THE ARRANGEMENT OF SUCCESSIVE CONVERGENTS IN ORDER OF ACCURACY.

By Alexander Brown.

(From the Applied Mathematics Laboratory, South African College.)

(Read September 15, 1915.)

§ 1. One of the most important uses of simple continued fractions is for the solution of the problem *to find the fraction, whose denominator does not exceed a given integer, which shall most closely approximate to a given number commensurable or incommensurable.* A practically complete solution was provided by Lagrange in 1769 in his paper "Sur la Resolution des Equations Numériques"* and his treatment, involving the use both of principal convergents (*fractions principales*) and of intermediate convergents (*fractions secondaires*), has become the classic exposition in text-books of algebra. His results give the fraction nearest in defect, and the fraction nearest in excess, satisfying the conditions. He does not consider the question of deciding which of these two fractions is nearest in absolute value to the given number.

Chrystal, in his *Algebra*,† vol. ii, p. 424, Ex. 10, gives a rule "enabling us in most cases to save calculation in deciding between the closeness" of the fraction in defect and that in excess, but the rule is not easy to apply; while Serret, in his *Cours d'Algèbre Supérieure*,‡ p. 24, leaves the question as Lagrange dealt with it. The present paper was written to give an easy method of deciding between the two fractions, and to arrange the successive convergents in such a scheme that the nearest in absolute value satisfying the stated condition could be at once picked out. Afterwards a short note by Muir § was found *On Convergents*, in which the same method of discrimination is given without proof. This note seems to have been overlooked by later writers.

* "Mémoires de l'Académie royale des Sciences et Belles-Lettres de Berlin," t. xxiii, 1769. "Œuvres de Lagrange" (Serret), 1868, vol. ii, p. 568.

† "Algebra" (Chrystal), 1859, part ii.

‡ Sixth edition, 1910.

§ Report of the British Association for the Advancement of Science for 1876.

I give below a proof of the rule and a method of arranging the convergents in one set so as to show the nearest in defect, the nearest in excess and the nearest in absolute value, satisfying the stated condition.

§ 2. Consider the simple continued fraction

$$x = a_1 + \frac{1}{a_2} + \frac{1}{a_3} + \ldots + \frac{1}{a_s} + \ldots$$

and let the successive principal convergents be $p_1/q_1, p_2/q_2, \ldots p_s/q_s \ldots$

Arrange the sets of quantities :

$$0/1, p_1/q_1, p_3/q_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A),$$

$$1/0, p_2/q_2, p_4/q_4 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B),$$

and between any successive two p_l/q_l and p_{l+2}/q_{l+2} interpolate the set

$$\frac{p_l + p_{l+1}}{q_l + q_{l+1}}, \frac{p_l + 2p_{l+1}}{q_l + 2q_{l+1}} \ldots \frac{p_l + (a_{l+1} - 1)\, p_{l+1}}{q_l + (a_{l+1} - 1)\, q_{l+1}}$$

(intermediate convergents).

When the sets are completed they give fractions increasing in complexity and respectively increasing and decreasing in magnitude, which continually approach x. These sets provide solutions to the problem of finding the fraction of given complexity (*i. e.* with denominator less than a given number) nearest in defect or nearest in excess to x. A doubt is left as to which of the two is nearest in absolute value.

Of the two fractions here mentioned it is known that one (at least) must be a principal convergent.* The final problem is thus restricted to a comparison between a principal convergent of one set and an intermediate convergent of the other set.

For precision let p_l/q_l and p_{l+2}/q_{l+2} be two successive principal *odd* convergents ; p_{l+1}/q_{l+1} is a principal even convergent, and the question is whether p_{l+1}/q_{l+1} or $(p_l + rp_{l+1})/(q_l + rq_{l+1})$ is nearer x, where r can have any integral value from 1 to $a_{l+2} - 1$.

Let x_{l+1} be defined by

$$x = a_1 + \frac{1}{a_2} + \ldots + \frac{1}{a_l} + \frac{1}{x_{l+1}}$$

so that
$$x_{l+2} = a_{l+2} + \frac{1}{a_{l+3}} + \ldots$$

Then
$$x = (p_l + x_{l+2} \cdot p_{l+1})/(q_l + x_{l+2} \cdot q_{l+1})$$

Also, since l is supposed odd,

$$p_{l+1}/q_{l+1} > x > (p_l + rp_{l+1})/(q_l + rq_{l+1})$$

The differences D_1 and D_2 of these fractions from x are

$$\begin{aligned} D_1 &= x - (p_l + rp_{l+1})/(q_l + rq_{l+1}) \\ &= \frac{p_l + x_{l+2} \cdot p_{l+1}}{q_l + x_{l+2} \cdot q_{l+1}} - \frac{p_l + rp_{l+1}}{q_l + rq_{l+1}} \\ &= \frac{x_{l+2} - r}{(q_l + x_{l+1}\, q_{l+1})\,(q_l + rq_{l+1})} \end{aligned}$$

* Chrystal, *loc. cit.*, pp. 421–2.

since
$$p_{l+1} \cdot q_l - p_l q_{l+1} = +1$$
$$D_2 = p_{l+1}/q_{l+1} - x$$
$$= \frac{1}{q_{l+1}(q_l + x_{l+2}\, q_{l+1})}$$

$\therefore$ $D_1 \gtrless D_2$ according as
$$\frac{x_{l+2} - r}{q_l + r\, q_{l+1}} \gtrless \frac{1}{q_{l+1}}$$

i. e. as
$$x_{l+2} - r \gtrless q_l/q_{l+1} + r$$

i. e. as
$$r \lessgtr \frac{1}{2}\,(x_{l+2} - q_l/q_{l+1})$$

(Chrystal's result, already referred to, follows easily.)

We now observe that
$$x_{l+2} = a_{l+2} + \frac{1}{a_{l+3}} + \ldots$$
and
$$\frac{q_l}{q_{l+1}} = \frac{1}{a_{l+1}} + \frac{1}{a_l} + \ldots + \frac{1}{a_2}$$

In selecting the convergents nearest in absolute value to x, we seek to exclude those intermediate convergents which are further from x in absolute value than is p_{l+1}/q_{l+1} : *i. e.* the r's to be excluded satisfy the inequality $D_1 > D_2$

$$i.e.\ \ r < \frac{1}{2}\left\{a_{l+2} + \frac{1}{a_{l+3}} + \ldots - \left(\frac{1}{a_{l+1}} + \frac{1}{a_l} \ldots + \frac{1}{a_2}\right)\right\}$$

The cases of a_{l+2} odd and a_{l+2} even must be distinguished.

(1) a_{l+2} odd $= 2m + 1$ (say).

Since $\frac{1}{a_{l+2}} + \ldots$ and $\frac{1}{a_{l+1}} + \ldots$ are both proper fractions, the inequality is satisfied by $r = 1, 2, \ldots m$ only.

(2) a_{l+2} even $= 2m$ (say).

It is certain that values of r up to $(m - 1)$ are to be excluded.

The value $r = m$ is to be excluded only if
$$\frac{1}{a_{l+3}} + \frac{1}{a_{l+4}} + \ldots > \frac{1}{a_{l+1}} + \frac{1}{a_l} \ldots + \frac{1}{a_2}$$

i.e. if $\quad a_{l+3} < a_{l+1}$

or if $\quad a_{l+3} = a_{l+1}$ and $a_{l+4} > a_l$

or if $\quad a_{l+3} = a_{l+1},\ a_{l+4} = a_l,\ a_{l+5} < a_{l-1}$

etc.

The test is an easy one to apply in practice, for we are comparing partial quotients respectively right and left of a particular partial quotient.

A difficulty arises when the comparison has to be carried so far that one of the partial fractions terminates ; this can be overcome by adding ∞'s as partial quotients at the end, as many as necessary. Thus we may write

$a_2 = a_2 + \frac{1}{\infty} + \frac{1}{\infty} \ldots$; and we note that for the purposes of the comparison we replace a_1 by ∞.

An interesting special case occurs when the two partial fractions compared are actually equal; this requires that all the corresponding partial quotients be equal in pairs:

$$i.\,e.\ a_{k+1}=a_{k-1},\ a_{k+2}=a_{k-2}\ \ldots\ a_{2k-2}=a_2.$$

In this case the intermediate convergent and the principal convergent under comparison are equidistant in value from x. The value of x is thus half the sum of these two convergents, and we may write

$$a_1 + \frac{1}{a_2} \cdots + \frac{1}{a_l} + \frac{1}{2m} + \frac{1}{a_l} \cdots + \frac{1}{a_2} = \frac{1}{2}\left\{\frac{p_{l+1}}{q_{l+1}} + \frac{p_l + mp_{l+1}}{q_l + mq_{l+1}}\right\}$$

a result which can be got by applying the properties of symmetrical continuants.*

The reasoning, used above for l odd, gives identical results for l even.

§ 3. To obtain a convenient scheme of arrangement take an example

$$2 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{2} + \tfrac{1}{5}.$$

Write down the principal odd and the principal even convergents, and interpolate the intermediate convergents. We have the two sets,

0/1; 2/1, (7/3), 12/5; 17/7, 90/37; 163/67.

1/0; 5/2, (22/9), (39/16), 56/23; 73/30, (236/97), (399/164), 562/231, 725/298; 888/365.

Certain of the intermediate convergents may now be removed as being further from the true value than is a principal convergent with a smaller denominator.

In the set between 2/1 and 17/7 $a_{l+1} = 3$, and we remove 7/3 only.

In the set between 5/2 and 73/30 $a_{l+1} = 4$, and we remove 22/9 certainly and possibly 39/16; here a_{l+2} is 2 and a_l is 3, *i. e.* $a_{l+2} < a_l$ and so 39/16 must be removed; and so on.

Those to be removed are enclosed by brackets in the above list, and the fractions left have the property that between two of them consecutive in complexity no simpler fraction can be inserted as near in absolute value to the fraction as the less complex of the two.

The following arrangement in groups which *end* instead of *begin* with a principal convergent has advantages:

Fractions in defect.	Fractions in excess.
0/1	1/0
(1/1), 2/1	(3/1), 5/2,
(7/3), 12/5, 17/7	(22/9), (39/16), 56/23, 73/30
90/37, 163/67	(236/97), (399/164), 562/231, 725/298, 888/365

The method of construction is as follows:

* Muir, Proc. Roy. Soc. Edin., 1873–4.

Write down 0/1 and 1/0

then $\frac{0+1}{1+0}, \frac{0+2\times 1}{1+2\times 0}$, *i. e.* 1/1, and 2/1

then $\frac{1+2}{0+1}, \cdot \frac{1+2\times 2}{0+2\times 1}$, *i. e.* 3/1 and 5/2

and so on, the number of terms in each set being determined by the corresponding partial quotient.

The terms to be removed are now the earlier members of a set.

When these are removed we obtain a series of fractions of least complexity nearest x in absolute value by reading along the successive lines completely from left to right, *e. g.* the fractions nearest 888/365 with denominators less than 10, 20, 40, 100 are 17/7, 17/7, 90/37, 163/67, all being in defect.

The particular fraction 1/0 is exceptional.

The actual differences are tabulated below in the order of the previous table:

2·43	∞
(1·43), ·43	(·57), ·067
(·10), ·033, ·0042	(·0115), (·0046), ·0019, ·00054
·00045, ·000041	(·00011), (·000050), ·000023, ·000009, 0.

A further example is added to show the special points mentioned in § 2:

$$2 + \tfrac{1}{4} + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{2}.$$

The table of convergents is

0/1	1/0
1/1, 2/1	(3/1), (5/2), 7/3, 9/4
11/3, 20/9	(29/13), 49/22, 69/31, 89/40
(109/49), 198/89	(287/129), 485/215, 683/307, 881/396.

The convergents 3/1, 29/13, and 287/129 can be removed at once.

In settling the doubtful ones remember that we use ∞ instead of the first partial quotient, and ∞ for any partial quotient after the last or before the first.

For exclusion of 1/1 $4 < \infty$. $\therefore$ 1/1 is excluded.

For 5/2 $2 < \infty$. $\therefore$ 5/2 is excluded.

For 11/3 $4 = 4, 2 > \infty$; this not true. $\therefore$ 11/3 is included.

For 49/22 $2 = 2, 4 = 4, \infty = \infty$. $\therefore$ $\frac{49}{22}$ and $\frac{20}{9}$ are equidistant.

For 109/49 $4 = 4, \infty > 2$. $\therefore$ 109/49 is excluded.

485/218 $\infty < 2$; this is not true. $\therefore$ 485/218 is included.