Some Properties of Co-normal Points on a Parabola.

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1. If the coordinates of a point on a parabola,

$$y^2-4ax=0,$$

be $(am^2, 2am)$, then the equations to the tangent and normal at the point are

$$x-my+am^2=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{i.}),$$

and $$mx+y-a\,(m^3+2m)=0 \;\ldots\ldots\ldots\ldots\ldots\ldots(\text{ii.}),$$

and to the chord through (m), (m'), is

$$y\,(m+m')-2x-2amm'=0\ldots\ldots\ldots\ldots\ldots\ldots(\text{iii.}).$$

If we write (ii.) in the form

$$am^3+(2a-x)\,m-y=0\ldots\ldots\ldots\ldots\ldots\ldots(\text{iv.}),$$

we see that from a given point (x, y), we can draw three normals to the curve, with the condition

$$\Sigma m=0.$$

Let O be the point, and $P\,(m_1)$, $Q\,(m_2)$, $R\,(m_3)$ the corresponding points on the parabola; then I call these latter *co-normal* points, and the circle through them a *co-normal* circle.

2. Since $$S_1\equiv\Sigma\ldots=0,$$
$$S_2\equiv\Sigma m^2=-2\,(m_1m_2+\ldots+\ldots)=-2\Sigma m_1m_2,$$
$$S_3=3m_1m_2m_3=3\mu,$$
$$S_4=S_2^2/2\,;$$

also $$m_1^2-m_2m_3=\ldots=\ldots=m_1^2+m_1m_2+m_2^2=S_2/2.$$

3. In the case when P, Q, R are *any* three points on the curve the circle PQR is

$$x^2+y^2-ax\left[S_2+\Sigma m_1m_2+4\right]+ay\left[S_1\,.\,\Sigma m_1m_2-\mu\right]/2-a^2\mu S_1=0\ldots(\text{i.}),$$

and the tangent-circle pqr is

$$x^2+y^2-ax\left[1+\Sigma m_1 m_2\right]-ay\left[S_1-\mu\right]+a^2\Sigma m_1 m_2 = 0 \ldots\ldots \text{(ii.)}.$$

If the points are co-normal points, then these equations take the form

$$x^2+y^2-ax\,(S_2+8)/2-ay\mu/2 = 0 \;\ldots\ldots\ldots\ldots\ldots \text{(iii.)},$$

$$x^2+y^2+ax\,(S_2-2)/2+ay\mu-a^2S_2/2 = 0 \;\ldots\ldots\ldots\ldots\ldots \text{(iv.)}.$$

4. The coordinates of p (§ 3), for co-normal points, are

$$(am_2m_3,\ -am_1)\,;$$

the coordinates of P', mid-point of QR, are

$$\left[a\,(m_2^2+m_3^2)/2,\ -am_1\right];$$

and the coordinates of p', mid-point of qr, are

$$(-am_1^2/2,\ am_1/2).$$

5. The equation to $P'Q'R'$ (N.P. circle of PQR) is

$$x^2+y^2-ax\,(3S_2-8)/4+ay\mu/4+a^2\,(S_4-4S_2)/4 = 0 \ldots\ldots \text{(i.)},$$

and to $p'q'r'$ (N.P. circle of pqr), is

$$x^2+y^2+ax\,(S_2+2)/4-ay\mu/2 = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii.)};$$

this circle evidently passes through the vertex.

6. The circle through the vertex, which touches the parabola at (m_1), is

$$x^2+y^2-a\left[3m_1^2+4\right]x+am_1^3y = 0 \ldots\ldots\ldots\ldots\ldots\ldots \text{(i.)},$$

if the parameter of the fourth point of section (P_1) be M, then, because

$$0+2m_1+M = 0,$$

we see that the points P_1, $(Q_1,\ R_1)$ are co-normal points.

7. The centre of curvature at P is

$$\left[a\,(3m_1^2+2),\ -2am_1^3\right],$$

therefore the circle of curvature at P is

$$x^2+y^2-2ax\,(3m_1^2+2)+4am_1^3y-3a^2m_1^4 = 0 \;\ldots\ldots\ldots\ldots \text{(i.)};$$

and, because $\qquad 3m_1+M' = 0,$

the fourth points of section P_2, $(Q_2,\ R_2)$ are co-normal points.

The equation to circle $P_2Q_2R_2$ is

$$x^2+y^2-ax\,(9S_2+8)/2+27ay\mu/2 = 0 \ldots\ldots\ldots\ldots\ldots(\text{ii.});$$

the radical axis of this and PQR is

$$2S_2x = 7\mu y \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{iii.}).$$

The tangent-circle $(p_2q_2r_2)$ is

$$x^2+y^2+ax\,(9S_2-2)/2-27ay\mu-9a^2S_2/2 = 0 \ldots\ldots\ldots(\text{iv.}).$$

The radical axis of this and pqr is

$$S_2x-7\mu y-aS_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{v.}).$$

8. The equation to $P'p'$ is

$$yS_2+3m_1x+am_1\,(m_1^2+2m_2m_3)/2 = 0,$$

whence we see that

$P'p'$, $Q'q'$, $R'r'$ cointersect in $(-aS_2/12, \; -3a\mu/2S_2)$.

9. From § 3 (iii.), § 5 (ii.), we see that the radical axis of PQR, $p'q'r'$ is the tangent at the vertex, and, from § 7, that the radical centre of the three circles of curvature is $(-aS_2/4, \; 3a\mu/2S_2)$.

10. The equation to QR is

$$2x+m_1y+2am_2m_3 = 0,$$

and the equation to qr is $\quad x-m_1y+am_1^2 = 0;$

hence these cut in $\left[a\,(m_1^2+2m_2m_3)/3, \; aS_2/3m_1\right]$, *i.e.*, in p_3 say; hence the centroid of all such triangles as $p_3q_3r_3$ lies on the tangent at the vertex.

11. The line Pp is

$$yS_2-6m_1x+a\,(6\mu+m_1S_2) = 0;$$

hence Pp, Qq, Rr cointersect in $(aS_2/6, \; -6a\mu/S_2)$.

12. The perpendicular from q on RP is

$$m_2x-2y = a\,(2m_2+\mu);$$

hence the perpendiculars from p on QR, q on RP, r on PQ, meet in $(2a, \; -a\mu/2)$.

13. The orthocentre of PQR is

$$\left[a\,(S_2-8)/2, \; -a\mu/2\right];$$

the orthocentre of pqr is $(-a,\ a\mu)$;
and the orthocentre of PQr (π_r) is

$$\left[-a\,(2+m_1m_2),\ -a\,(2m_3+\mu)\right];$$

hence $A\pi_r$ is parallel to the tangent at R,

$A\pi_p$ „ „ „ P,

and $A\pi_q$ „ „ „ Q.

The centroid of $\pi_p\pi_q\pi_r$ is

$$\left[a\,(S_2-12)/6,\ -a\mu\right].$$

14. The centroid of PQR is $(aS_2/3,\ 0)$,
the centroid of pqr is $(-aS_2/6,\ 0)$,
the centroid of PQr (g_3) is $(aS_2/6,\ -am_3)$;
hence g_1, g_2, g_3 are collinear, and their centroid is

$$(aS_2/6,\ 0).$$

15. From § 1, the coordinates of O are

$$\left[a\,(S_2+4)/2,\ a\mu\right].$$

The isoclinal (through P) to PN with the normal is

$$y-m_1x+a\,(m_1^3-2m_1)=0 \quad\text{......................(i.)};$$

hence the similar lines through Q, R meet it in

$$O',\ \left[a\,(S_2-4)/2,\ -a\mu\right].$$

The equation to OO' is

$$4y-2\mu x+a\mu S_2=0 \quad\text{........................(ii.)}.$$

16. Since PO cuts the axis at

$$\left[a\,(m_1^2+2),\ 0\right],$$

the perpendicular to PO, through this point, is

$$m_1y-x+a\,(m_1^2+2)=0 \quad\text{.......................(i.)};$$

this meets the similar line for Q in r_6, say

$$\left[a\,(2-m_1m_2),\ am_3\right].$$

The circle $p_6q_6r_6$ is

$$x^2+y^2-ax\left[S_2+6\right]/2-a\mu y+a^2\,(S_2+4)/2=0 \quad\text{.........(ii.)},$$

of which the diameter is SO. By its construction the triangle $p_5q_5r_5$ has its sides parallel to the sides of pqr, and, as the radii of the circles are equal, therefore the triangles are equal. The circles touch at S. (§ 16 furnishes an easy way of finding O when P, Q, R are given.) S is the centre of perspective of the triangles. Since, with the usual notation, $ST = SG$, therefore PO cuts the circle in the mid-point of PG, and so for the other normals.

17. Perpendiculars drawn to the tangents at P, Q, R, at the points where they meet the axis, since their equations are

$$y+m_1x+am_1^3=0, \;\ldots,$$

meet in Π,
$$\left[-aS_2/2,\; -a\mu\right].$$

This point is the extremity of the diameter of pqr through S. The polar of Π, with respect to the parabola, is

$$2x+\mu y = aS_2 \quad\text{.............................(i.)};$$

hence it is perpendicular to OO', and meets that line in

$$\left[aS_2/2,\; 0\right],$$

i.e., on the axis.

18. The circle round $P(m_1)$, $P_1(-2m_1)$, $P_2(-3m_1)$ cuts the curve again (because $m_1-2m_1-3m_1+4m_1=0$) in $P_3(4m_1)$; hence P_3, (Q_3, R_3) are co-normal points.

The equation to PP_1P_2 is

$$x^2+y^2-ax\,(15m_1^2+4)-5am_1^3y+24a^2m_1^4=0 \quad\text{...........(i.)},$$

and the radical centre of it and the circles for Q, R is

$$\left[4aS_2/5,\; 48a\mu/5S_2\right].$$

19. Through P, Q, R draw parallels to

(i.) the tangents at Q, R, P;

(ii.) „ „ R, P, Q;

and let P_q, Q_r, R_p; P_r, Q_p, R_q be the points where the sets (i.), (ii.) respectively meet the parabola; their parameters are

$$(2m_2-m_1,\; 2m_3-m_2,\; 2m_1-m_3), \quad (2m_3-m_1,\; 2m_1-m_2,\; 2m_2-m_3);$$

hence the two sets are co-normal points.

The mid-point of P_qP_r, &c. is on the diameter through P_3, &c.

20. The circle PQP_q, because

$$m_1+m_2+(2m_2-m_1)-3m_2=0,$$

passes through Q_2; therefore QRQ_r, RPR_p pass through R_2, P_2, respectively.

Similarly, PRP_r, QPQ_p, RQR_q pass through R_2, P_2, Q_2, respectively.

21. Again, because

$$m_1+(2m_2-m_1)-2m_2+0=0,$$

therefore
$$\left.\begin{array}{llll} PP_qA & \text{passes through} & Q_1 \\ QQ_rA & \quad ,, & \quad ,, & R_1, \\ RR_pA & \quad ,, & \quad ,, & P_1, \end{array}\right\}$$

and
$$\left.\begin{array}{llll} PP_rA & \text{passes through} & R_1, \\ QQ_pA & \quad ,, & \quad ,, & P_1, \\ RR_qA & \quad ,, & \quad ,, & Q_1. \end{array}\right\}$$

22. The circles $P_qQ_rR_p$, $P_rQ_pR_q$ are given by

$$x^2+y^2-ax\,(7S_2+8)/2-a\nu y/2=0 \quad \text{...............(i.),}$$

$$x^2+y^2-ax\,(7S_2+8)/2-a\nu' y/2=0 \quad \text{...............(ii.),}$$

where
$$\nu \equiv (2m_1-m_3\,.\,2m_2-m_1\,.\,2m_3-m_2),$$
$$\nu' \equiv (2m_1-m_2\,.\,2m_2-m_3\,.\,2m_3-m_1),$$

and
$$\nu+\nu'=20\mu.$$

Hence (i.), (ii.), intersect on the axis, and abscissæ of their centres are the same.

23. $$\Delta P_qQ_rR_p+\Delta P_rQ_pR_q=162\mu a^2.$$

24. PP_q, QQ_p meet in R_r,

$$\left[a\,(S_2-4m_1m_2)/2,\ -am_3\right],$$

i.e., on the diameter through r; and the centroid of P_p, Q_q, R_r is

$$\left[a\,(5S_2/6),\ 0\right].$$

25. The circle through mid-points of P_qP_r, Q_pQ_r, R_pR is

$$x^2+y^2-ax\,(9S_2+32)/2+a\mu y/4+a^2\,(10S_4+32S_2)=0.$$

26. The centroids of $P_q Q_r R_p$, $P_r Q_p R_q$ coincide in

$$[7aS_2/3,\ 0].$$

Their orthocentres are $[a(7S_2-8)/2,\ -a\nu/2]$

and $[a(7S_2-8)/2,\ -a\nu'/2]$.

27. The tangents at Q_r, R_p meet in π_q,

$$[a(2m_3-m_2)(2m_1-m_3),\quad -a(2m_2-m_1)],$$

and at Q_p, R_q in π'_q,

$$[a(2m_1-m_2)(2m_2-m_3),\quad -a(2m_3-m_1)];$$

hence the centroids of $\pi_p\pi_q\pi_r$, $\pi'_p\pi'_q\pi'_r$ coincide in

$$[-7aS_2/6,\ 0].$$

28. Let PS, QS, RS meet the curve again in t_1, t_2, t_3; then, since t_1 is given by

$$[a/m_1^2,\quad -2a/m_1],$$

the circle $t_1t_2t_3$ is

$$x^2+y^2-a[\Sigma 1/m^2+4]x+ay/2\mu-a^2\Sigma 1/m/\mu = 0\ldots\ldots\ldots\ldots(\text{i.}).$$

The tangents at t_1, t_2, t_3, meet in t'_1, t'_2, t'_3, where t'_1 is the point

$$[am_1/\mu,\quad am_1^2/\mu];$$

hence the circle $t'_1t'_2t'_3$ is

$$x^2+y^2-ax-ay(S_2+2)/2\mu = 0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{ii.}),$$

which passes through the vertex as well as the focus.

29. In fact, if the tangents at the co-normal points P, Q, R cut the line given by $x = ka$, and from the points of section the second set of tangents be drawn, touching the curve at T_1, T_2, T_3; then, since T_1 is given by (k/m_1), $\Sigma m_1m_2 = 0$; hence, by § 3 (ii.), the tangent-circle for $T_1T_2T_3$ is

$$x^2+y^2-ax+ay(k\,.\,S_2+2k^3)/2\mu = 0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{i.}).$$

30. The "N.P." circle of $t'_1t'_2t'_3$ is

$$x^2+y^2+ax/2+ay(2-3S_2)/4\mu+a^2(S_4-S_2)/2\mu^2 = 0\ \ldots\ldots\ldots\ (\text{i.}).$$

The centroid of the triangle is

$$[0,\ aS_2/3\mu].$$

31. The following method of obtaining sets of co-normal points is of some interest:—

Draw PS and produce it to meet the curve in L; join LA and draw Aa_1 perpendicular to AL. The parameter of a_1 is $4m_1$ (and it is therefore the point P_8 of § 17); similarly, from a_1 we can get a_2, and so on. Let b_1, c_1 be the corresponding points for Q_1, R_1; then (a_1, b_1, c_1), (a_2, b_2, c_2), ... are co-normal sets. Again, reversing the above process, *i.e.*, join PA, draw AL' perpendicular to AP, then join L' to S and produce to meet the curve in a_1', ..., and we have sets

$$(a_1', b_1', c_1'), \quad (a_2', b_2', c_2'), \ldots,$$

with parameters $m_1/4$, $m_1/16$, &c. a_1a_1', a_2a_2', &c. meet the axis where it is cut by the tangent at P, &c., and the tangents at a_1, a_1', meet on the ordinate of P, and so on.

32. The circle Pa_1a_1' is

$$x^2+y^2-\left(4+\frac{357}{16}m_1^2\right)ax+\frac{425}{32}am_1^3y-\frac{21}{4}a\,m_1^4=0 \ldots\ldots \text{(i.)},$$

and it cuts the curve in a fourth point $(-21m_1/4)$; hence this point and the analogous ones form a co-normal system.

33. The equations to $a_1b_1c_1$, $a_1'b_1'c_1'$ are

$$x^2+y^2-ax\,(8S_2+4)-32a\mu y=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i.)},$$

and $$x^2+y^2-ax\,(S_2+128)/32-ay\mu/128=0 \ldots\ldots\ldots\ldots\ldots \text{(ii.)}.$$

34. The sum of the squares of the tangents, taken once, from a_1, b_1, c_1 to pqr, PQR, are respectively

$$a^2\,(272S_4+93S_2/2) \quad \text{and} \quad 240S_4a^2.$$

35. Since the parameters of P_q, P_r, a_1 are $2m_2-m_1$, $2m_3-m_1$, and $4m_1$, their sum $=0$; hence the three points form a co-normal system, as do Q_r, Q_p, b_1; R_p, R_q, c_1. (*Cf.* § 17.)

36. The equation to any parabola passing through the co-normal system P, Q, R is

$$(y+\lambda x)^2=ax\left[8+\lambda^2S_2\right]/2+\lambda ay\left[2S_3+\mu\lambda\right]/2+4\lambda\mu a^2 \ldots\ldots \text{(i.)},$$

where λ is arbitrary.

If we make the curve pass through O, we find $\lambda^2=2$, *i.e.*, the two parabolas are

$$(y\pm\sqrt{2}\,x)^2=ax\,(S_2+4)+ay\,(\mu\pm\sqrt{2}\,S_3)\pm4\sqrt{2}\,\mu a^2 \ldots\ldots \text{(ii.)}.$$

This latter may be written in the form

$$y^2+2x^2-ax(S_2+4)-ay\mu = \pm\sqrt{2}\left[ayS_2+4\mu a^2-2xy\right]\ldots\ldots\text{(iii.)}$$

Hence we see that the ellipse through the four points and through the vertex is given by

$$y^2+2x^2-ax(S_2+4)-ay\mu = 0\ldots\ldots\ldots\ldots\ldots\ldots\text{(iv.)},$$

and the rectangular hyperbola through the four points is

$$2xy = ayS_2+4\mu a^2\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(v.)}$$

37. From (ii.) of § 36, we see that the mid-point of each of the intercepts made by the parabolas on the axis of the given parabola is the projection of O on that axis.

38. From § 15 and § 36 (iv.), we see that the centre of the ellipse is at the mid-point of AO.

39. The centre of the hyperbola is $(aS_2/2,\ 0)$; *cf.* § 17. The tangent at O is

$$2y+x\mu = a\mu(S_2+8)/2,$$

which shows that it coincides with the perpendicular through O on the polar of O with regard to the parabola.*

40. The second parabola, which passes through A and the co-normal points, is given by

$$2x^2 = axS_2+a\mu y.$$

41. $SP = a(1+m_1^2),\ \ldots,\ \ldots;$

$$\therefore\quad SP\,.\,SQ\,.\,SR = a^3\left[1+S_2+\frac{S_2^2}{4}+\mu^2\right] = a\,.\,SO^2.$$

42. Many results may be got by taking μ = constant; thus (§ 9) in this case the locus of the radical centres of the three circles of curvature at P, Q, R is the rectangular hyperbola

$$8xy = -3a^2\mu.$$

[43. We refer to *Milne and Davis*, the Parabola, pp. 37 and 56, for geometrical proofs of §§ 1 and 3 (iii.). From this last it is obvious that, given any two points on the curve, the co-normal system can be determined by finding the 4th point of section of a circle through them and the vertex with the curve.

* *Cf.* also § 15 (ii.).

44. A referee states that § 5, (ii.), was set as a problem in the June 1889 Examination at Caius College, Cambridge; and I find a long proof of § 6, (i.), in the *Nouvelles Annales des Mathématiques*, March, 1890, by M. Lemaire.

45. *Milne*, p. 34, illustrates § 16.

46. On §§ 19–21, a referee remarks that the result may be generalized thus: Let PQR be one set of co-normal points, and $P'Q'R'$ another set; through P, Q, R, draw parallels to the tangents at P', Q', R' meeting the curve again in L, M, N, respectively, then L, M, N are also co-normal points. Again, the circle $PP'L$ meets the curve in P'_3, and the circle PLA meets the curve in P'_2, P'_2 being (-2μ), $P'_3\,(-3\mu)$, where μ is the parameter of P'.

47. If p_t, q_t, r_t are the points of contact of tangents parallel to QR, RP, PQ, then these are given by

$$(p_t)\ \left[am_1^2/4,\ \ -am_1\right],\ \ldots,\ \ldots.$$

The tangent circle for p_t, q_t, r_t is

$$x^2+y^2+ax\left[S_2-8\right]/8-a\mu y/8-a^2S_2/8=0.$$

The perpendicular from r_t on PQ is

$$m_3x+2y=2am_3+am_3^3/4\,;$$

hence the three like perpendiculars meet in

$$a\left[2+S_2/8\right],\quad -a\mu/8.$$

The orthocentre of $p_tq_tr_t$ is

$$a\left[-4+S_2/8\right],\quad a\mu/16.$$

48. Since the co-normal points (P, Q, R) are on a circle through the vertex, the inverse points, with respect to A, which lie upon a cissoid, are on a straight line. Its equation is

$$\mu y+(S_2+8)\,x=2k^2/a,$$

where k is the modulus of inversion.

49. If
$$\mu_1\equiv p_1m_1+q_1m_2+r_1m_3,$$
$$\mu_2\equiv p_2m_1+q_2m_2+r_2m_3,$$
$$\mu_3\equiv p_3m_1+q_3m_2+r_3m_3,$$

then μ_1, μ_2, μ_3 will be a co-normal system, provided

$$\Sigma p=\Sigma q=\Sigma r.]$$

The following presents have been received during the recess :—

Cabinet likeness of Mr. A. B. Basset, for the Society's Album.

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APPENDIX.

The following notes, received from Mr. J. Griffiths* (December, 1889), are of interest in connexion with the papers on "Isoscelians" (*Proc. Lond. Math. Soc.*, Vol. XIX.), and on "Isoscelian Hexagrams" in this Volume (p. 4).

I. Using the Fig. A (*Proc.*, Vol. XIX., p. 166), and calling the angles

$$\theta,\ \phi,\ \psi,\ [\theta+\phi+\psi = 180^\circ] \qquad \text{(i.)},$$

in place of A, B, C, he gives the trilinear coordinates of P by the equations

$$\alpha \sin\theta \sin(\theta+A) = \beta \sin\phi \sin(\phi+B) = \gamma \sin\psi \sin(\psi+C) \ \ldots \ \text{(ii.)}.$$

These relations are unaltered when

$$\pi-\theta-A,\ \pi-\phi-B,\ \pi-\psi-C$$

are written for θ, ϕ, ψ respectively; hence we arrive at the same point with the negative θ-, ϕ-, ψ-lines.

(1) When $\theta = A$ or $\pi-2A$, $\phi = B$ or $\pi-2B$, $\psi = C$ or $\pi-2C$,

we arrive at the Isoscelian results (*loc. cit.*, p. 166).

(2) When $\theta = \frac{\pi}{2}-\frac{A}{2},\ \phi = \frac{\pi}{2}-\frac{B}{2},\ \psi = \frac{\pi}{2}-\frac{C}{2}$,

the positive and negative circles coincide: the coordinates of P, in

* Mr. Griffiths was a referee for the paper on "I. Hexagrams," and has kindly put the following notes, amongst others, at our service.