A single-photon source prepares quantum states with tunable path amplitudes, and detectors at the output arms record relative frequencies.

To test for deviations, one can prepare a known input state $|\Psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$, vary the phase ϕ, and record output detection rates. The expected counts are:

$$P_{\text{ideal}} = \left|\alpha + \beta e^{i\phi}\right|^2.$$

By comparing empirical frequencies against this ideal distribution across a sweep of ϕ, one can detect or constrain non-quadratic scaling in the effective probability rule.

Current photon-counting experiments can detect deviations in balanced configurations ($|\alpha| = |\beta| = 1/\sqrt{2}$). No such deviations have been observed, but improved precision could probe closer to the symmetry-derived limit.

Another example, Experimental Implementation in Superconducting Qubit Arrays.

Here high-fidelity unitary operations and repeated state preparation are now routine. These systems allow precise engineering of states $|\Psi\rangle \in \mathbb{C}^d$, with control over initialization, entanglement, and measurement axes via microwave gate sequences.

To test the volume-based prediction $\mu \propto |\Psi|^2$, one can:

- Prepare superpositions of known amplitudes (e.g. $|\Psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$),
- Measure in multiple bases via tomographic rotation gates,
- Compare the empirical frequencies of outcomes across repeated trials to the expected Born-rule predictions.

If the underlying measure deviated to $\mu \propto |\Psi|^{2+\epsilon}$, the frequencies in some rotated frames would differ by measurable amounts.

This experimental route tests the core claim of the paper: that the $|\Psi|^2$ rule follows from geometric symmetry constraints, not from axiomatic postulates or probabilistic collapse.

5.1 What Could Be Falsified

There are several concrete ways in which the framework could, in principle, fail:

- **Asymmetry in Microstate Sampling:** If physical preparation procedures do not yield an invariant ensemble of microstates, but instead bias toward specific regions of S, the resulting outcome frequencies would deviate from those predicted by $\mu(\Omega_i)$. High-precision preparation and tomography could test this.
- **Violation of Volume Preservation:** If the dynamics φ_t fail to preserve volume (e.g. due to dissipation, decoherence, or external interaction), then weights computed from $\mu(\Omega_i)$ would not remain stable. Time-dependent statistical drift could indicate such effects.

- **Failure of Coarse-Graining Consistency:** If the macroscopic observable does not yield a sharp partition of S, or if microscopic ambiguity leads to smeared outcomes, then deviations from the sharp-volume rule may arise. This could be probed by increasing measurement resolution.
- **Incompatibility in Composite Systems:** If experiments on entangled systems reveal correlations inconsistent with any $SU(d)$-invariant volume assignment across tensor products of S, the framework would require modification.

5.2 Prospects and Constraints

At present, no empirical deviation from the Born rule has been confirmed. The volume-based derivation aligns perfectly with standard outcome statistics wherever they have been tested. However, the framework provides a conceptual advantage: its assumptions are visible, structural, and testable. This is in contrast to postulates treated as fundamental or axiomatic.

The challenge for future work is to identify realistic experimental contexts, possibly involving decoherence, gravitational effects, or high-precision state preparation, where deviations from the assumed geometry might occur.

6 Conclusion

This paper has shown that the Born rule, the assignment of outcome frequencies proportional to $|\Psi|^2$, can be derived from deterministic principles and geometric symmetry. By assuming a finite-dimensional Hilbert space and identifying two natural invariances, complex-scaling symmetry (which subsumes global phase transformations) and unitary covariance, we proved that the squared-norm measure is the only volume form consistent with these constraints.

The derivation relies on no stochastic assumptions, no collapse mechanisms, and no interpretive overlays. Instead, it demonstrates that the statistical structure of quantum theory follows directly from the geometry of amplitude space under symmetry-preserving dynamics. When combined with a deterministic branching model in which macroscopic outcomes correspond to disjoint regions of the state space, this measure explains why long-run frequencies align with the standard quantum rule.

This result reframes the Born rule as a consequence of measure invariance under structural symmetries, not as an axiom of probability. It supports a deterministic foundation in which observed randomness emerges from the typicality of volume flow through well-defined outcome partitions.

The framework presented here is intentionally minimal: it makes no commitment to any particular ontological picture, and no assumptions beyond continuity, compactness, and symmetry. Its strength lies in its generality: the Born rule is not inserted, it is required. Future work may extend this approach to entangled systems, continuous spectra, and dynamical observer models. But even in its present form, the result affirms a striking conclusion: quantum statistics arise from symmetry and structure, not chance.

Appendix A Comparison with Other Approaches

Numerous attempts have been made to derive or justify the Born rule from deeper principles. While differing in assumptions, scope, and interpretation, these approaches share the goal of explaining quantum outcome statistics without postulating them. Below, we briefly compare the present volume-based framework with several leading alternatives.

Everettian (Many-Worlds) Derivations

In Everettian frameworks, all outcomes occur, and probabilities are interpreted as branch weights. Modern versions appeal to decision theory [Deutsch, 1999, Wallace, 2012] or typicality over branches [Vaidman, 2022].

- **Strength:** Retains the full formalism of quantum theory.
- **Limitation:** Requires interpretational commitments (e.g. branching reality, rational agents).
- **Comparison:** The present framework avoids both branching and agents; it derives weights directly from geometric structure.

Envariance and Decoherence-Based Arguments

Zurek's envariance program [Zurek, 2003] uses entanglement symmetry to argue that amplitudes must square to produce consistent reduced states. Decoherence is invoked to explain effective outcome separation.

- **Strength:** Tied to physical processes like entanglement and environment.
- **Limitation:** Envariance is subtle, and derivations often assume partial trace properties or reduced-state normalisation.
- **Comparison:** The volume-based approach is more abstract but avoids assumptions about environment-induced structure. It derives weights from intrinsic geometry and symmetry, not environmental context.

Gleason-Type Theorems

Gleason's theorem [Gleason, 1975] and its generalisations show that, under certain conditions, any probability measure on projectors must match the Born rule.

- **Strength:** Rigorous mathematical constraint from Hilbert space axioms.
- **Limitation:** Assumes the Hilbert space formalism and measure additivity from the outset.
- **Comparison:** This paper rederives the same rule, but from volume and symmetry, without relying on Hilbert space probability theory.

Hidden-Variable Theories

In deterministic models like Bohmian mechanics [Bohm, 1952a,b], the Born rule is typically imposed as an initial condition on the distribution of hidden variables.

- **Strength:** Provides a deterministic ontology.
- **Limitation:** Requires a postulated equilibrium distribution, typically not derived.
- **Comparison:** Our framework is also deterministic but shows that outcome weights arise necessarily from volume structure, no initial "quantum equilibrium" is assumed.

The volume-typicality framework uniquely combines:

- Determinism,
- No hidden structure,
- No appeal to rational agents,
- And a self-contained derivation of the Born rule from geometry and symmetry.

Comparison of Born Rule Derivations

Approach	Assumptions
This work	Complex homogeneity, phase indifference, unitary covariance (no probability axioms)
Gleason's Theorem	Hilbert space structure; additive, non-contextual probability measure
Zurek's Envariance	System–environment symmetry (envariance); partial trace assumptions
Deutsch–Wallace Decision	Rational agent preferences in many-worlds; subjective utility theory

Glossary of Key Terms

Ψ (psi): A complex vector representing the state of a finite-dimensional quantum system. In this paper, $\Psi \in \mathbb{C}^n$ is treated as an operational descriptor of system preparation, not a fundamental ontological object.

Outcome region, Ω_i: A measurable, disjoint subset of the unit sphere in state space corresponding to a macroscopically distinguishable outcome. Each Ω_i is constructed to align with a specific observable result.

Volume-typicality: The principle that, in a deterministic system with branching, long-run frequencies of outcomes are determined by the invariant volume of each outcome region. A "typical" trajectory lies in the largest such region compatible with its history.

Branching: The partitioning of state space into disjoint regions as the system evolves, representing distinct future outcomes. Branching is deterministic and geometric, not probabilistic.

Invariant measure, μ: A volume measure on state space preserved under the system's deterministic dynamics. Frequencies of outcomes are given by $\mu(\Omega_i)/\mu(\Omega_0)$, where Ω_0 is the full accessible state region.

Symmetry constraints: The physical invariances imposed on the system, specifically:

- Complex-scaling symmetry: $\Psi \to \lambda\Psi$ for $\lambda \in \mathbb{C}^*$
- Unitary covariance: $\Psi \to U\Psi$ for $U \in SU(n)$

These ensure the measure respects physical indistinguishability of rescaled, rotated, or phase-shifted states.

Deterministic flow, φ_t: A smooth, volume-preserving evolution map $\varphi_t : \Sigma \to \Sigma$ that governs how states evolve over time. Used in Blore [2025] to track branching and outcome region volumes.

Appendix B Infinite-Dimensional Extension via Inductive Embedding

We briefly sketch how the two-symmetry derivation and $k = 2$ result extend to any Hilbert space $\mathcal{H}$ by constructing an inductive limit of finite-dimensional projective spaces.

Appendix B.1 Inductive Embedding of Projective Spaces

Consider the natural embedding of complex projective spaces:

$$\mathbb{C}P^n \hookrightarrow \mathbb{C}P^{n+1}$$

given by appending a zero coordinate:

$$[\Psi_1, \Psi_2, \ldots, \Psi_n] \to [\Psi_1, \Psi_2, \ldots, \Psi_n, 0].$$

Each finite-dimensional subspace thereby nests inside a larger one. The standard Fubini-Study measure on $\mathbb{C}P^n$ pushes forward to $\mathbb{C}P^{n+1}$ consistently under this embedding.

Appendix B.2 Tightness and Existence of Limit Measure

Because each Fubini-Study measure is normalised and probability measures on compact spaces are tight, Prokhorov's theorem [Prokhorov, 1956] guarantees the existence of a unique projective limit measure on:

$$\mathbb{C}P^{\infty} \equiv \lim_{n\to\infty} \mathbb{C}P^n.$$

This limit measure inherits the $U(n)$ invariance properties in the finite case, extended to the infinite-dimensional unitary group $U(\mathcal{H})$.

Appendix B.3 Preservation of Homogeneity Condition

The homogeneity condition:

$$\mu(\lambda\Psi) = |\lambda|^k \mu(\Psi)$$

is dimension-independent: scalar rescaling affects every finite subspace identically. Thus, the same constraint argument fixing $k = 2$ on each finite truncation applies unchanged in the limit.

Consequently, the derivation of the Born weight $|\Psi|^2$ holds on the full separable Hilbert space by continuity and consistency of the measure in this limit construction.

References

Zayn Blore. Volume-based probability: Outcome frequencies from deterministic geometry, 2025. URL https://zenodo.org/doi/10.5281/zenodo.15564079.

Sheldon Goldstein. *Typicality and Notions of Probability in Physics*, page 59–71. Springer Berlin Heidelberg, August 2011. ISBN 9783642213298. doi:10.1007/978-3-642-21329-8_4. URL http://dx.doi.org/10.1007/978-3-642-21329-8_4.

Kenneth G. Wilson. Confinement of quarks. *Physical Review D*, 10(8):2445–2459, October 1974. ISSN 0556-2821. doi:10.1103/physrevd.10.2445. URL http://dx.doi.org/10.1103/PhysRevD.10.2445.

David Deutsch. Quantum theory of probability and decisions. *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*, 455(1988):3129–3137, August 1999. ISSN 1471-2946. doi:10.1098/rspa.1999.0443. URL http://dx.doi.org/10.1098/rspa.1999.0443.

David Wallace. *The Emergent Multiverse: Quantum Theory according to the Everett Interpretation*. OUP Oxford, May 2012.

Lev Vaidman. Why the many-worlds interpretation? *Quantum Reports*, 4(3):264–271, 2022. ISSN 2624-960X. doi:10.3390/quantum4030018. URL https://www.mdpi.com/2624-960X/4/3/18.

Wojciech Hubert Zurek. Environment-assisted invariance, entanglement, and probabilities in quantum physics. *Physical Review Letters*, 90(12), March 2003. ISSN 1079-7114. doi:10.1103/physrevlett.90.120404. URL http://dx.doi.org/10.1103/PhysRevLett.90.120404.

Andrew M. Gleason. *Measures on the Closed Subspaces of a Hilbert Space*, page 123–133. Springer Netherlands, 1975. ISBN 9789401017954. doi:10.1007/978-94-010-1795-4_7. URL http://dx.doi.org/10.1007/978-94-010-1795-4_7.

David Bohm. A suggested interpretation of the quantum theory in terms of "hidden" variables. i. *Physical Review*, 85(2):166–179, January 1952a. ISSN 0031-899X. doi:10.1103/physrev.85.166. URL http://dx.doi.org/10.1103/PhysRev.85.166.

David Bohm. A suggested interpretation of the quantum theory in terms of "hidden" variables. ii. *Physical Review*, 85(2):180–193, January 1952b. ISSN 0031-899X. doi:10.1103/physrev.85.180. URL http://dx.doi.org/10.1103/PhysRev.85.180.

Yu. V. Prokhorov. Convergence of random processes and limit theorems in probability theory. *Theory of Probability & Its Applications*, 1(2):157–214, January 1956. ISSN 1095-7219. doi:10.1137/1101016. URL http://dx.doi.org/10.1137/1101016.