A SIEVE CONNECTING THE WEAK GOLDBACH CONJECTURE WITH THE STRONG ONE

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Abstract. While a full proof of the Goldbach Conjecture remains beyond my reach, this work presents the construction of an arithmetic sieve that connects the already proven weak Goldbach Conjecture to any even natural number greater than 2. Through this structure, we show how the strong Goldbach Conjecture may be derived by expressing any even number as the difference between a ternary prime sum and one of its components.

Preliminaries

We recall the statement of the weak (ternary) Goldbach Conjecture:

Theorem 1 (Weak Goldbach Theorem [1]). *Every odd integer greater than 5 can be written as the sum of three prime numbers.*

Our goal is to show that, assuming the weak Goldbach Conjecture holds, then every even integer greater than 2 can be expressed as the sum of two primes — i.e., the strong Goldbach Conjecture holds.

Sieve Construction

Let $P = \{2, 3, 5, 7, 11, 13, \dots\}$ be the set of prime numbers. Define $N_{\text{odd}} = 2s + 1$ for $s \in \mathbb{N}$, and consider the operation:

$$N_{\text{even}} = N_{\text{odd}} - p, \quad \text{for } p \in P.$$

By constructing the table $N_{\text{odd}} - p$, where N_{odd} ranges over odd numbers and p over primes, we show that every even number eventually appears as an entry in this table for each column that correspond to a subtraction with a prime greater than 3, see Table 1. Because the values in the row $i + 1$ for $i \in \mathbb{N}$ in the sieve table are the values in the row i increased by adding 2.

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TABLE 1. Sieve connecting any even number N_{even} with $Nodd - P$

		$N_{odd} - P$						
s	N_{odd}	$N_{odd} - 2$	$N_{odd} - 3$	$N_{odd} - 5$	$N_{odd} - 7$	$N_{odd} - 11$	$N_{odd} - 13$	...
1	3	1	0	-2	-4	-8	-10	...
2	5	3	2	0	-2	-6	-8	...
3	7	5	4	2	0	-4	-6	...
4	9	7	6	4	2	-2	-4	...
5	11	9	8	6	4	0	-2	...
6	13	11	10	8	6	2	0	...
7	15	13	12	10	8	4	2	...
8	17	15	14	12	10	6	4	...
9	19	17	16	14	12	8	6	...
10	21	19	18	16	14	10	8	...
11	23	21	20	18	16	12	10	...
12	25	23	22	20	18	14	12	...
13	27	25	24	22	20	16	14	...
14	29	27	26	24	22	18	16	...
15	31	29	28	26	24	20	18	...
⋮								

We prove that the set of even natural numbers

$$2\mathbb{N} := \{ 2k \mid k \in \mathbb{N} \}$$

is contained in each column of the sieve shown in Table 1 for all $p > 2$, $p \in P$.

Base case: Let p be any fixed prime number. Let s_0 be the smallest natural number such that $2s_0 + 1 > p$. Then:

$$2s_0 + 1 - p = 2 \in 2\mathbb{N}.$$

Induction hypothesis: Assume that for some $s_i \in \mathbb{N}$ with $s_i \geq s_0$ there exists $k \in \mathbb{N}$ such that:

$$2s_i + 1 - p = 2k.$$

Induction step: Consider the next index $s_{i+1} = s_i + 1$. Then:

$$\begin{aligned} 2(s_i + 1) + 1 - p &= (2s_i + 2) + 1 - p \\ &= (2s_i + 1 - p) + 2 \\ &= 2k + 2 \in 2\mathbb{N}. \end{aligned}$$

Hence, if the property holds for s_i, it also holds for s_{i+1}.

Conclusion: By the principle of mathematical induction, all elements of $2\mathbb{N}$ appear in the column defined by the difference with p.

Main Argument

The goal is to show that the strong Goldbach Conjecture:

> *Every even integer greater than 2 can be expressed as the sum of two prime numbers*

can be deduced from the weak Goldbach Theorem by means of the sieve construction defined earlier.

Proof. Let N_{even} be an arbitrary even number such that $N_{\text{even}} > 2$.

Since the set $\{N_{\text{odd}} - p \mid N_{\text{odd}} \text{ odd}, p \in P\}$ includes all even numbers, there exists at least one odd number N_{odd} and one prime $p_3 \in P$ such that:

$$N_{\text{even}} = N_{\text{odd}} - p_3.$$

By the weak Goldbach Conjecture, since $N_{\text{odd}} > 5$, there exist primes p_1, p_2, p_3 such that:

$$N_{\text{odd}} = p_1 + p_2 + p_3.$$

Then,

$$N_{\text{even}} = N_{\text{odd}} - p_3 = p_1 + p_2.$$

Since p_1, p_2 are primes, this expresses N_{even} as a sum of two primes. $\square$

Example

Let $N_{\text{even}} = 28$. In the sieve, this value appears when $N_{\text{odd}} = 31$ and $p = 3$, since:

$$31 = 3 + 11 + 17 \Rightarrow 28 = 31 - 3 = 11 + 17.$$

Conclusion

This argument shows that, by applying the weak Goldbach Theorem and utilizing the sieve structure $N_{\text{odd}} - p$ the strong Goldbach Conjecture follows in a natural way.

It is important to note that this sieve construction does not directly identify the two primes that sum to a given even number. However, it ensures that any even number can be expressed as the difference between an odd number and a prime. Since, by Theorem 1, every odd number greater than 5 can be written as the sum of three primes, subtracting one of those primes yields a decomposition of the even number into the sum of the remaining two primes. This reinforces the structural link between the weak and strong Goldbach conjectures.

References

[1] Harald Andres Helfgott. *The ternary Goldbach problem.* arXiv:1501.05438, 2015. https://arxiv.org/abs/1501.05438.