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On a General Method of Solving Partial Differential Equations.

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[*Read February* 14, 1878.]

It is well known that the general method of solving a partial differential equation of the first order consists in forming n equations, including the given one, such as to render

$$dz - p_1 dx_1 - \ldots - p_n dx_n = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (a)$$

an integrable equation. These equations are found by solving certain partial differential equations usually written, for short,

$$[F_i, F_j] = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (b).$$

The object of the following paper is to deduce a system equivalent to (b) directly from the conditions of integrability of (a). Such a system is, in fact, obtained, and precisely the same form of equation serves to integrate equations of the higher orders.

It is shown that only one system of the kind just mentioned has to be integrated in order to get a final integral of an equation or system of equations, although some of them may be of an order higher than the first.

When we have to deal with equations of the second or higher orders, we find it necessary to solve a second set of auxiliary equations which have no analogue in the theory of equations of the first order. One system of this kind must be solved before passing to the first integral, another before we can get a second integral, and so on. No such system has to be solved before passing from the penultimate integral to the solution.

I have thought it desirable to prove that, in the case of equations of the first order, the system of auxiliary equations of which use is made in this paper, is completely equivalent to the system (*b*) commonly employed. But I have not succeeded in effecting a similar transformation in the case of equations of the second and higher orders, though such transformations are no doubt possible, and would be of the utmost practical value.

It may be added that the results are applicable to systems of simultaneous equations of the same or different orders.

Equations of the First Order.

1. Let z be a function of $x_1 \dots x_n$; viz., suppose there is a relation

$$\phi(z, x_1 \dots x_n) = 0 \quad \dots\dots\dots\dots\dots\dots(1);$$

then, writing p_i for $\frac{dz}{dx_i}$, we have

$$p_i = -\frac{d\phi}{dx^i} : \frac{d\phi}{dz} \quad (i = 1 \dots n).$$

These equations give, without difficulty,

$$\frac{dp_i}{dx_j} + p_j \frac{dp^i}{dz} = \frac{dp_j}{dx^i} + p_i \frac{dp_j}{dz} \quad (i, j = 1 \dots n) \dots\dots\dots\dots(2),$$

which, being independent of the form of ϕ, must be satisfied whatever ϕ may be. Moreover they give

$$\frac{d\phi}{dz}(dz - p_1 dx_1 - \dots - p_n dx_n) = d\phi,$$

an exact differential.

Conversely, if (2) be satisfied, $p_1 \dots p_n$ are the derivatives with respect to $x_1 \dots x_n$ of a function z defined by some equation of the form (1). For (2) are just sufficient to ensure that

$$dz - p_1 dx_1 - \dots - p_n dx_n = 0 \dots\dots\dots\dots\dots\dots(3)$$

should be integrable; viz., that the expression on the left, when multiplied by a certain integrating factor, should become an exact differential. Any integral of (3) will be an equation of the form (1) consistent with the given forms of p.

2. Suppose now that $p_1 \dots p_n$, instead of being given as functions of z, $x_1 \dots x_n$, are given by n equations of the form

$$F_i(p_1 \dots p_n, z, x_1 \dots x_n) = a_i, \ (i = 1 \dots n) \ \dots\dots\dots(4),$$

where a_i is a constant separated from F_i for convenience. We seek the relations between F_i which are necessary and sufficient to ensure the existence of an equation (1) consistent with them.

Let ξ represent any one of the $n+1$ quantities z, $x_1 \dots x_n$. Then we have n equations of the form

$$\frac{dF_i}{dp_1}\cdot\frac{dp_1}{d\xi}+\ldots+\frac{dF_i}{dp_n}\cdot\frac{dp_n}{d\xi}+\frac{dF_i}{d\xi}=0.$$

The differential coefficients which form the second factors of the terms in these equations are obtained on the supposition that $p_1 \ldots p_n$ are by means of (4) expressed as functions of $z, x_1 \ldots x_n$. From these equations eliminate every $\frac{dp}{d\xi}$ except $\frac{dp_j}{d\xi}$. Thus, using Donkin's notation,

we get
$$\frac{d(F_1 \ldots F_n)}{d(p_1 \ldots p_n)}\cdot\frac{dp_i}{d\xi}=-\frac{d(F_1 \ldots F_j \ldots F_n)}{d(p_1 \ldots \xi \ldots p_n)}.$$

We assume that
$$\frac{d(F_1 \ldots F_n)}{d(p_1 \ldots p_n)}$$

does not vanish; for, if it did, (4) would not define $p_1 \ldots p_n$. Substituting in (2), we obtain

$$\frac{d(F_1 \ldots F_i \ldots F_n)}{d(p_1 \ldots x_j \ldots p_n)}+p_j\frac{d(F_1 \ldots F_i \quad F_n)}{d(p_1 \ldots z \ldots p_n)}$$
$$=\frac{d(F_1 \ldots F_j \ldots F_n)}{d(p_1 \ldots x_i \ldots p_n)}+p_i\frac{d(F_1 \ldots F_j \ldots F_n)}{d(p_1 \ldots z \ldots p_n)}.$$

If, for convenience, we indicate $\frac{du}{dx_i}+p_i\frac{du}{dz}$ by $\frac{du}{d\bar{x}_i}$, these equations may be written

$$\frac{d(F_1 \ldots F_i \ldots F_n)}{d(p_1 \ldots \bar{x}_j \ldots p_n)}=\frac{d(F_1 \ldots F_j \ldots F_n)}{d(p_1 \ldots \bar{x}_i \ldots p_n)} \quad \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Or again, if we represent the Jacobian formed from

$$\frac{d(F_1 \ldots F_n)}{d(p_1 \ldots p_n)},$$

by substituting $\bar{x}_i$ for p_j, by the symbol

$$\begin{vmatrix} j \\ i \end{vmatrix}$$

then (5) may be written

$$\begin{vmatrix} i \\ j \end{vmatrix}=\begin{vmatrix} j \\ i \end{vmatrix} \quad (i, j=1 \ldots n) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

The equations (5) or (6) are the conditions we seek; for they simply express that, when (4) are solved for $p_1 \ldots p_n$, the values of $p_1 \ldots p_n$ satisfy the equations (2) which are both necessary and sufficient to ensure the existence of an integral (1).

3. The equations (5), (6) may be put in another form. The transformation is easily effected by the use of the identity

$$\frac{d(F_1 \ldots F_r)}{d(\eta_1 \ldots \eta_r)}\cdot\frac{d(F_1 \ldots F_n)}{d(\xi_1 \ldots \xi_n)}=\sum_{j=1}^{j=n}\frac{d(F_1 \ldots F_r)}{d(\eta_1 \ldots \xi_j)}\cdot\frac{d(F_1 \ldots F_j \ldots F_n)}{d(\xi_1 \ldots \eta_r \ldots \xi_n)} \ldots(7),$$

where the j^{th} term on the right is formed from the term on the left by transposing η_r, ξ_j: and r is not greater than n. To establish (7) we notice that

$$\frac{dF}{d\eta}\cdot\frac{d(F_1 \dots F_n)}{d(\xi_1 \dots \xi_n)} - \sum_{j=1}^{j=n}\frac{dF}{d\xi_j}\cdot\frac{d(F_1 \dots F_j \dots F_n)}{d(\xi_1 \dots \eta \dots \xi_n)}$$
$$= \frac{dF}{d\eta}\cdot\frac{d(F_1 \dots F_n)}{d(\xi_1 \dots \xi_n)} + \sum_{j=1}^{j=n}(-)^j\frac{dF}{d\xi_j}\cdot\frac{d(F_1 \dots F_n)}{d(\eta . \xi_1 \dots \xi_{j-1}\,\xi_{j+1} \dots \xi_n)}$$
$$= \frac{d(F, F_1 \dots F_n)}{d(\eta, \xi_1 \dots \xi_n)},$$

and this vanishes when F is any one of the quantities $F_1 \dots F_n$. Now, bringing the right-hand member of (7) to the left side, and partially expanding the first factor of each term, we may write it

$$\frac{d(F_1 \dots F_{r-1})}{d(\eta_1 \dots \eta_{r-1})}\cdot\left\{\frac{dF_r}{d\eta_r}\cdot\frac{d(F_1 \dots F_n)}{d(\xi_1 \dots \xi_n)} - \sum_{j=1}^{j=n}\frac{dF_r}{d\xi_j}\cdot\frac{d(F_1 \dots F_j \dots F_n)}{d(\xi_1 \dots \eta_r \dots \xi_n)}\right\}$$
$$-\frac{d(F_1 \dots F_{r-2}, F_r)}{d(\eta_1 \dots \eta_{r-2}, \eta_{r-1})}\cdot\left\{\frac{dF_{r-1}}{d\eta_r}\cdot\frac{d(F_1 \dots F_n)}{d(\xi_1 \dots \xi_n)} - \sum_{j=1}^{j=n}\frac{dF_{r-1}}{d\xi_j}\cdot\frac{d(F_1 \dots F_j \dots F_n)}{d(\xi_1 \dots \eta_r \dots \xi_n)}\right\}$$
$$+\ \&c. \qquad = 0 \ldots\ldots (8).$$

But the second factor of each line has just been proved to vanish provided $F_1 \dots F_r$ are any r of $F_1 \dots F_n$; this being implied in the condition that r is not greater than n, the truth of (8) or (7) is manifest.

If ξ_j should be identical with any of the $r-1$ quantities $\eta_1 \dots \eta_{r-1}$, the Jacobian

$$\frac{d(F_1 \dots F_{r-1}, F_r)}{d(\eta_1 \dots \eta_{r-1}, \xi_j)}$$

vanishes identically; and the corresponding term in (7) disappears. Hence the summation in (7) need only be extended to those quantities $\xi_1 \dots \xi_n$, which do not appear amongst the $r-1$ quantities $\eta_1 \dots \eta_{r-1}$.

4. Multiply (6) by $\dfrac{d(F_k, F_l)}{d(p_i, p_j)}$. The result may be written

$$\frac{d(F_k, F_l)}{d(p_i, p_j)}\begin{vmatrix} i \\ j \end{vmatrix} + \frac{d(F_k, F_l)}{d(p_j, p_i)}\begin{vmatrix} j \\ i \end{vmatrix} = 0.$$

Sum the results obtained by giving i, j all values from 1 to n inclusive. It is clear that the sums derived from the two terms will be identical; so that we may confine our attention to one of the terms—say the first. The summation with respect to i gives

$$\sum_{i=1}^{i=n}\frac{d(F_k . F_l)}{d(p_i . p_j)}\cdot\begin{vmatrix} i \\ j \end{vmatrix} = -\sum_{i=1}^{i=n}\frac{d(F_k . F_l)}{d(p_j . p_i)}\cdot\frac{d(F_1 \dots F_i \dots F_n)}{d(p_1 \dots \bar{x}_j \dots p_n)}$$
$$= -\frac{d(F_k, F_l)}{d(p_j, \bar{x}_j)}\cdot\frac{d(F_1 \dots F_n)}{d(p_1 \dots p_n)} \quad \text{by (7).}$$

Then, summing with respect to j,

$$\frac{d(F_1 \dots F_n)}{d(p_1 \dots p_n)} \cdot \sum_{j=1}^{j=n} \frac{d(F_k, F_l)}{d(\bar{x}_j, p_j)}.$$

Now, if all the equations (6) are satisfied, this sum must vanish; but the first factor cannot vanish, so that

$$\sum_{j=1}^{j=n} \cdot \frac{d(F_k, F_l)}{d(\bar{x}_j, p_j)} = 0 \; (k, l = 1 \;..\; n) \quad \dots\dots\dots\dots\dots\dots \quad (9).$$

These equations will be indicated as usual by the notation

$$[F_k, F_l] = 0 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots (10).$$

5. It has been proved that the system (9) is a consequence of (5). To prove the complete equivalence of these systems, it is still necessary to show that (5) is a consequence of (9). For this purpose, multiply (9) by the Jacobian formed from

$$\frac{d(F_1 \dots F_n)}{d(p_1 \dots p_n)},$$

by omitting F_k, F_l $(k < l)$ and two p's—say p_κ, p_λ. Give the product a positive or negative sign according as $k+l$ is odd or even. Then sum the results for $k, l = 1 \dots n$, κ, λ being the same throughout. Call the result $\Sigma_{\kappa\lambda}$. Consider the sum derived from a single term of (9), say

$$\frac{d(F_k \,.\, F_l)}{d(\bar{x}_j, p_j)}.$$

This is

$$\frac{d(F_1, F_2)}{d(\bar{x}_j, p_j)} \cdot \frac{d(F_3 \dots F_n)}{d(p_1 \dots p_n)} - \frac{d(F_1, F_3)}{d(\bar{x}_j, p_j)} \cdot \frac{d(F_2 \,.\, F_4 \dots F_n)}{d(p_1 \dots p_n)} + \&c.$$

$$= \frac{d(F_1, F_2, F_3 \dots F_n)}{d(\bar{x}_j, p_j, p_1 \dots p_n)},$$

where p_κ, p_λ are supposed to be omitted in the range $p_1 \dots p_n$. Now this determinant vanishes identically if two constituents of the denominator are the same; that is, unless j is either κ or λ. Hence

$$\Sigma_{\kappa\lambda} = \frac{d(F_1 \,.\, F_2 \dots F_n)}{d(\bar{x}_\kappa, p_\kappa, p_1 \dots p_n)} + \frac{d(F_1 \dots F_n)}{d(\bar{x}_\lambda, p_\lambda, p_1 \dots p_n)}$$

$$= \frac{d(F_1 \dots F_n)}{d(\bar{x}_\kappa, p_\kappa, p_1 \dots p_n)} - \frac{d(F_1 \dots F_n)}{d(p_\lambda, \bar{x}_\lambda, p_1 \dots p_n)}.$$

Now in each determinant place the second constituent of the denominator (p_κ or x_λ) under F_κ and the first under F_λ: which changes will clearly produce the same change of sign in each determinant. Thus

$$\pm \Sigma_{\kappa\lambda} = \frac{d(F_1 \dots F_\lambda \dots F_n)}{d(p_1 \dots \bar{x}_\kappa \dots p_n)} - \frac{d(F_1 \dots F_\kappa \dots F_n)}{d(p_1 \dots \bar{x}_\lambda \dots p_n)}.$$

But if (9) be true, $\Sigma_{\kappa\lambda}$ vanishes for all values of κ, λ from 1 to n inclusive, so that this last result is simply (5). Hence (9), (5)—and therefore (2)—are completely equivalent.

6. The application of these results to the integration of a system of equations of the first order is easy. It may be possible from a given system to eliminate $p_1 \dots p_n$ so as to get one or more relations between $z, x_1 \dots x_n$. These cannot be regarded as integrals; and we shall suppose the system freed from such irrelevant members by suitable transformations. The reduced system consists of not more than n equations—say the number is r. These r equations may be put in the place of $F_1 \dots F_r$ in (4), and we have to find $F_{r+1} \dots F_n$ so as to satisfy (5) or (9). This, in general, implies the solution of linear partial differential equations, the theory of which is well known. But in some cases, as Prof. Nanson has observed,* it is possible to find some of the unknown functions F without integration. We have in fact, using the system (9), certain relations

$$[F_i F_j] = 0 \quad (i, j = 1 \dots r)$$

between known functions. Now these relations may be satisfied identically, or in virtue of the given system; or they may be new equations which must be satisfied if there be any integral to the system. If in this way, or otherwise, we determine $F_{r+1} \dots F_n$, such that (10) are all satisfied, the values of $p_1 \dots p_n$ derived from the complete system will render the equation

$$dz - p_1 dx_1 - \dots\dots - p_n dx_n = 0 \quad \dots\dots\dots\dots\dots\dots \quad (3)$$

integrable, and the integral will be a solution of the given system. If we can find all the solutions of (10), then we can find all the solutions of the given system; and there is no solution to the system when there is no solution of (10).

The number of arbitrary constants in the solution is generally $n-r+1$. For each of the $n-r$ functions $F_{r+1} \dots F_n$, being determined from its differential coefficients, involves one constant, and there is a constant of integration from (3). If, however, any of $F_{r+1} \dots F_n$ are determined without integration, as above, the number of arbitrary constants is reduced.

These results agree substantially with those given by Prof. Nanson in his paper cited above, though the form, and the manner of obtaining them, are somewhat different.

* On the Theory of the Solution of a System of Simultaneous Non-linear Partial Differential Equations of the First Order. Proc. R. S., Vol. xxiv. p. 337. [This theory does not appear to be essentially different from that previously given by Bour. See Mansion, "Théorie des équations aux dérivées partielles du premier ordre," §§ 21, 22.]

Equations of the Second Order.

7. In discussing equations of the second order, we shall start from the results obtained in the former part of this paper. It has been proved, it will be remembered, that the existence of an integral

$$\phi(z, x_1 \dots x_n) = 0 \quad \dots\dots\dots\dots (1)$$

requires that the first differential coefficients of z should satisfy n equations

$$F_j(p_1 \dots p_n, z, x_1 \dots x_n) = a_j \quad \dots\dots\dots\dots (4)$$

such that

$$\frac{d(F_1 \dots F_i \dots F_n)}{d(p_1 \dots \bar{x}_j \dots p_n)} = \frac{d(F_1 \dots F_j \dots F_n)}{d(p_1 \dots \bar{x}_i \dots p_n)} \quad \dots\dots\dots\dots (5).$$

Differentiating (4) with respect to x_i, and writing s_{ij} for $\dfrac{d^2 z}{dx_i\, dx_j}$, we get

$$\frac{dF_j}{dp_1} s_{1i} + \dots + \frac{dF_j}{dp_n} s_{ni} + \frac{dF_j}{dz} p_i + \frac{dF_j}{dx_i} = 0.$$

From the n equations formed by putting $j = 1, 2, \dots n$ in turn, eliminate every s but s_{ij}. The result is

$$\frac{d(F_1 \dots F_n)}{d(p_1 \dots p_n)} s_{ij} = -\frac{d(F_1 \dots F_j \dots F_n)}{d(p_1 \dots x_i \dots p_n)} - p_i \frac{d(F_1 \dots F_j \dots F_n)}{d(p_1 \dots z \dots p_n)}.$$

If we put

$$\left.\begin{aligned} X_{ij} &= \frac{d(F_1 \dots F_j \dots F_n)}{d(p_1 \dots x_i \dots p_n)} : \frac{d(F_1 \dots F_n)}{d(p_1 \dots p_n)} \\ Z_j &= \frac{d(F_1 \dots F_j \dots F_n)}{d(p_1 \dots z \dots p_n)} : \frac{d(F_1 \dots F_n)}{d(p_1 \dots p_n)} \end{aligned}\right\} \quad \dots\dots\dots\dots (11),$$

we may write this result

$$s_{ij} = -X_{ij} - p_i Z_j \quad \dots\dots\dots\dots (12).$$

In the same notation (5) may be written

$$X_{ji} + p_j Z_i = X_{ij} + p_i Z_j \quad \dots\dots\dots\dots (13),$$

so that $s_{ji} = s_{ij}$, as it should.

8. The equations

$$\frac{d(F, F_1 \dots F_n)}{d(x_i, p_1 \dots p_n)} = 0, \quad \frac{d(F, F_1 \dots F_n)}{d(z, p_1 \dots p_n)} = 0,$$

are identically satisfied when F is any one of the quantities $F_1 \dots F_n$. Expanding them, and substituting from (11), we get them in the form

$$\Delta_i F \equiv \frac{dF}{dx_i} - X_{i1} \frac{dF}{dp_1} - \dots - X_{in} \frac{dF}{dp_n} = 0,$$

$$\Delta F \equiv \frac{dF}{dz} - Z_1 \frac{dF}{dp_1} - \dots - Z_n \frac{dF}{dp_n} = 0.$$

Giving i all values from 1 to n, we have a system of $n+1$ independent equations, having $2n+1$ independent variables, and n common solutions, $F_1 \dots F_n$. This implies the conditions

$$\left.\begin{aligned} \Delta_j X_{ik} &= \Delta_i X_{jk} \\ \Delta\, X_{ik} &= \Delta_i Z_k \end{aligned}\right\} \quad \dots\dots\dots\dots\dots\dots\dots\dots (14)$$

for all values of i, j, k from 1 to n. Also, these conditions being satisfied, the n solutions are the integrals of the integrable system,

$$dp_i + X_{1i}\,dx_1 + \dots + X_{ni}\,dx_n + Z_i\,dz = 0 \quad (i = 1 \dots n) \quad \dots\dots (15).$$

If in this equation we write $p_1 dx_1 + \dots + p_n dx_n$ for dz [which is permissible, because we have assumed (5) or (13) to hold good], and reduce by means of (12), the system (15) takes the form

$$dp_i - s_{1i}\,dx_1 - \dots - s_{ni}\,dx_n = 0 \quad \dots\dots\dots\dots\dots\dots (16).$$

9. We may also express (14) in terms of s_{ij}, &c. Observe that, in virtue of the definition of Δ_i, Δ, taking account of (12), we have

$$\Delta_i + p_i\Delta = \frac{d}{dx_i} + p_i\frac{d}{dz} + s_{i1}\frac{d}{dp_1} + \dots + s_{in}\frac{d}{dp_n};$$

and, whatever u may be,

$$\Delta_i\, p_j u = p_j\,\Delta_i u - X_{ij}u,$$
$$\Delta\, p_j u = p_j\,\Delta\, u - Z_j\, u.$$

With the help of these relations it is easy to verify that

$$(\Delta_i + p_i\Delta)(X_{jk} + p_j Z_k) = (\Delta_j + p_j\Delta)(X_{ik} + p_i Z_k),$$

that is,

$$\left(\frac{d}{dx_i} + p_i\frac{d}{dz} + s_{i1}\frac{d}{dp_1} + \dots + s_{in}\frac{d}{dp_n}\right) s_{jk}$$
$$= \left(\frac{d}{dx_j} + p_j\frac{d}{dz} + s_{j1}\frac{d}{dp_1} + \dots + s_{jn}\frac{d}{dp_n}\right) s_{ik}.$$

This may be written $$\frac{ds_{jk}}{d\bar{x}_i} = \frac{ds_{ik}}{d\bar{x}_j} \quad \dots\dots\dots\dots\dots\dots\dots\dots (17),$$

if we take these differential coefficients to imply differentiation with respect to x, not only as it occurs explicitly in s, but also as involved in z and the first differential coefficients of z.

10. It has been shown that (17) are necessary consequences of the existence of an integral $\phi\,(z, x_1 \dots x_r) = 0$.

We shall now investigate how far these conditions are sufficient to ensure the existence of an integral and to enable us to find it. The first step in the discussion will be to show that, when (17) are satisfied, the equations $$dp_i - s_{1i}\,dx_1 - \dots - s_{ni}\,dx_n = 0 \quad (i = 1 \dots n) \quad \dots\dots\dots (18)$$

are an integrable system. But it is obvious that in the form (18) this cannot be true, save under certain conditions; for the coefficients of the equations involve z, while the differential dz does not occur anywhere. That (18) in their present form should be an integrable system, it is necessary that each s should be free from z. But the results of Arts. 7—9 suggest the procedure to be adopted when the s's do involve z.

11. Assume $$s_{ij} = -X_{ij} - p_i Z_j = -X_{ji} - p_j Z_i = s_{ji} \quad \text{..............(19)},$$

and write $$\Delta_i = \frac{d}{dx_i} - X_{i1}\frac{d}{dp_1} - \ldots - X_{in}\frac{d}{dp_n},$$
$$\Delta = \frac{d}{dz} - Z_1\frac{d}{dp_1} - \ldots - Z_n\frac{d}{dp_n};$$

so that (17) may be written

$$(\Delta_i + p_i\Delta)(X_{jk} + p_j Z_k) = (\Delta_j + p_j\Delta)(X_{ik} + p_i Z_k) \quad \text{........(20)}.$$

Here X_{ij}, Z_i, &c. are only defined by (19). If we determine Z, then every X is given, and *vice versâ*. We adopt the former course, and suppose that $Z_1 \ldots Z_n$ are determined by the equations

$$\frac{dZ_i}{dx_j} + \frac{ds_{ij}}{dz} - Z_1\frac{ds_{ij}}{dp_1} - \ldots - Z_n\frac{ds_{ij}}{dp_n} - Z_i Z_j = 0 \quad \text{...........(21)},$$

or, as it may be written,

$$(\Delta_j + p_j\Delta) Z_i = \Delta(X_{ji} + p_j Z_i) + Z_i Z_j \quad \text{...............(22)}.$$

Of these equations we shall have to say a few words hereafter (Art. 15). For the present we suppose that $Z_1 \ldots Z_n$ are determined by their help; (21) involving, besides $Z_1 \ldots Z_n$, only quantities given.

Now (22) gives on reduction

$$\Delta_j Z_i = \Delta X_{ji} \quad \text{...................................(23)};$$

and in virtue of this (20) reduces to

$$\Delta_i X_{jk} = \Delta_j X_{ik} \quad \text{.......................................(24)},$$

as may easily be seen by expanding it so that the Δ's act only on the X's and Z's; and then reducing it by the help of (23) and (19).

The relations (23), (24) show that, whatever u may be,

$$\left.\begin{aligned} \Delta_i\Delta_j u &= \Delta_j\Delta_i u \\ \Delta_i\Delta\, u &= \Delta\,\Delta_i u \end{aligned}\right\} \quad \text{.................................(25)}.$$

For, taking the first of these, consider the coefficients of $\frac{du}{dp_k}$. They are $-\Delta_i X_{jk}$ and $-\Delta_j X_{ik}$ respectively, and by (24) are equal. Again, the coefficient of $\frac{d^2u}{dp_k\,dp_l}$ on each side is $X_{ik}X_{jl} + X_{jk}X_{il}$. Hence the two sides are equal term to term. Similarly for the second of (25).

12. Form the $n+1$ equations,

$$\Delta F = 0, \quad \Delta_i F = 0 \quad (i = 1 \dots n) \quad \dots\dots\dots\dots\dots (26),$$

having $2n+1$ independent variables. Since (25) are true, these equations have the maximum number (n) of common solutions. If we write for Δ, Δ_i their forms, given at the beginning of Art. 11, it is clear that the n solutions of (26) are to be found by integrating the system of ordinary differential equations,

$$dp_i + X_{1i}\, dx_1 + \dots + X_{ni}\, dx_n + Z_i\, dz = 0 \quad \dots\dots\dots\dots\dots (27).$$

These n equations form an integrable system. We do not say that any equation of the system is an exact differential, or is even integrable; but that from the system we can form, by suitable combinations, n independent exact differential equations. For, if not, (26) would have less than n common solutions; that is, (25) would not be true.

This system (27) it is which must take the place of (18); and it will be noticed that the difficulty which was found in employing (18) is now entirely removed.

13. Let the n common solutions of (26), or the n integrals of (27), be indicated by

$$F_i(p_1 \dots p_n,\ z,\ x_1 \dots x_n) = a_i \quad (i = 1 \dots n) \quad \dots\dots\dots (28).$$

The general solution of (26) is

$$\phi\,(F_1, \dots F_n) = 0,$$

where ϕ is arbitrary. If from this by differentiation we eliminate ϕ, we reproduce the equations (26); and comparison of coefficients gives

at once
$$\left.\begin{aligned} X_{ji} &= \frac{d\,(F_1 \dots F_i \dots F_n)}{d\,(p_1 \dots x_j \dots p_n)} : \frac{d\,(F_1 \dots F_n)}{d\,(p_1 \dots p_n)} \\ Z_i &= \frac{d\,(F_1 \dots F_i \dots F_n)}{d\,(p_1 \dots z \dots p_n)} : \frac{d\,(F_1 \dots F_n)}{d\,(p_1 \dots p_n)} \end{aligned}\right\} \quad \dots\dots\dots\dots (29).$$

Hence (19) becomes

$$\frac{d\,(F_1 \dots F_j \dots F_n)}{d\,(p_1 \dots x_i \dots p_n)} + p_i \frac{d\,(F_1 \dots F_j \dots F_n)}{d\,(p_1 \dots z \dots p_n)}$$
$$= \frac{d\,(F_1 \dots F_i \dots F_n)}{d\,(p_1 \dots x_j \dots p_n)} + p_j \frac{d\,(F_1 \dots F_i \dots F_n)}{d\,(p_1 \dots z \dots p_n)}.$$

But these are precisely the conditions which are necessary and sufficient to ensure that the values of $p_1 \dots p_n$ derived from (28) are such as to render
$$dz - p_1 dx_1 - \dots - p_n dx_n = 0$$
integrable.

14. The results obtained may be summed up thus:—Let $s_{11} \dots s_{ij} \dots s_{nn}$ be given as functions of $x_1 \dots x_n$, z, $p_1 \dots p_n$, such that

$$\frac{ds_{jk}}{d\bar{x}_i} = \frac{ds_{ik}}{d\bar{x}_j} \quad \dots\dots\dots\dots (17).$$

Determine $Z_1 \dots Z_n$ from the n^2 equations

$$\frac{dZ_i}{d\bar{x}_j} + \frac{ds_{ij}}{dz} - Z_1 \frac{ds_{ij}}{dp_1} - \dots - Z_n \frac{ds_{ij}}{dp_n} - Z_i Z_j = 0 \quad \dots\dots\dots\dots (21),$$

and X_{ji} from the equations

$$X_{ji} = p_j Z_i - s_{ji}.$$

Then the system

$$dp_i + X_{1i} dx_1 + \dots + X_{ni} dx_n + Z_i dz = 0 \quad \dots\dots\dots\dots (27)$$

is integrable; and the values of $p_1 \dots p_n$ determined from the n integrals of this system are such as to make

$$dz - p_1 dx_1 - \dots - p_n dx_n = 0 \quad \dots\dots\dots\dots (3)$$

integrable. Any integral of this equation is a solution of the given system.

15. We revert for a moment to the consideration of the system (21) by means of which $Z_1 \dots Z_n$ are determined. This system, consisting of n^2 equations, involves n dependent variables, and $2n+1$ independent variables. Now it may very well be that the system does not always admit of being satisfied by any values of $Z_1 \dots Z_n$; in other words, these equations may not be consistent unless certain conditions are fulfilled which are not implied in (17). It will then be well to point out that such conditions, should they exist, are necessary conditions for the existence of a solution of the proposed system of equations, and have not been introduced by any arbitrary procedure peculiar to the method of this paper. The results of Art. 13 show that the X, Z of Arts. 11—14 are actually identical with the X, Z of Arts. 7—9; and, therefore, that the Δ, Δ_i of these Articles are also the same. Now the equation (21) written in the form

$$(\Delta_j + p_j \Delta) Z_i = \Delta (X_{ji} + p_j Z_i) + Z_i Z_j \quad \dots\dots\dots\dots (22),$$

or

$$\Delta_j Z + p_j \Delta Z_i = \Delta X_{ji} + p_j \Delta Z_i,$$

is simply one of the equations (14). But (14) have been proved to be a consequence of the existence of an integral. Hence the equations (21) must be satisfied if there be any integral; and if there be no solutions of (21), neither can there any integral corresponding to the given values of $s_{11} \dots s_{ij} \dots s_{nn}$.

It may be desirable to show how to obtain an equation for each of the quantities $Z_1, \dots Z_n$ freed from all the others. This may be done by writing (21) in the form

$$\frac{dZ_i}{d\bar{x}_j}+\frac{ds_{ij}}{dz}-Z_i\frac{ds_{ij}}{dp_i}-Z_1\frac{ds_{ij}}{dp_1}-\ldots-Z_j\left(\frac{ds_{ij}}{dp_j}+Z_i\right)-\ldots-Z_n\frac{ds_{ij}}{dp_n}=0.$$

In this give j all values from 1 to n inclusive; and from the n equations thus formed eliminate every Z but Z_i. The result will be a partial differential equation of the first order for Z_i which is linear in the differential coefficients of Z_i.

It will be observed that in (21) all the terms save the first are symmetrical with respect to i, j. Hence we get

$$\frac{dZ_i}{d\bar{x}_j}=\frac{dZ_j}{d\bar{x}_i},$$

equations which are equivalent to

$$Z_i=\frac{dZ}{d\bar{x}_i}.$$

Hence it appears that the problem of finding $Z_1, \ldots Z_n$ is reducible to the problem of finding one quantity only. In fact, I have not found any advantage resulting from this remark; but it may possibly be of value in considering the theory of equations of higher orders.

16. In the preceding paragraphs it has been supposed that $s_{11} \ldots s_{ij} \ldots s_{nn}$ have been given in terms of $p_1 \ldots p_n, z, x_1 \ldots x_n$. Let us now take the more general case in which these functions are given by $\frac{n(n+1)}{2}$ equations of the form

$$F_{ij}(s_{11}\ldots s_{ij}\ldots s_{nn},\ p_1\ldots p_n,\ z,\ x_1\ldots x_n)=b_{ij} \quad\ldots\ldots\ldots\ldots(30).$$

It is required to find the relations between F_{ij} which are equivalent to the conditions (17). This necessitates a transformation exactly like that of Art. 2, and the result is

$$\frac{d(F_{11}\ldots F_{jk}\ldots F_{nn})}{d(s_{11}\ldots \bar{x}_i\ldots s_{nn})}=\frac{d(F_{11}\ldots F_{ik}\ldots F_{nn})}{d(s_{11}\ldots \bar{x}_j\ldots s_{nn})}\quad\ldots\ldots\ldots\ldots(31),$$

the differentiations with respect to x_i, x_j being made on the supposition that $z, p_1 \ldots p_n$ are functions of x_i, x_j.

These equations may be indicated by an abbreviation similar to that used in (6); viz.

$$\left|\begin{matrix} jk \\ i \end{matrix}\right| = \left|\begin{matrix} ik \\ j \end{matrix}\right| \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(32),$$

where the first member, for example, indicates the result of substituting $\bar{x}_i$ for s_{jk} in the Jacobian

$$\frac{d(F_{11}\ldots F_{jk}\ldots F_{nn})}{d(s_{11}\ldots s_{jk}\ldots s_{nn})}.$$

17. These results may be applied to the solution of a system of equations of the second order in a manner similar to that employed in Art. 6. The system may be such that it is possible to form from it equations between $z, x_1 \dots x_n$; but as these cannot be regarded as integrals, they will be neglected, and we shall suppose the system freed from them. Again, it may be possible to get relations between $p_1 \dots p_n, z, x_1 \dots x_n$; or such equations may explicitly occur as members of the system. The treatment of such cases will be noted in the next Article; and for the present we assume that the system comprises r $\left(\text{not more than } \frac{n(n+1)}{2}\right)$ equations from which the differential coefficients of the second order cannot be eliminated.

We have then to find $\frac{n(n+1)}{2}-r$ functions F, either by artifices like those mentioned in Art. 6, or by actual integration of (31) in some one of its forms. If this can be accomplished, nothing remains but to solve for $s_{11} \dots s_{ij} \dots s_{nn}$ from the completed system, and proceed as in Art. 14 to the final integral.

18. If any equations of the first order be included in the system; the procedure is not essentially changed. Each such equation should be differentiated with respect to $x_1, \dots x_n$ in turn; thus furnishing n equations of the second order. Of course some or all of these may be included in the given equations of the second order. The system thus formed is to be treated as in Art. 17, but the arbitrary constants of the first integrals must be determined so as to agree with the given equations of the first order.

19. As to the number of arbitrary constants. Observe that $F_{11} \dots F_{nn}$ each involve one arbitrary constant (b) because each is determined from its differential coefficients. Now suppose there are r_1 equations of the first order, and r_2 of the second order (taking account of those derived from the r_1 equations of the first order). We shall then have

$$\frac{n(n+1)}{2}-r_2$$

constants from the system (30). We shall have

$$n-r_1$$

constants, of integration, from (27); and one from the integration of (3). Thus altogether there will be

$$\frac{n(n+1)}{2}+n+1-r_2-r_1$$

arbitrary constants in the solution.

I have used the phrase "arbitrary constants" in the above passage and elsewhere; but without qualification this would be wrong. The quantities indicated are not constants at all, though they include constants as a particular case. They really are functions of the variables defined by certain differential equations, which are well known when the equations in whose solution they occur are of the first order; but have not been determined, so far as I know, for equations of higher orders save in special cases.

Equations of the Third and Higher Orders.

20. It would be possible, and perhaps desirable, to discuss the equation of the third order in a manner similar to that in which equations of the second order were dealt with. But I purpose changing the method so as to make it applicable at once to equations of higher orders, and the remarks will be brief.

Evidently the equation

$$\frac{d^4 z}{dx_i\,dx_j\,dx_k\,dx_l} = \frac{d^4 z}{dx_j\,dx_i\,dx_k\,dx_l} \quad \text{.................}(33)$$

is identically true. Let us see what it becomes in terms of the differential coefficients of the third order, supposed to be given in terms of s, p, z, x. Write these differential coefficients a_{ijk}, &c., the subscripts indicating the independent variables which are concerned. Then (33) may be written

$$\frac{da_{jkl}}{dx_i} = \frac{da_{ikl}}{dx_j} \quad \text{...............................}(34),$$

where $x, p_1 \ldots p_n, s_{11} \ldots s_{nn}$ are all regarded as functions of x_i, x_j. If a should be given by $\dfrac{n\,(n+1)\,(n+2)}{1\,.\,2\,.\,3}$ equations of the form

$$F_{ijk}\,(\ldots\, a \ldots s \ldots p \ldots z \ldots x \ldots) = c_{ijk} \quad \text{..............}(35)\ ;$$

then F_{ijk} must satisfy the relations

$$\begin{vmatrix} jkl \\ i \end{vmatrix} = \begin{vmatrix} ikl \\ j \end{vmatrix} \quad \text{.............................}(36),$$

where the first member indicates the result of substituting x_i for a_{jkl} in

$$\frac{d\,(F_{111} \ldots F_{jkl} \ldots F_{nnn})}{d\,(a_{111} \ldots a_{jkl} \ldots a_{nnn})},$$

and, as before, every s, p, z is regarded as a function of x_i.

21. Suppose that we know $F_{111} \ldots F_{nnn}$. They may be given; or some of them may be given, and the rest discovered by the integration of some system equivalent to (36). Without distinguishing between these cases, suppose F known, so that we can express a_{jkl}, &c. in terms of s, p, z, x. Form the equations

$$ds_{ij} - a_{ij1}\,dx_1 - \ldots - a_{ijn}\,dx_n = 0 \quad \text{................}(37).$$

This system must be equivalent to an integrable system if there be any solution consistent with the values of F_{ijk}. To find this equivalent system, write

$$-a_{ijk} = X_{ijk}+p_i Z_{jk}+p_j Z_{ki}+p_k Z_{ij}+s_{ij} P_k+\dots \quad \text{.........(38)},$$

leaving X, Z, P indeterminate. Substitute these values in (37), and reduce by means of

$$p_1 dx_1+\dots+p_n dx_n = dz,$$
$$s_{1i} dx_1+\dots+s_{ni} dx_n = dp_i,$$
$$\text{\&c.},$$

and we obtain a system of equations of the form

$$ds_{ij}+\dots+X_{ijk}\,dx_k+\dots+Z_{ij}\,dz+P_i dp_j+P_j\,dp_i = 0 \quad \text{......(39)}.$$

Let us now write down the conditions that this should be an integrable system, and regard the equations thus obtained, combined with (38) and with

$$a_{ijk} = a_{jik} = \text{\&c.} \quad \text{..............................(39)},$$

as equations for the determination of X, Z, P. If values can be obtained, then we can find a system of first integrals; and these in virtue of (39) are sufficient to determine s_{ij} so as to yield a system of second integrals. But the determination of this second system requires a transformation analogous to that of the present article.

If, on the other hand, the equations by which X, Z, P are to be found should be inconsistent, we may be sure that there is no integral of the given equations; because all the equations for X, Z, P are necessary consequences of the existence of an integral

$$\phi\,(z,\, x_1 \dots x_n).$$

22. As was indicated above, these remarks are applicable at once to equations of the m^{th} order.

It may be permissible here to notice that, in the solution of a system of equations of the m^{th} order, the number of arbitrary constants is

$$\frac{\lfloor m+n-1}{\lfloor m\,\lfloor n-1} + \frac{\lfloor m+n-2}{\lfloor m-1\,\lfloor n-1} + \text{\&c.} + \frac{(n+1)\,n}{1\,.\,2} + n + 1 - \sum_{i=1}^{i=m} r_i,$$

where r_i is the number of independent equations of the i^{th} order, counting not only those actually given, but also such as are derived by differentiation from given equations of lower orders.