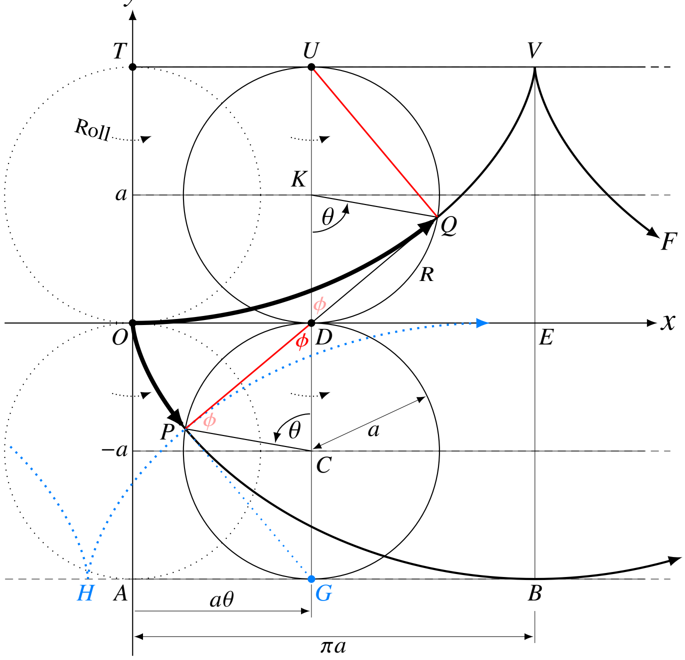
Quick Proofs of Properties of the Cycloid

Gavin R. Putland; version 0.5, July 25, 2025.



A **cycloid** is the plane curve traced by a point (**generating point**) on a circle (**generating circle**) that rolls along a fixed line (**generating line**) without slip. If the line is horizontal with the circle hanging vertically below it, the cycloid is "**inverted**". The diagram shows two circles with radius a. The lower circle has rolled along segment OD of the x-axis, turning through angle θ and traveling distance $a\theta$, so that point P, starting from the initial contact point O, has traced the arc OP of the inverted cycloid OPB. The upper circle, turning through the same angle, has rolled the same distance along the top line TV, so that point Q, starting from O (*opposite* the initial contact point T), has traced the arc OQ of the inverted cycloid $OQVF$. When θ reaches π, P will reach B and Q will reach V. *The arcs OPB and VQO are congruent because they are mirror images*, as is apparent if the "rolling" of either is played backwards.

I. Normal chords

The infinitesimal part of the lower circle at D has no velocity component in the x direction (no slip) or in the y direction (because it has reached maximum y). So its instantaneous velocity is zero, with no component along PD, so that the motion of P must be *normal to PD*. Thus *PD is the normal to cycloid OPB at P*; and in general, *the normal to a cycloid at a point is given by the chord of the generating circle from that point (as generating point) to the point of contact with the generating line*.

II. Normal cycloids

If a second circle centered on C, superimposed on the first, rolls along the bottom line AB, the points of that circle trace cycloids, one of which, namely HP, passes through P, where cycloid HP has normal chord PG and cycloid OPB has normal chord PD. As the two chords make a right angle with each other (angle in a semicircle), *the cycloids are normal to each other at P*. The same applies to later positions of P and HP, including (by symmetry) positions for which the part of HP left of H intersects the part of OPB right of B. Thus *cycloid OPB is an **orthogonal trajectory** to the family of cycloids generated by circle GPD and line AB*—and to the family of **cycloidal cylinders** generated by the latter cycloids if they move in the z direction (normal to the xy plane).

III. Horizontal and normal velocities

Let angle PDC be ϕ. Let cycloid HP travel in the x direction at speed u, and let v be its resulting *normal* speed at P (measured along PG). Then

$$v/u = \cos\phi\,. \tag{1}$$

At P, $y = -DP\cos\phi$. But $DP = 2a\cos\phi$ (from the right triangle DPG). Back-substituting for DP we have, at P,

$$y = -2a\cos^2\phi\,. \tag{2}$$

Eliminating $\cos\phi$ between eqs. (1) and (2) gives

$$\tfrac{1}{2}v^2 + \tfrac{u^2}{4a}\,y = 0\,. \tag{3}$$

IV. Brachistochrone

The first term in eq. (3) is the kinetic energy, per unit mass, of a particle with speed v. In a uniform gravitational field g in the $-y$ direction, the particle's gravitational potential energy per unit mass is gy (relative to $y=0$). Hence, if the particle is released from rest at $y=0$ and moves under gravity at speed v along a frictionless path, conservation of energy gives

$$\tfrac{1}{2}v^2 + gy = 0\,. \tag{4}$$

Given an origin O and a destination point Z (not illustrated), with Z no higher than O, let us choose the radius a and the x direction so that cycloid OPB passes through Z. Then eq. (4) determines v as a *function of* y, say $v(y)$, on all frictionless paths from O to Z—including cycloid OPB—if the particle is released from rest at O. Now let the horizontal speed u of cycloidal cylinder HP be

$$u = \sqrt{4ga}\,. \tag{5}$$

With this substitution, (3) becomes (4), so that $v(y)$ becomes the *normal* speed of the cycloidal cylinder HP, i.e. the speed of its intersection with its orthogonal cycloid OPB. Now let the particle be released from O when cylinder HP crosses O. If the frictionless path is cycloid OPB, the particle will *just* keep up with cylinder HP, and will reach Z when cylinder HP crosses Z. But if the path from O to Z departs from cycloid OPB, then for part of the path the same $v(y)$ over the same range of y will *not* be normal to cylinder HP, so that the particle will need to travel *further at the same speed* between successive positions of cylinder HP and will therefore fall behind it, taking more time to reach Z. Thus *the path of least time (**brachistochrone**) from O to Z is an inverted cycloid meeting its generating line at O and passing through Z*.

V. Self-evolute / self-involute

As triangles DCP and DKQ are congruent (by two sides and the included angle θ), the angle KDQ is ϕ, so that *PDQ is a straight line segment*. This segment, PQ, is *normal to cycloid OPB* at P (because PQ includes the normal chord), and *tangential to cycloid OQV* at Q (because PQ makes an angle-in-a-semicircle with the normal chord QU)—just as if P were the free end of a thin, taut, inextensible string, anchored at V and wrapped against cycloid VQO as far as O, and then unwrapped as far as Q. And by the mirror-symmetries we may continue cycloid OPB past B by wrapping against VF. Thus *cycloid OQVF is the **evolute** of cycloid OPB*; i.e., *cycloid OPB is an **involute** (also called an **evolvent**) of cycloid OQVF.*

VI. Total and partial areas

By the same congruency, QD is $1/2$ of the length of QP. Hence, if QP is "wrapped" by an infinitesimal angle $d\phi$, sweeping out an infinitesimal triangular area, the part of that area above the x-axis is $1/4$ of the whole. So the part *below* the x-axis is $3/4$ of the whole. Adding the contributions, **area *OPBEDO*** is $3/4$ of area *OPBEVQO*. But the latter area is that of rectangle *OEBA*, because the portion *ODEVQO* has the same area as *BGAOPB*; and that rectangle has area $2\pi a^2$, of which $3/4$ is $\frac{3}{2}\pi a^2$, which is $3/2$ of the area of the generating circle. Multiplying by 2, we find that *the area between a full cycloidal arch and its generating line is three times the area of the generating circle*.

[Continued ...]

When ϕ increases by $d\phi$, the swept area above the x-axis is that of a triangle with base QD and altitude $QD\,d\phi$, where $QD = 2a\cos\phi$. The swept area below the axis is three times that. Areas for ranges of ϕ can thence be found by integration.

VII. Total and partial arc lengths

As arc OQV has the involute OPB, the length of the former (and therefore of the latter, which is congruent thereto) is the length of VB, namely $4a$. Multiplying by 2, we find that *the arc length of a full cycloidal arch is four times the diameter of the generating circle*.

More generally, the length of the cycloidal arc OQ is the length of the straight segment to which it unwraps, namely

$$QP = 4a\cos\phi\,. \tag{6}$$

From this we may obtain partial lengths of the involute OPB. Let s be the arc length BP. If ϕ increases by $d\phi$, the infinitesimal change in s is not affected by the infinitesimal movement of the pivot Q along VQO, because that movement has no component in the direction of ds, i.e. normal to QP.[1] So we have simply $ds = QP\,d\phi$ or, using (6),

$$ds = 4a\cos\phi\;d\phi\,. \tag{7}$$

Integrating, and fixing the additive constant to give $s=0$ when $\phi=0$, we get

$$s = 4a\sin\phi\,. \tag{8}$$

VIII. Tautochrone (isochrone)

In the aforesaid gravitational field, if a particle moves under gravity, with no resistance, along the cycloidal path OPB, its (tangential) acceleration in the s direction is

$$\ddot{s} = -g\sin\phi\,; \tag{9}$$

i.e., by comparison with (8),

$$\ddot{s} = -\tfrac{g}{4a}\,s\,. \tag{10}$$

This describes simple harmonic motion *along the path*, with the radian frequency

$$\omega = \sqrt{\tfrac{g}{4a}}\,, \tag{11}$$

hence the period

$$T \;=\; 2\pi/\omega \;=\; 2\pi\sqrt{4a/g} \tag{12}$$

—which is the familiar formula for the period of a simple pendulum of length $4a$, except that *it does not involve any small-angle approximation*; that is, within the limits of the involute, *the period T does not depend on the peak amplitude of the oscillation*. This is the **tautochrone** (same-time) or **isochrone** (equal-time) property. If the bob of the modified pendulum is released from rest at any point on its cycloidal path, it reaches the bottom of its swing in time $T/4 = \pi\sqrt{a/g}$, which is likewise independent of the length of the swing, depending only on a and g.

And the evolute property tells us how to modify a simple pendulum so as to make the bob swing in the cycloidal arc OPB instead of a circular arc centered on V: confine the string between the jaws of the cycloid $OQVF$.

IX. Generating circle of brachistochrone

Question: *If the motion of the generating point is brachistochronal, what does the generating circle do?*

Dividing (7) by the corresponding time interval dt gives

$$\dot{s} = 4a\cos\phi\;\dot{\phi}\,. \tag{13}$$

As the question concerns the generating circle, whose angle of rotation is θ, let us rewrite $\dot{s}$ in terms of θ. In the isosceles triangle DCP, we have $\theta + 2\phi = \pi$, hence $\phi = \frac{\pi}{2} - \frac{\theta}{2}$, hence $\cos\phi = \sin\frac{\theta}{2}$ and $\dot{\phi} = -\dot{\theta}/2$. Substituting for $\cos\phi$ and $\dot{\phi}$ in (13) gives

$$\dot{s} \;=\; -2a\dot{\theta}\sin\tfrac{\theta}{2}\,. \tag{14}$$

But $v = -\dot{s}$, so

$$v^2 \;=\; \dot{s}^2 \;=\; 4a^2\,\dot{\theta}^2\sin^2\tfrac{\theta}{2}\,; \tag{15}$$

i.e., by a standard trigonometric identity,

$$v^2 \;=\; 2a^2\,\dot{\theta}^2\big(1-\cos\theta\big)\,. \tag{16}$$

From the diagram, at P,

$$y \;=\; -a + a\cos\theta \;=\; -a(1-\cos\theta)\,. \tag{17}$$

Substituting this and (16) into the energy equation (4), we can cancel the factor $a(1-\cos\theta)$ and solve for $\dot{\theta}$, obtaining

$$\dot{\theta} = \sqrt{g/a} \tag{18}$$

(where we want only one square root, because least-time motion evidently does not include backtracking). So $\dot{\theta}$ is time-independent:

Answer: *The generating circle rolls at a constant speed!*

This angular velocity $\dot{\theta}$ is *twice* the isochronal radian frequency given by (11), because a *full turn* of the circle gives a full maximum swing of the isochronal pendulum, which is only *half a cycle* of that pendulum.

X. Parametric equations [at last]

At P, eq. (17) gives y in terms of θ. For x, from the diagram, we have

$$x \;=\; a\theta - a\sin\theta \;=\; a(\theta - \sin\theta)\,. \tag{19}$$

So, if we let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be the unit vectors in the x, y, z directions respectively, we can write the position vector $\mathbf{r}$ of P as

$$\mathbf{r} \;=\; a(\theta-\sin\theta)\,\mathbf{i} \;+\; a(\cos\theta-1)\,\mathbf{j}\,. \tag{20}$$

XI. Electromagnetic interpretation

If $\theta = \omega t$ where ω is constant, so that the generating circle rolls with constant angular speed ω, the position vector in (20) becomes

$$\mathbf{r} \;=\; a(\omega t - \sin\omega t)\,\mathbf{i} \;+\; a(\cos\omega t - 1)\,\mathbf{j}\,, \tag{21}$$

whence

$$\dot{\mathbf{r}} \;=\; a(\omega - \omega\cos\omega t)\,\mathbf{i} \;-\; a\omega\sin\omega t\;\mathbf{j}\,; \tag{22}$$

$$\ddot{\mathbf{r}} \;=\; a\omega^2\sin\omega t\;\mathbf{i} \;-\; a\omega^2\cos\omega t\;\mathbf{j}\,. \tag{23}$$

From (22),

$$\omega\mathbf{k}\times\dot{\mathbf{r}} \;=\; a(\omega^2 - \omega^2\cos\omega t)\,\mathbf{j} \;+\; a\omega^2\sin\omega t\;\mathbf{i}\,. \tag{24}$$

Recognizing the right-hand side as $\ddot{\mathbf{r}} + a\omega^2\mathbf{j}$, then multiplying through by m (for *mass*) and rearranging, we obtain

$$m\ddot{\mathbf{r}} \;=\; -ma\omega^2\,\mathbf{j} \;+\; \dot{\mathbf{r}}\times(-m\omega\mathbf{k})\,. \tag{25}$$

This matches the **Lorentz force**

$$m\ddot{\mathbf{r}} \;=\; q(-E\mathbf{j}) \;+\; q\dot{\mathbf{r}}\times(-B\mathbf{k}) \tag{26}$$

on a particle with (non-relativistic) mass m and charge q in a uniform electric field $-E\mathbf{j}$ and a uniform magnetic field (flux density) $-B\mathbf{k}$, *if*

$$qE = ma\omega^2 \quad\text{and}\quad qB = m\omega\,. \tag{27}$$

Solving the second of these equations for ω and thence solving the first for a, we obtain

$$a \;=\; {}^{mE}/_{qB^2} \quad\text{and}\quad \omega = qB/m\,. \tag{28}$$

Eqs. (28) make (26) agree with (21), which gives the initial conditions $\mathbf{r}=\mathbf{0}$ and $\dot{\mathbf{r}}=\mathbf{0}$ at $t=0$. Thus, if the particle is released from rest at O in the crossed electric and magnetic fields, its path will be a cycloid whose generating circle has the radius and angular speed given by (28).

Acknowledgments

The brachistochrone proof is essentially Johann Bernoulli's, reduced to first principles for the earliest possible delivery. The other proofs are not consciously based on anyone else's; but of course, given the volume of prior literature, I would be amazed if any of them were substantially new.

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[1] The same cannot be said for the movement of the pivot D due to the rolling of the generating circle through $d\theta$, wherefore we have used the chord DP to determine only the *direction*, not the magnitude, of the movement of P! (It is possible, and apparently conventional, to determine the both the magnitude and the direction of the movement of P by taking the movement of D into account; but here we have sidestepped that complication.)