A New Approximation of the Factorial Function

Anton Heringh

July 2025

Abstract

This paper presents a novel method for approximating the factorial function $n!$ based on powers and asymptotic expansions. By manipulating powers of $n!$ and extracting suitable roots, we derive both Gosper's and Ramanujan's approximations as special cases. A new approximation is proposed, offering significantly reduced relative error across the interval $n > 0$, including near zero. Comparative analysis and graphical results are included.

Keywords: factorial function, approximation, asymptotic expansion, Stirling's formula, Ramanujan, numerical analysis, special functions, error estimation

1 Introduction

The approximation of the factorial function $n!$ is typically based on its asymptotic expansion. Although accuracy improves by including higher-order terms, the approximation error increases noticeably within the interval $n \in [0, 1]$. This is expected, as the expansion is derived under the assumption that $n \to \infty$.

However, several formulas eliminate divergence at $n \to 0$, allowing accurate evaluation across the domain $n > 0$. Well-known approximations include:

Gosper's approximation:

$$n! \approx \sqrt{2\pi n + \tfrac{\pi}{3}} \left(\frac{n}{e}\right)^n \tag{1}$$

Ramanujan's approximation:

$$n! \approx \sqrt{\pi} \left(8n^3 + 4n^2 + n + \tfrac{1}{30}\right)^{1/6} \left(\frac{n}{e}\right)^n \tag{2}$$

This paper demonstrates a unified approach from which both approximations follow and proposes a new formula with even greater accuracy.

2 Asymptotic Expansion and Stirling's Formula

The asymptotic series for $n!$ is

$$n! \sim \sqrt{2\pi} \cdot n^{n+1/2} e^{-n} \left(1 + \frac{1}{12n} + \frac{1}{288n^2} + \dots\right)$$

Neglecting correction terms yields Stirling's approximation:

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

3 Approximations from Factorial Powers

3.1 Square

Squaring $n!$ gives:

$$n!^2 \sim \frac{1}{3}(6\pi n + \pi)n^{2n}e^{-2n}$$

$$\Rightarrow n! \approx \sqrt{2\pi n + \tfrac{\pi}{3}}\left(\frac{n}{e}\right)^n$$

This result corresponds to Gosper's approximation (1).

3.2 Fourth Power

Expanding $n!^4$:

$$n!^4 \sim \frac{1}{9}(36\pi^2 n^2 + 12\pi^2 n + 2\pi^2)n^{4n}e^{-4n}$$

$$\Rightarrow n! \approx \frac{\sqrt{3}\cdot\sqrt[4]{2}\cdot\sqrt{\pi}}{3}(18n^2 + 6n + 1)^{1/4}\left(\frac{n}{e}\right)^n$$

This expression is structurally analogous to Ramanujan's approximation (2).

3.3 Discussion

Approximation (3) performs better than (1) for large n, but worse for $n < 0.3$.

4 Ramanujan via Sixth Power

The sixth power yields:

$$n!^6 \sim \frac{1}{30}(240\pi^3 n^3 + 120\pi^3 n^2 + 30\pi^3 n + \pi^3)n^{6n}e^{-6n}$$

$$\Rightarrow n! \approx \sqrt{\pi}\left(8n^3 + 4n^2 + n + \tfrac{1}{30}\right)^{1/6}\left(\frac{n}{e}\right)^n$$

This coincides exactly with approximation (2).

5 New Approximation via $(n+1)!^4$

Let us consider

$$(n+1)!^4 \sim P(n)\left(\frac{n}{e}\right)^{4n}\pi^2$$

where

$$P(n) = 4n^6 + \frac{52}{3}n^5 + \frac{266}{9}n^4 + \frac{10072}{405}n^3 + \frac{25697}{2430}n^2 + \frac{105277}{51030}n + \frac{515969}{4592700}$$

Final approximation:

$$n! \approx \frac{\sqrt{\pi}}{n+1} \cdot n^n e^{-n} \cdot P(n)^{1/4} \tag{3}$$

6 Accuracy Comparison

n	Relative Error (3)	Relative Error (2)
1	8.13×10^{-6}	2.48×10^{-4}
5	2.59×10^{-8}	1.84×10^{-7}
10	1.53×10^{-9}	2.16×10^{-8}
20	9.44×10^{-11}	2.67×10^{-9}
50	5.93×10^{-12}	4.23×10^{-10}
100	7.39×10^{-13}	1.05×10^{-10}

Table 1: Relative errors of Ramanujan's (2) and the new approximation (3).

7 Graphical Comparison

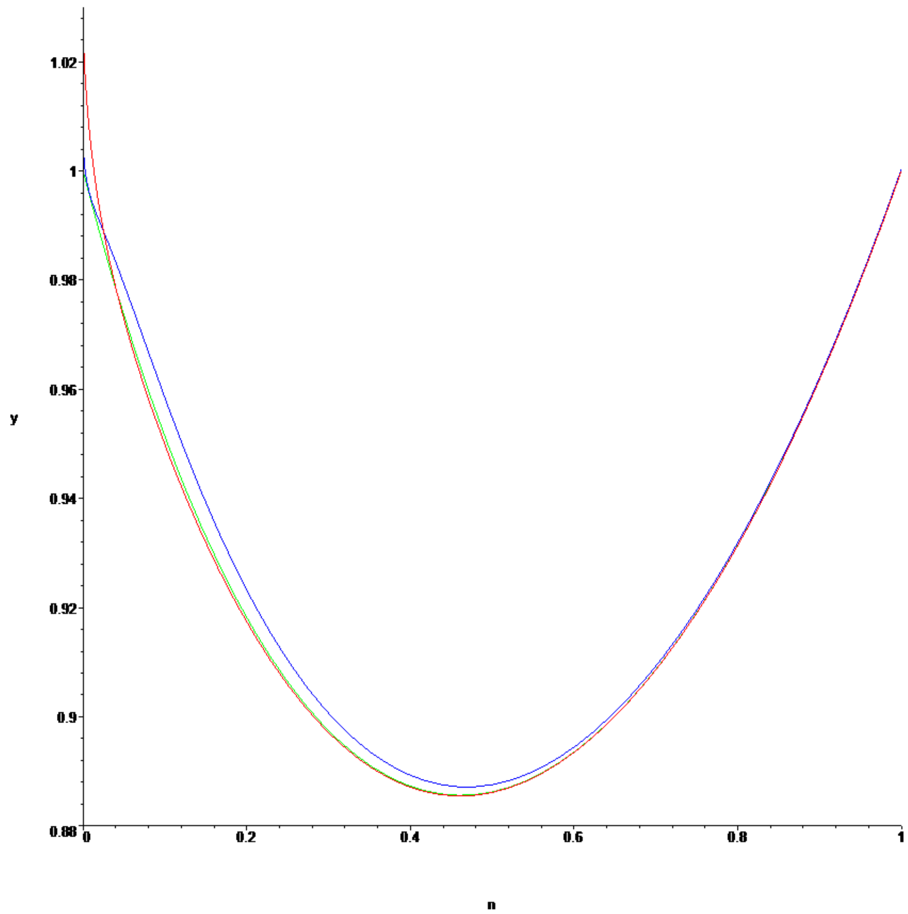

Figure 1: The red curve represents the proposed approximation (3), the blue curve corresponds to Ramanujan's formula (2), and the green curve shows the exact factorial function $n!$ used as the reference over the interval $n \in [0, 1]$.

8 Conclusion

We have introduced a unified method for constructing factorial approximations using powers and roots. This approach not only recovers known approximations such as Gosper's (1) and Ramanujan's (2), but also yields a new approximation (3) with demonstrably superior accuracy across the entire interval $n > 0$.

The method can be generalized further by selecting alternative generating expressions, striking a balance between polynomial complexity and numerical precision.

References

1. R. W. Gosper, "Decision procedure for indefinite hypergeometric summation," *Proceedings of the National Academy of Sciences*, vol. 75, no. 1, pp. 40–42, 1978. DOI:10.1073/pnas.75.1.40

2. S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, edited by S. Raghavan and S. S. Rangachari, Narosa Publishing House, New Delhi, 1987.