THE EQUIVALENT MASS OF A SPRING VIBRATING LONGITUDINALLY.

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§ 1. When a mass M is oscillating under gravity at the end of a spiral spring, it is usual to make allowance for the mass m of the spring itself by adding a quantity $\frac{1}{3}m$ to M and treating the spring as if it were light. This result is correct only if m is small compared to M; and in this case it is possible to give an elementary solution by supposing the displacement of any point on the spring to be proportional to its distance from the fixed end. In ordinary laboratory practice M is comparable with m in magnitude, and the above approximation no longer holds; for small values of M it is found that $\frac{2}{5}m$ is a better value for the equivalent mass of the spring than $\frac{1}{3}m$. It seems worth while to determine how this quantity changes as M changes. Rayleigh, in his book on Sound, vol. i, §§ 155–6, works out certain results in the longitudinal vibrations of bars which can be applied to this problem. The effect of a very small mass and of a very large mass added to a bar vibrating longitudinally are there shown; and from them the results $\frac{4}{\pi^2}m$ and $\frac{1}{3}m$ can be deduced for the equivalent mass of m when M is very small and very large respectively. For moderate values of M the equivalent mass varies between the extremes mentioned, and for any actual m it is important to know the effect of the mass of the spring more closely.

§ 2. We assume that the spring behaves like a uniform thin elastic cord. Let E be its elastic coefficient, ρ its line density, and l its length; M is the mass attached to the free end.

Let ξ be the distance of a particular element of the spring from the fixed end at any time; x the distance when the spring is unstretched;

ξ_0 the distance when the spring hangs in equilibrium under its own weight and that of M.

T is the tension of the spring in the neighbourhood of the element.

The element originally of length Δx is stretched to $\Delta\xi$ under tension T.

$$\therefore \frac{\Delta\xi - \Delta x}{\Delta x} = \frac{T}{E} \text{ or } \frac{d\xi}{dx} = 1 + \frac{T}{E}$$

The equation of motion of the element is

$$\rho \,.\, \Delta x \,.\, \ddot{\xi} = \rho g \,.\, \Delta x + \left(T + \frac{\partial T}{\partial x} \,.\, \Delta x\right) - T$$

$$i.e. \quad \rho\ddot{\xi} = \rho g + \frac{\partial T}{\partial x} = \rho g + E\frac{\partial^2 \xi}{\partial x^2}$$

Since ξ_0 marks an equilibrium position

$$0 = \rho g + E\frac{\partial^2 \xi_0}{\partial x^2}$$

$\therefore$ if z be the displacement of the element from its equilibrium position $z = \xi - \xi_0$, and we have

$$\rho\ddot{z} = E\frac{\partial^2 z}{\partial x^2}$$

Try a solution of the type $z = X \begin{matrix}\cos\\ \sin\end{matrix}\, pt$ where X is a function of x only.

$$\therefore \quad -\rho p^2 X = E\frac{X}{\partial x^2} \quad \therefore \; X = A \cos xp\sqrt{\frac{\rho}{E}} + B \sin xp\sqrt{\frac{\rho}{E}}$$

The condition $z=0$ when $x=0$ for all time cuts out the cosine terms in x.

If the system is started from rest (as happens in the ordinary experiment), $\dot{z}=0$ when $t=0$ for all values of x; this cuts out the cosine term in t.

We are left with

$$A \sin pt \sin px \sqrt{\frac{\rho}{E}}$$

The value of p is now got by considering the conditions at the free end.

The element Δx is stretched to length $\Delta\xi_0$ under the tension $\rho g(l-x) + Mg$.

$$\therefore \quad \frac{\Delta\xi_0 - \Delta x}{\Delta x} = \frac{\rho g(l-x) + Mg}{E}$$

$$\therefore \quad \xi_0 = x + \frac{g\rho lx}{E} - \frac{1}{2} \cdot \frac{g\rho x^2}{E} + \frac{Mgx}{E}$$

Then

$$T = E\left(\frac{\partial\xi}{\partial x} - 1\right) = E\left(\frac{\partial z}{\partial x} + \frac{\partial \xi_0}{\partial x} - 1\right)$$

$$= E\left(\frac{\partial z}{\partial x} + 1 + \frac{g\rho l}{E} - \frac{g\rho x}{E} + \frac{Mg}{E} - 1\right)$$

$\therefore$ the end value of T is

$$E\left(\frac{\partial z}{\partial x}\right)+Mg$$

The acceleration of M is

$$-Ap^2 \sin pt \,.\, \sin pl\sqrt{\frac{\rho}{E}}$$

$$\therefore \quad -MAp^2 \sin pt \,.\, \sin pl\sqrt{\frac{\rho}{E}}=Mg-T_e=-E\,.\left(\frac{\partial z}{\partial x}\right)_l$$

$$=-EAp\sqrt{\frac{\rho}{E}}\cdot \sin pt\,.\,\cos pl\sqrt{\frac{\rho}{E}}$$

$$\textit{i.e.}\quad Mp \tan pl\sqrt{\frac{\rho}{E}}=\sqrt{\rho E}$$

Let m be the whole mass of the spring, and K the force necessary to produce unit extension of it, then

$$\rho=\frac{m}{l} \text{ and } E=Kl$$

$$\therefore \quad Mp \tan pl\sqrt{\frac{m}{K}}=\sqrt{mK}$$

To deal with this equation write $p=\theta\sqrt{\frac{K}{m}}$; the equation becomes

$$\frac{M}{m}=\frac{\cot\theta}{\theta}$$

Let m' be the equivalent mass of m for the oscillation; m' is defined by

$$\text{period}=2\pi\sqrt{\frac{M+m'}{K}}=\frac{2\pi}{p}$$

$$\therefore \quad M+m'=\frac{K}{p^2}=\frac{m}{\theta^2}$$

$$\therefore \quad m'=\frac{m}{\theta^2}-M=m\left[\frac{1}{\theta^2}-\frac{\cot\theta}{\theta}\right]$$

§ 3. For particular values of M and m the equation for θ can be solved to any stated degree of accuracy and the corresponding value of m' found.

The two extreme cases may be noted:

(a) M small—

Approximately $\theta=\frac{\pi}{2}$ $\therefore$ $m'=m\,.\,\frac{4}{\pi^2}$

This is very nearly $\frac{2}{5}\,m$.

(b) M great—

$$\text{Here } \theta=0 \quad m'=m\left[\frac{1}{\theta^2}-\frac{1-\frac{\theta^2}{2}}{\theta\left(\theta-\frac{\theta^3}{6}\right)}\right]=\frac{m}{3}$$

This is the result generally used.

It is interesting to observe that the addition of a mass M to the end of the spring does not increase the equivalent mass of the apparatus by M. To show this we obtain a further approximation to the equivalent mass when $\frac{M}{m}$ is small—equal to e, say.

$$e=\frac{\cot\theta}{\theta} \text{ gives } \theta=\frac{\pi}{2}-\lambda \text{ where } \lambda \text{ is small.}$$

$$i.e.\quad e=\frac{\tan\lambda}{\frac{\pi}{2}-\lambda}=\frac{\lambda}{\frac{\pi}{2}} \text{ approx. whence } \lambda=\frac{\pi e}{2} \text{ and } \theta=\frac{\pi}{2}-\frac{\pi e}{2}$$

$$\therefore\quad m'=m\left[\frac{1}{\frac{\pi^2}{4}(1-e)^2}-\frac{\tan\frac{\pi e}{2}}{\frac{\pi}{2}}\right]=m\left[\frac{4}{\pi^2}-e\left(1-\frac{8}{\pi^2}\right)\right]$$

$$=\frac{4m}{\pi^2}-\frac{M}{5} \text{ roughly.}$$

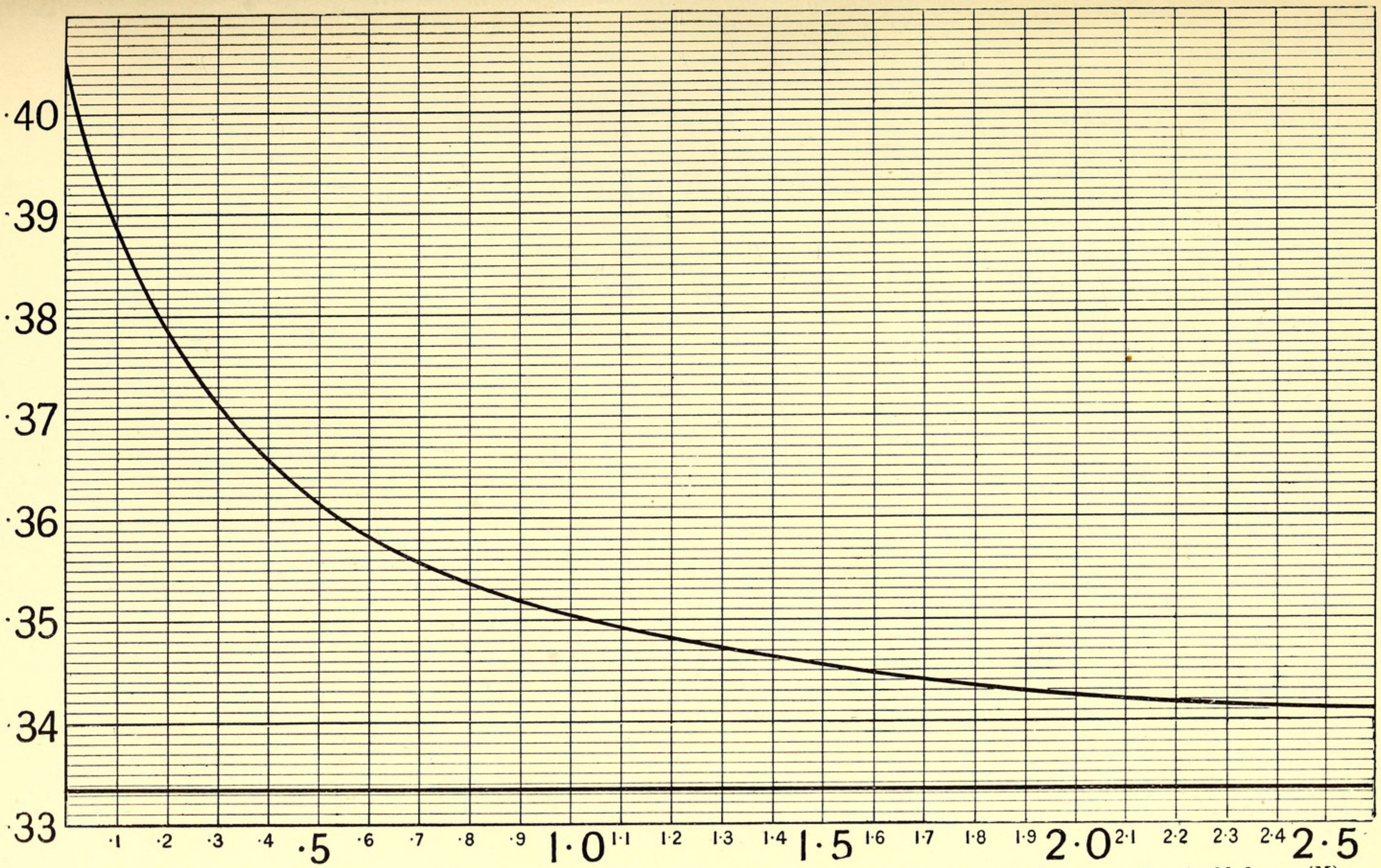
Thus for small additions the equivalent mass of the spring diminishes by about one-fifth of the mass added.

§ 4. To show how m' varies between its extreme values a graph has been drawn with $\frac{M}{m}$ as abscissa and $\frac{m'}{m}$ as ordinate; the method of construction of the graph was to tabulate values of $\frac{\cot\theta}{\theta}$ and of $\frac{1}{\theta^2}-\frac{\cot\theta}{\theta}$ for values of θ from 0° to 90° at intervals of 5°. The line showing the value $\frac{m'}{m}=\frac{1}{3}$ has also been drawn in the same figure to show the deviation of the approximate result for M large from the true result as M varies.

For reference the tabulated values are given here.

θ	$\frac{\cot\theta}{\theta}$	$\frac{1}{\theta^2}-\frac{\cot\theta}{\theta}$	θ	$\frac{\cot\theta}{\theta}$	$\frac{1}{\theta^2}-\frac{\cot\theta}{\theta}$
0°	∞	·3333	50°	·9616	·3515
5	130·9788	·3335	55	·7295	·3555
10	32·4940	·3340	60	·5516	·3604
15	14·2553	·3349	65	4111	·3659
20	7·8710	·3363	70	·2980	·3722
25	4·9149	·3375	75	·2047	·3793
30	3·308	·3396	80	·1263	·3866
35	2·3378	·3418	85	·0590	·3954
40	1·7071	·3448	90	0	·4054
45	1·2732	·3479			

§ 5. To test the theory experiments were carried out on a spiral spring of mass 164 grammes, length 38 cms., and diameter 4 cms., made of steel wire of circular section 1·35 mm. diameter. The statical value of K (force necessary to produce unit extension) was found to be 16·96 grammes'



The curve shows the variation of the equivalent mass of the spring as the added mass changes (see §4). Abscissae are ratios of added mass (M) to actual mass of spring (m). Ordinates are ratios of equivalent mass of spring (m') to actual mass of spring.

weight for a range of weights from 50 to 400 grammes. The period of oscillation of the spring with various weights attached was determined, and the equivalent mass of spring and added weight calculated from the formula

$$t = 2\pi\sqrt{\frac{M'}{K}}$$

Subtraction of the added weight from M′ now gives the equivalent mass of the spring alone.

The last column gives the equivalent mass of the spring as derived from the above theory.

TABLE OF RESULTS.

$m = 164$ grammes. $K = 16{\cdot}96 \times 979{\cdot}7$ dynes.

M.	T.	M′.	m' (obs.).	m' (theor.).
0	·3991	67·1	67·1	65·5
10	·4223	75·1	65·1	64·8
20	·4469	84·1	64·1	63·5
50	·5139	111·2	61·2	61·0
100	·6138	158·6	58·6	58·8
200·1	·7817	257·3	57·3	57·1
250·3	·8537	306·8	56·5	56·7
400·1	1·0390	454·2	54·1	55·7

What difference there is between the observed and the theoretical value may be put down to the change in the value of K for different loads; the value increases with the load, but more elaborate experiments will be necessary in order to assign definite values of K for different loads.

The experiments show that (1) the assumption of uniformity of the spring introduces no error, (2) the rough approximation usually given for the equivalent mass is inadequate, (3) no difference appears between the static and the dynamic value of K.