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THE AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE

THE PROBLEM OF SEVERAL BODIES: RECENT PROGRESS IN ITS SOLUTION[1]

I

THE DIFFERENTIAL EQUATIONS AND THEIR TRANSFORMATIONS

WHITTAKER has formulated the classic problem of three bodies as follows: Three bodies attract each other according to the Newtonian law so that between each pair of particles there is an attractive force which is proportional to the product of the masses of the particles and the inverse square of their distances apart: they are free to move in space and are initially supposed to be moving in any given manner; to determine their subsequent motion.

In mathematical phraseology the problem is to integrate a certain system of the eighteenth order of differential equations which at present are usually written in the so-called canonical form

$$dx_i = \frac{\partial F}{\partial p_i}\,dt, \quad dp_i = -\frac{\partial F}{\partial x_i}\,dt,$$

in which t is the time, x_i a coordinate, p_i a component of momentum, and F a certain function of all the x_i and p_i.

In recent investigations, especially those originating in the researches of Poincaré, the canonical equations are preferred to other types because of their simplicity of

[1] Abstract of the address of the vice-president and chairman of Section A—Astronomy and Mathematics—American Association for the Advancement of Science, Baltimore, 1908.

form, ease of transformation and perspicuity in showing how the variables enter the problem. It becomes of advantage, then, to expand the function F in terms of canonical elements; Charlier has given in his lectures a method of expansion which Norén and Wallberg carried out to terms of the second degree. Stone has published a simple direct derivation of the canonical elements introduced into the three-body problem by Delaunay and Jacobi; while the formulæ of transformation from Cartesian to Jacobian coordinates in the n-body problem have been derived by Pizzetti with the aid of linear substitutions.

From the standpoint of the theory of integral invariants introduced by Lie and Poincaré the characteristic property of the canonical system is the existence of the relative integral invariant $\int \Sigma x_i dp_i$, and the covariantive correspondence between the canonical system and the differential expression $\Sigma x_i dp_i - F dt$ forms the connection between their theory and that of contact transformations. A systematic study of integral invariants has been published by De Donder, including his own researches and those of Appell, Hadamard and Koenigs. Morera has shown in a series of memoirs how this transformation theory gains in generality, simplicity and elegance when at its foundation we lay the bilinear covariantive correspondence to which allusion has just been made; Morera rediscovers and generalizes the theorems of Lie on the invariance of canonical systems under contact transformations. The importance of these results for the problem in hand is recognized when we recall that Lagrange's method of the variation of arbitrary constants in the theory of perturbations leads to equations of the canonical form; Lie's theory thus stamps the history of a perturbation problem as the history of a contact transformation, a relation exhibited on the geometrical side by the true orbit enveloping the successive approximate ones. The notion of an intermediate orbit has been extended to canonical systems by Charlier, who has employed it in a generalization of Jacobi's theorem.

Lagrange showed that the eighteenth order system in the three-body problem can be reduced to one of the sixth order; this reduction has been effected in a variety of ways by other mathematicians. Poincaré employed a contact transformation to reduce the problem to the twelfth order, and Whittaker has used an extended point transformation to carry the reduction on to the eighth order. Whittaker has also exhibited in explicit form the contact transformations involved in Radau's direct reduction from the eighteenth to the sixth order. Routh's transformation known as the ignoration of coordinates has recently been generalized by Woronetz to a form which includes as special cases Poincaré's equations of motion, and the reductions of Lagrange, Jacobi, Bour and Brioschi. The discovery of the existence of a force center in the three-body problem has enabled Delaunay to write its equations in a special form. Scholz has shown that under certain assumptions regarding the perturbative function the three-body problem can be reduced to the integration of a single differential equation, and a new reduction of the plane problem has been given by Perchot and Ebert. The corresponding reduction of twelve units in the order of the n-body problem has been effected by Bennett through Poincaré's transformation and a generalization of the one employed by Whittaker in the three-body problem.

In the transformation and reduction of the problem a principal rôle has been played by the ten known integrals, namely, the six integrals of motion of the center of gravity, three integrals of angular momentum, and the integral of energy. The question of further progress in this reduction is vitally related to the non-existence theorems of Bruns, Poincaré and Painlevé. Bruns demonstrated that the n-body problem admits of no algebraical integral other than the ten classic ones, and Poincaré proved the non-existence of any other uniform analytical integral. A strikingly instructive example illustrating these non-existence theorems has been given by Perchot and Ebert. Painlevé has generalized Bruns's theorem by showing that, in addition to the classical integrals of energy and momentum, there exists neither integral nor integral equation algebraic in the velocities, and the theorem of Poincaré, by proving that there exists no new analytical integral uniform with respect to the velocities. Gravé showed that the three-body problem under forces varying as any function of the distance possesses no new integral independent of the law of attraction, and this theorem has been generalized for the n-body problem. Bohlin has very recently added to the non-existence theorems by demonstrating that the mutual distances in the problem of three bodies can not be expressed as the roots of an algebraical equation of the fifth degree with transcendental coefficients.

II

PARTICULAR SOLUTIONS AND THEIR GENERALIZATIONS

In 1772 the prize of the Académie Royale des Sciences de Paris was awarded to Lagrange for an "Essai sur le Problème des Trois Corps." In this celebrated memoir Lagrange "shows that the complete solution of the problem requires only that we know at each instance the sides of the triangle formed by the three bodies; the coordinates of each may then be determined without difficulty. As for the solution of the triangle, it depends upon three differential equations, of which two are of the second order, the third of order three." He determined all the solutions of the problem in which the ratios of the mutual distances of the bodies remain constant. In one of the two distinct configurations the bodies are always at the vertices of an equilateral triangle; in the other they lie always on a straight line. In both of these cases the motion of each body relative to either of the others is the elliptic motion of the two-body problem. Tscherny has constructed these solutions geometrically; he has also shown that the only cases of the three-body problem for which known mathematical and mechanical means suffice are those which reduce to the problem of two bodies. Lagrange's solutions were originally discovered in his problem of the mutual distances; the latter, called by Hesse the reduced problem, has recently assumed a new form under Charlier's treatment, in which the mutual distances are replaced by the distances from the center of gravity. From Lagrange's discussion certain imaginary considerations were omitted; Whittemore has filled this gap, but the completed discussion yields no other real solution. The equilateral triangular solution is possible for all distributions of the masses; their distribution on the straight line is defined by the real positive root of a certain quintic equation; Frederigo has given a new derivation of this equation and Bohlin has formulated four developments, of which three represent the roots of the quintic in three distinct domains, and the fourth for an isolated value. The question of the stability of the solutions furnishes Levi-Civita an example of his theory of stationary motion in which reappear the results of Liouville and Routh, namely, the triangular solution is stable if

$$(m_1 + m_2 + m_3)^2 > 27 (m_1m_2 + m_2m_3 + m_3m_1),$$

while the rectilinear solution is always unstable.

Theoretical interest in the Lagrangian solutions has been increased by Sundman's theorem that the more nearly all three bodies in the general problem tend to collide simultaneously, the more nearly do they tend to assume one or the other of Lagrange's configurations; and on the other hand practical interest in them has been revived by the discovery of three small planets, 1906 T.G., 1906 V.Y., 1907 X.M., near the equilateral triangular points of the Sun-Jupiter-Asteroid system. Linders has begun the investigation of the motion of the first of these by starting from a periodic solution of the differential equations and developing the Jupiter perturbations from the osculating elements.

Lehmann-Filhés, Hoppe and Dziobek have generalized the exact solutions to cases of more than three bodies placed on a line or at the vertices of a regular polygon or polyhedron, and isosceles triangular solutions have been studied by Fransen, Gorjatschew and Woronetz, while Longley in an investigation of the plane problem of invariable configuration pays special attention to the rhombus. The cases considered by Dziobek and Lehmann-Filhés have been generalized by Pizzetti in a direct study of the homographic motion of n bodies. Among the most interesting extensions of Lagrange's theorem are those due to Banachiewitz and Moulton. The former considers a non-equilateral triangular system with fixed center of gravity and under attractions according to the inverse cube of the distance. He finds a particular solution in which the triangle rotates around the x-axis, its angles remaining constant, and each point describing a curve on a cone of revolution about the same axis which projects into a spiral on the base of the cone. This is the first case of an exact solution in which three finite bodies describe curves of double curvature. Moulton's case is that of the four-body problem consisting of three arbitrary masses, in motion according to either of Lagrange's solutions, and an infinitesimal body; there are eighteen solutions of arbitrary period in which the finite bodies lie on a line, and ten in which they are at the vertices of an equilateral triangle. Periodic solutions analogous to those in the restricted three-body problem have been constructed for Moulton's problem.

The method of Lagrange's memoir has been extended to the four-body problem by Seydler and more recently by Woronetz; the latter has pointed out particular solutions in which three of the bodies are equal; these solutions are given by quadratures if the law of force is inversely as the cube of the distance and are capable of direct extension to the case of any number of bodies.

III

PERIODIC SOLUTIONS AND THEIR APPLICATIONS

The Lagrangian solutions remained the only known periodic solution of the problem of three bodies for one hundred and five years until 1877, when Hill, in his epoch-making researches on the lunar theory, demonstrated the existence of a periodic solution which could serve as the starting point for a study of the moon's orbit. With these memoirs he broke ground for the erection of the new science of dynamical astronomy whose mathematical foundations were laid broad and deep by Poincaré. Up to the time when Hill's work appeared, mathematical astronomers were accustomed to assume a solution of the problem of two bodies as a first approximation in the lunar theory; which intermediate orbit includes none of the inequalities due to the sun's disturbing force. Hill proposed to take as this first approximation an orbit which would include all the inequalities depending upon the mean motions of the sun and moon. The old theories consisted essentially in suitably varying a solution of the problem of two bodies, while Hill's theory seeks the true orbit by attempting to vary appropriately the restricted problem of three bodies. During the last fifteen years, Brown has published a series of papers, concluding with the 1907 Adams Prize Essay of the University of Cambridge, which extend Hill's work to the construction of the most perfect of all the ten or eleven theories of the moon which have appeared since Newton's "Principia." Hill found periodic solutions of the motion of a particle in a plane under the influence of two bodies which revolve round each other in circular orbits and whose distance apart is infinite. In its initial stages Brown's theory modified Hill's solution in two particulars, first by reducing the distance of the two bodies to finite dimensions, and thus introducing the inequalities which involve the solar parallax, and second by including those inequalities which are due to the moon's eccentricity. Adequate accounts of these theories are given in the presidential addresses delivered on the occasions of the award of the gold medal of the Royal Astronomical Society to Hill in 1887, and to Brown in 1907, while the relations of Brown's perfected work to the highly original pioneer work of Hill are exhibited in the introduction which Poincaré has written to Hill's "Collected Works." Brown has recently finished his complete numerical theory, and lunar tables based upon it are to be published by Yale University. His numerical results furnish an interesting confirmation of the validity of Newton's law. Newcomb proposed an explanation of the motion of Mercury's perihelion by changing the exponent 2 in the Newtonian law to $2 + 0.00000016$. Brown finds in his theory of the moon's motion that the exponent can differ from 2 only by ± 0.00000004.

The work reviewed up to this point in our discussion has found its sources in Hill's periodic solution, the memoir of Lagrange, the non-existence theorems of Bruns and Poincaré, and Lie's theory of contact transformations; that which follows may trace its origins to Poincaré's theoretical and Darwin's numerical investigations on periodic solutions, Newcomb's and Lindstedt's solutions in trigonometric series, Gyldén's theory of absolute orbits, and Painlevé's theorems on the singularities of the problem.

Although periodic and asymptotic solutions do not exist in nature their services to astronomy have been two-fold: to the practical astronomer in supplying first approximations to orbits under investigation, and to the mathematical astronomer in opening the way to further theoretical researches through what Poincaré has characterized as "la seule brèche par où nous puissons essayer de pénétrer dans une place jusqu'ici reputée inabordable." Darwin has constructed a splendid collection of examples of these orbits, planetary and lunar; among his most curious satellite orbits are perhaps those which present three new moons in a month, and another which has five full moons in one period. Darwin's orbits were subjected to a searching analytical examination by Poincaré who showed that two sets of curves which Darwin treated as continuous can not be considered as such; the true sequence of the orbits in question has been exhibited by Hough. Certain of Darwin's results have been derived analytically by Charlier, and specially with reference to the families of oscillating satellites in the vicinity of the five centers of libration corresponding to the exact Lagrangian solutions. In Charlier's paper no account was taken of the imaginary centers of libration; the analytical treatment was completed in this particular by a note which showed that there are imaginary centers about which real orbits exist. Plummer has extended Charlier's analysis to arbitrary fields of force, and to terms of the second and third orders in the developments. Schlitt has reckoned five orbits to whose construction Darwin referred as not belonging to the category of simply periodic orbits, and for that reason disregarded by him. With Darwin's orbits Moulton has compared certain of his own, established by Poincaré's method of analytical continuation, and arranged in power series rather than Fourier series. Finally to Darwin's orbits Strömgren has applied his conditions for cusps and loops in the restricted three-body problem; Strömgren has shown that these singularities may be encountered in every point in the plane, in the absolute motion as well as in that referred to moving axes.

Periodic orbits have been variously classified. If the curves are reentrant after a single period Darwin calls the orbits "simply periodic"; all the orbits considered by him have this property. Hill has grouped them broadly into two classes: the first contains those in which a rotation of the whole system has taken place; the second, those in which no such rotation has occurred, but the longitudes of the bodies and their distances have returned to the same values. Poincaré has classified them elaborately into species, classes and kinds, but as Charlier has pointed out this classification is not exhaustive. The great majority of the orbits referred to here belong to the first two kinds, as distinguished by Poincaré, that is, they either have inclination and eccentricity zero or inclination zero and eccentricity not zero. Von Zeipel has published a thorough study of the solutions of the third kind—that is, those having both inclination and eccentricity different from zero—in which they are grouped in no fewer than ten types and their stability discussed by the aid of their characteristic exponents. Whittaker has designed a criterion for the discovery of periodic orbits analogous to those theorems which indicate the positions of the roots of an algebraic equation.

A matter of vital theoretical and practical import in the domain of periodic solutions is the question of their stability. Following Poincaré's lead, Brown has formulated the sufficient conditions for stability in the n-body problem as follows: first, that the bodies never become infinitely distant from one another; second, that their mutual distances never descend below a certain limit; third, that each body passes an infinite number of times as near as we wish to any point through which it has once passed; fourth, that a small external disturbance shall not affect the fulfillment of these conditions. Poincaré stated the first three and investigated the third in detail; numerical limits for the first and second have been found by Haffel in a particular case of the sun-earth-moon system. Levi-Civita has worked out criteria in which the stability is made to depend upon that of a certain point transformation associated with the periodic solution; these criteria show the instability of certain orbits which in the first approximation appear to be stable; they indicate further that contrary to accepted opinion a purely imaginary characteristic exponent α does not always single out a stable solution—the solution is unstable if $\alpha/\sqrt{-1}$ is not commensurable with the mean motion $2\pi/T$. Applying his method to the restricted problem Levi-Civita has found that solutions differing little from circles and having a mean motion $1+3/h$ are certainly unstable, thus proving the existence of zones of instability surrounding Jupiter's orbit which may extend throughout the plane. The above conditions are approximately satisfied for the small planets (167) Urda, (243) Ida, and (396) whose mean motion is near $1 + 3/2$ and whose eccentricities and inclinations are very small; the planet (188) Menippo has a mean motion near 5/2, but an inclination and an eccentricity too large for these considerations to be immediately applicable. Kobb has called the attention of astronomers to the fact that he found the orbit of (153) Hilda to be stable but that the conditions for stability are not satisfied by the motion of (279) Thule; the same writer has shown the motion of the seventh satellite of Jupiter to be stable and that of the eighth unstable, while Moulton has established limits of temporary stability for satellite motion. Levi-Civita's criteria have been studied by Cigala, and those of Lehmann-Filhés for circular motion have been generalized by Frank. Gray has given a résumé of the work of Charlier, Hill, Picart, Roche and Schiaparelli on the stability of a swarm of meteorites and of a planet and satellite, and Routh has discussed the motion and stability of a swarm of particles whose center of gravity describes an elliptic orbit of small eccentricity about the sun. Considering a system composed of a planet, a rigid ring, and a satellite Bohl has proved that under certain initial conditions the motion can be terminated only by a ring planet collision; further, that the possibility of the latter collision may be excluded and permanent stability secured.

The new methods in celestial mechanics have proved their usefulness in computing the perturbations of those small planets whose period of revolution is approximately commensurable with that of Jupiter. To enumerate: Simonin has applied Poincaré's methods to the case of Hecuba and has succeeded in obtaining a very close solution by means of simple expressions; Hill has devoted two memoirs to examples of periodic solution in studies primarily concerned with cases of mean motions respectively triple and double (Hecuba type) that of Jupiter; Poincaré has shown the essential agreement between his own results and those of Brendel's "Theorie der kleinen Planeten" constructed along the lines of Gyldén's method; Hill and Andoyer have applied Delaunay's method to the Hecuba group; Poincaré has exhibited the relations of Simonin's results to the applications of Gyldén's method made by Ludendorff to Hecuba, by Brendel to Hestia, and by Harzer to Hecuba; Schwarzschild has made a numerical investigation of periodic solutions in the vicinity of the Hecuba orbit; Wilkens has applied his asymmetric solutions to orbits of the Hecuba type, establishing their stability by Poincaré's method; and finally Wilkens and De Sitter have studied solutions of the Hestia type. A class of periodic solutions was designed by Moulton, and successfully applied to the lunar theory; independently Gyldén and Moulton utilized periodic orbits to explain the Gegenschein; and McCallie, following a suggestion of Hill's, constructed an example of periodic solutions from the theory of Jupiter and Saturn. Strömgren found that asymptotic motion towards one of the equilateral triangular centers of libration takes place only under exceptional circumstances, for as a rule the body describes a periodic orbit around this center or recedes indefinitely from it.

In an exhaustive treatment including all the limitation and libration motions of the special case of the three-body problem when two of the bodies are fixed Charlier has noted two applications: first to the case where a small body passes at great speed through a double star system, and second to the generalization of those conditions which result in the moon revolving about the sun if the earth and sun become fixed centers. Charlier has also considered the relations of the two-body problem to the two-center problem, and has pointed out the advantages of solutions of the latter as intermediate orbits in the asteroid problem. The case where one fixed center attracts and the other repels was worked out in detail by Wöller; and Hiltebeitel applied the method of Charlier's work to the qualitative discussion of the most general two-center problem admitting of separation of the variables.

Hill has prepared a number of examples of Gyldén's periplegmatic orbits, some of which are periodic. The construction of the solutions calls for elliptic functions, Lindstedt's series, and sequences of Delaunay transformations. These examples of Hill have been generalized in several directions, in one of which certain of Painlevé's new transcendental functions find application. For the case of two nearly equal bodies and a third infinitesimal body Pavannini found a new category of periodic solutions which have been extended to the restricted problem of four bodies. Andoyer's memoir on the relative equilibrium of n bodies has been made by him the basis of a study of periodic solutions in the vicinity of positions of relative equilibrium under forces varying as the masses and any power of the distances. Longley has constructed the only orbits (one direct the other retrograde) of preassigned period in the plane n-body problem which consist of an infinitesimal body revolving around one of $n-1$ finite masses which are in periodic motion. For the plane n-body problem having the same distribution of masses as the solar system, Griffin found a class of periodic solutions of which he has made numerical application to the three inner satellites of Jupiter.

FORMAL AND QUALITATIVE RESOLUTION OF THE PROBLEM

By generalizing somewhat the theory of periodic and asymptotic solutions by which Poincaré established the divergence of Lindstedt's series von Zeipel has been able to study the series, however great the mutual inclination of the orbits may be. He derived the following necessary and sufficient conditions for the existence of the series: first, that the orbits be nearly circular; second, that a certain biquadratic equation have real and unequal roots. Von Zeipel found that if the inclination of an asteroid exceeds a certain limit (about 30°, slightly variable) the series of Lindstedt cease to exist; and he remarked that it is perhaps permissible to see in this theorem, although Lindstedt's series are only semi-convergent, the cause of the surprising fact that among five hundred asteroids there exists but one (Pallas) whose inclination exceeds 30°.

Hill has extended Delaunay's method to the general problem of planetary motion, and, employing the fundamental conceptions of Gyldén, he has indicated a practicable way for its application, in two memoirs on integrals of planetary motion, suitable for an indefinite length of time. Charlier has discussed the properties of the general solution in trigonometric series by supposing it to have been derived from the integration of the Hamilton-Jacobi equation. For constructing solutions in the form of trigonometric series, Whittaker has devised a method, closely analogous to Delaunay's, and consisting essentially in the repeated application of contact transformations which ultimately reduce the problem to the equilibrium problem. Bohlin has just published the concluding memoirs of a remarkable series of investigations which have culminated in a non-existence theorem quoted in a previous paragraph, and in his new astronomical series for the distances and coordinates; these series, both in their terms and in their coefficients, are built up from certain developments which Bohlin has derived for roots of the fundamental quintic met with in Lagrange's problem of the mutual distances.

The question of the validity of certain methods of Gyldén has been the source of considerable discussion among mathematical astronomers during the period under review. The appearance of a long memoir by Buchholz on Gyldén's horistic method, and its convergence, brought forth from Backlund a protest against the manner in which the material of the memoir had been accumulated and presented. To this protest Buchholz replied with a defense of the course he had pursued in preparing the work; and a little later he published another note objecting to a statement by Schwarzschild that Poincaré, in his prize memoir, had proved the divergency of the series employed by astronomers. About this time Poincaré examined in detail the second of Gyldén's two horistic methods, the first being open to grave objections as had been shown by himself and Backlund. As a result of his investigation Poincaré found that the second method, conveniently modified, is a legitimate one, not for the search of the general solution, but for the determination of one of those particular solutions which he himself had termed periodic. He pronounced futile the effort to derive from the horistic method developments uniformly convergent in the geometric sense of the word, and declared false Gyldén's conclusion that the terms of high order in the perturbative function can never produce libration. Poincaré's results were questioned by Backlund and an interesting controversy ensued, some points of which were elaborated upon in a later extensive memoir which Poincaré devoted to Gyldén's theory, where he pointed out Gyldén's great service to science in creating a number of new methods which have been applied with success to certain problems of mathematical astronomy, as for instance, in the theory of the small planets developed by Harzer and Brendel. He found the methods proposed in Gyldén's earlier memoirs to be correct in the main, but possessed of little more than historic interest, having been superseded by less inconvenient methods such as those of Hill and Brown. Gyldén's later theories Poincaré subjected page by page to a searching critical examination which resulted in a declaration that they are invalidated throughout by errors arising in the initial stages of Gyldén's analysis.

Thanks to the recent researches of Levi-Civita, Bisconcini, Sundman and Block, inspired as they were by an earlier theorem of Painlevé the qualitative solution has been attained in the field of the formal resolution of the mathematical problem of three bodies, and some progress has been made towards the same end in the astronomical problem. Painlevé demonstrated that starting from given initial conditions singularities occur only if one at least of the mutual distances tends towards zero, when t converges to a finite value t_1. When these singularities have been located, the recent theorems of Mittag-Leffler, on the representation of monogenic branches of analytical functions, warrant the assertion that the coordinates are expressible in every case, and throughout the duration of the motion, in series possessing the fundamental properties of Taylor's series. It may be remarked in passing that Volterra has given examples of the applicability of Mittag-Leffler's developments to certain cases of the general n-body problem. From the standpoint of the qualitative resolution of the problem, it becomes of paramount importance, then, to define with precision the initial conditions which lead to a collision. Painlevé in his Stockholm lectures announced the opinion that the initial conditions which constrain a collision of at least two of the three bodies at the end of a finite time, satisfy two distinct analytical relations, which reduce to one in the case of plane motion. These analytical relations whose existence Painlevé divined have been disclosed by the brilliant researches of the two Italian mathematicians, Levi-Civita and Bisconcini. Levi-Civita blazed the trail in the restricted problem and found an unique, invariant relation, algebraical in the velocities, periodic and uniform, which he developed in a power series. It may be noted that simple modifications of Levi-Civita's analysis render it immediately applicable to the restricted problem of four bodies. There appears again a single uniform periodic condition for collisions of two of the bodies, and this condition is algebraic in the velocities. The result thus constitutes an exception to Painlevé's theorem that when three of the masses are different from zero the conditions which must be satisfied in the n-body problem in order that after a finite interval of time two of the bodies may collide, can not be *algebraical conditions*. In the general three-body problem Bisconcini, following the route marked out by Levi-Civita in the restricted problem, has arrived at two distinct relations whose analytical form he has determined. Bisconcini has thus been able to characterize all the singular motions of the system in which any two of the bodies collide, and to determine the analytical conditions under which we may be certain that the motion will proceed regularly. One of the assumptions made by Bisconcini in the course of this work has since been demonstrated by Sundman. In a new elaboration of his original memoir Levi-Civita has been able to extend certain of his results to the astronomical, restricted problem. Sundman has found the condition for the simultaneous collision of all three bodies to consist in a vanishing of all three integrals of areas in the motion of the bodies with respect to their common center of gravity; if the constants of areas are not all zero, Sundman has assigned a positive limit below which, of the three distances, the greatest always remains so. The same writer has announced the extension of his results to the n-body problem, including explicit expressions for the coordinates in the vicinity of equilibrium. In the meantime Block has presented to the Swedish Academy of Sciences a memoir in which he has given the developments in powers of the time in Sundman's case of collision; these power series contain terms of three different forms in whose exponents the masses of the bodies appear. The recent memoirs of von Zeipel on intransitive motion in the three-body problem and the indeterminate singularities in the case of n bodies are treated in the report reviewed here. Mittag-Leffler is preparing a memoir soon to be published in the *Acta Mathematica* in which there will appear a digest of Weierstrass's correspondence in its bearing on the problem of three bodies. The memoir will be concerned especially with the relations of this correspondence to the setting of the problem for the prize, offered by the late King Oscar II., of Sweden; to the report on which the award of the prize was based; and to the recent work on the singular trajectories of the general problem of three bodies, and its resolution in power series.

V

GENERALIZATIONS OF THE PROBLEM AND ITS INVERSION

During the period under discussion the problem has been variously generalized. Ebert has formulated an equivalent problem to that of n bodies, with an additional integral; and a similar generalization has been made by extending the Bour-Bertrand treatment of the three-body problem. Esclagnon and Bohl have indicated applications of quasi-periodic functions to the ordinary problem; special cases in which the masses vary with the time have been considered by Mestchersky; Laves has studied the integrals when the forces depend upon the coordinates and their derivatives of the first two orders; and Ebert has taken up the problem in space of any number of dimensions.

Bertrand inverted the problem of two bodies by proposing to find the law of force under which a body, whatever may be its initial position and velocity, always describes a conic section. This inverse problem was solved independently by Bertrand, Darboux and Halphén; and extended by Dainelli to general curve trajectories. Stephanos has recently given another generalization of Bertrand's problem by including in the discussion the case in which the force has not necessarily an unique direction at every point of the conic section. This problem in turn has been generalized to conditions which include the conic section trajectories as special cases. Griffin observed that the law of force under which a given curve is described as a central orbit can not be determined uniquely if only the position of the center of force be known. Oppenheim gave to Bertrand's problem a new treatment which included the case of finding the central conservative forces under which three bodies of arbitrary mass describe given plane curves.

A further generalization of Bertrand's problem presents itself in the problem of finding the forces of a central conservative system capable of maintaining a system of m particles on as many prescribed but arbitrary orbits in a space of n dimensions. The resolution of this problem shows that the central conservative character of the motion and the equations of the orbits are necessary and sufficient to determine the components of the velocities, only in the case of $\frac{1}{2}n(n+1)$ bodies, and the components of the forces only in that of $2n-1$ bodies. From this point of view the plane three-body problem possesses an unique generality of its own, in that it is the only case in which all the elements of the mechanics of the problem are completely determinate when the arbitrary plane curves described by the bodies under central conservative forces are given. This circumstance has been turned to account in the construction of new integrable problems of three bodies under laws of force involving only the masses and the mutual distances of the bodies.

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THE PHYLETIC IDEA IN TAXONOMY[1]

TO-DAY every botanist is an evolutionist. It may well be that we have not yet agreed as to the details—as to the particular manner in which modifications were effected—whether they were by slow and almost imperceptible deviations from the parental type, or those more marked variations that we are in the habit to-day of calling "mutants." Some of us may lay more stress upon the "survival of the fittest," others upon the "survival of the unlike." For some the "struggle for existence" may account for the diversity of plant forms, while others see in "adaptation" the explanation of the same diversity. To some the "inherent tendency" in plants to vary is a potent factor, while for others all variation is a result of "environment." Yet with all this diversity of opinion as to details there is a practical unanimity as to the acceptance of the general doctrine of evolution. It may be asserted without fear

[1] Address of the vice-president and chairman of Section G—Botany—of the American Association for the Advancement of Science, Baltimore, 1908.