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room. May his mantle descend on our future presidents, and his spirit long continue to preside over our councils, and animate our exertions in the cause he had so much at heart!

On the conclusion of the reading of the preceding Memoir, the thanks of the Society were proposed by the Dean of Ely, and unanimously voted to Sir John Herschel.

It was then moved by Mr. De Morgan, seconded by Mr. Donkin, and unanimously resolved,

"That the Society feels it impossible to express in adequate terms its obligations to its late President; and it desires to impress on the minds of all the Fellows, that such imitation of his example as their occupations will allow is the mode of testifying their gratitude and respect for his memory with which he would have been most pleased."

VI. *On* Jacobi's *Elliptic Functions*.
By the Rev. Brice Bronwin.

I DID not expect to hear again from Mr. Cayley on this subject, as my paper printed in this Journal in August 1843 made the matter in dispute between us exceedingly plain. Every step of it is clear. The most difficult is—But for the third and fourth forms of ω the denominator

$$\mathrm{s}\,a\,(\mathrm{K}-2\,\omega)\,\mathrm{s}\,a\,(\mathrm{K}-4\,\omega)\ldots\ldots$$

cannot be reduced to the form $\mathrm{s}\,a\,\omega\,\mathrm{s}\,a\,.\,3\,\omega\ldots\ldots$ And this is perfectly easy (see my paper in this Journal, April 1843, p. 260). The next step—u cannot take any of the forms $2\,r\,\omega-\mathrm{K}$—is evident; for the imaginary part of u must be equal to the imaginary part of $2\,r\,\omega$, and then the real part also will be equal to the real part of $2\,r\,\omega$, and not equal to the real part minus K. This is all exceedingly easy, and very unlike the slippery path Mr. Cayley has trod in his last paper, a path in which a false step is easily made.

Besides, I called Mr. Cayley's attention to a transformation depending on the third form of ω, at p. 54 of Jacobi's work, and requested him to make trial of it in its simplest case. I pointed out to him the means of very easily testing this form. The result would have been a stubborn fact on one side or the other. I suppose the readers of this Journal, whether they understand Jacobi's functions or not, will know how to estimate this circumstance.

I will now enlarge on the other mode of deciding this matter, which I just hinted at in the conclusion of my paper of August 1843. In order to determine the constant M, Jacobi makes $x=\mathrm{s}\,a\,.\,u=1$ (see p. 41). No form of ω therefore which does not satisfy this condition can consist with his theory. As ω has n for a denominator, we must have $u = n\omega$, or some odd integer multiple of this quantity. But $\mathrm{s}\,a\,(n\,\omega) = 1$ for the first form of ω only. I expect Mr. Cayley will object to this, and think that some other value of u may be found to satisfy the required condition. Let then $p\,n\,\omega$ be such a value, and let us determine it so as to satisfy the required condition for the third and fourth forms of ω if possible, in order to oblige that gentleman.

By pages 32 and 34 of Jacobi,

$$\mathrm{s}\,a\,(p\,n\,\omega) = \mathrm{s}\,a\,(p\,m\,\mathrm{K} + p\,m'\,i\,\mathrm{K}'),$$

$$= \frac{\left\{\begin{array}{l} \mathrm{s}\,a\,(p\,m\,\mathrm{K})\,\mathrm{c}\,a\,(p\,m'\,i\,\mathrm{K}')\,\Delta\,.\,a\,(p\,m'\,i\,\mathrm{K}') \\ + \,\mathrm{s}\,a\,(p\,m'\,i\,\mathrm{K}')\,\mathrm{c}\,a\,(p\,m\,\mathrm{K})\,\Delta\,.\,a\,(p\,m\,\mathrm{K}) \end{array}\right\}}{1 - k^2\,\mathrm{s}^2\,a\,(p\,m\,\mathrm{K})\,\mathrm{s}^2\,a\,(p\,m'\,i\,\mathrm{K}')}$$

$$= \frac{\left\{\begin{array}{c} \mathrm{s}\,a\,(p\,m\,\mathrm{K})\,\Delta\,.\,a\,(p\,m'\,\mathrm{K}') + \\ i\,\mathrm{s}\,a\,(p\,m'\,\mathrm{K}')\,\mathrm{c}\,a\,(p\,m'\,\mathrm{K}')\,\mathrm{c}\,a\,(p\,m\,\mathrm{K})\,\Delta\,.\,a\,(p\,m\,\mathrm{K}) \end{array}\right\}}{\mathrm{c}^2\,a\,(p\,m'\,\mathrm{K}') + k^2\,\mathrm{s}^2\,a\,(p\,m\,\mathrm{K})\,\mathrm{s}^2\,a\,(p\,m'\,\mathrm{K}')}. \quad (a.)$$

The modulus of ω, and therefore of $i\,\mathrm{K}'$, is k; but that of K' in $(a.)$ is k', its own modulus.

The imaginary quantity i must vanish from $(a.)$. Let then

$$\mathrm{s}\,a\,(p\,m'\,\mathrm{K}') = 0, \text{ or } p\,m' = 2\,r,\ p = \frac{2\,r}{m'},$$

r as well as m and m' being any integers, positive or negative. The result is that

$$(a.) = \mathrm{s}\,a\,(p\,m\,\mathrm{K}) = \mathrm{s}\,a\left(\frac{2\,r\,m}{m'}\,\mathrm{K}\right).$$

If $\frac{2\,r\,m}{m'}$ be a fraction, this cannot answer; if it be an integer, it is an even one, and gives $\mathrm{s}\,a\,(p\,m\,\mathrm{K}) = 0$, which does not answer; for m is even, m' odd, or both are odd.

Next make $\mathrm{c}\,a\,(p\,m'\,\mathrm{K}') = 0$, or $p\,m' = 2\,r + 1$, $p = \frac{2\,r+1}{m'}$. The result in this case is

$$(a.) = \frac{1}{k\,\mathrm{s}\,a\,(p\,m\,\mathrm{K})},$$

which must be greater than unity, and therefore cannot answer.

Make $\mathrm{c}\,a\,(p\,m\,\mathrm{K}) = 0$, or $p\,m = 2\,r + 1$, $p = \frac{2\,r+1}{m}$, the result is

$$(a.) = \frac{1}{\Delta\,.\,a\,(p\,m'\,\mathrm{K}')},$$

which cannot be unity for m even, m' odd, or for both odd. The third and fourth forms of ω therefore never can give s $a.u=1$, as required by Jacobi's theory. It is not necessary, I presume, to say anything about the second form of ω.

Mr. Cayley has not condescended to state his objections to the reasoning in my last paper; had he done so, I should have had a chance of obviating them on his own ground, but as it is I am left entirely in the dark respecting them; and unhappily the darkness is in no measure dispelled by the cloud of mystery in which his last paper has enveloped the subject.

I must now notice Mr. Cayley's logic in the 16th No. of the Cambridge Mathematical Journal. He was at liberty to make $u=\omega$, or any other quantity; and in so doing he must determine the true value of C, if he proceeded by a right method. How then does it happen that for two forms of ω he obtains faulty or indeterminate results, no results as he calls them? Plainly because the forms of ω were faulty, not from any fault in the denominator. If Jacobi's denominator would have set all right, how did it happen that he did not fall upon it in these two cases? In the last case also he has in reality arrived at no result; it was quite ridiculous to argue against my denominator when he had obliterated it. If Jacobi's had been the true form, he would in every case have fallen upon it; and had there been no fault in the forms of ω, he would in no case have been led to faulty results. Had he carried his method out fully, and drawn from it the proper inferences, he would have proved all that I have asserted with regard to Jacobi's functions.

Gunthwaite Hall, Nov. 19, 1844. B. BRONWIN.

VII. *Proceedings of Learned Societies.*

CAMBRIDGE PHILOSOPHICAL SOCIETY.

Nov. 27, 1843. "ON the Foundation of Algebra," No. III. By Augustus De Morgan, of Trinity College, Professor of Mathematics in University College, London, &c.

In the second paper of this series a general definition of the operation A^B was laid down, A and B being each of them any form of $p+q\sqrt{-1}$. The logarithm (or as Mr. De Morgan calls it, the *logometer*) of a line is thus described:—a line whose projection on the unit-axis is the logarithm of the length, and whose projection on the perpendicular is the angle made with the unit-axis (or its arc to a radius unity). Thus a line r inclined at an angle θ has for its logometer a line $\sqrt{(\log^2 r+\theta^2)}$ inclined at an angle whose tangent is $\theta : \log r$ This being premised, the universal definition of A^B is the line whose logometer is $B \times \text{logom.} A$.

The object of this third paper is to show that the preceding defi-