Supplementary Note on Matrices. By J. Brill, M.A. Received December 12th, 1896. Read January 14th, 1897.

1. The present communication is intended to be supplementary to the "Note on Matrices" printed in the last volume of the *Proceedings* of the Society.* In that paper I shewed that the most general form of the differential of a matrix, such that it is commutative with the matrix itself, is given by the formula

$$dm = \Sigma \frac{(m-\lambda_2)(m-\lambda_3)\dots(m-\lambda_n)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\dots(\lambda_1-\lambda_n)}\, d\lambda_1.$$

This formula is sufficient, as stated in the paragraph appended to the note, as a test whether the differential of a matrix, due to the infinitesimal variation of some scalar element or elements contained in its expression, be commutative with the matrix itself. When, however, we attempt to replace it by an integral formula, we find that it gives rise to n conditions. These are obtained from the n equations

$$(dm-d\lambda_1)(m-\lambda_2)(m-\lambda_3)\dots(m-\lambda_n) = 0, \tag{1}$$

$$(dm-d\lambda_2)(m-\lambda_1)(m-\lambda_3)\dots(m-\lambda_n) = 0, \tag{2}$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$(dm-d\lambda_n)(m-\lambda_1)(m-\lambda_2)\dots(m-\lambda_{n-1}) = 0, \tag{n}$$

made use of in proving the above theorem. That the single equation is equivalent to the set of n equations is evident, since they may all be obtained from it with the aid of the characteristic equation of the matrix. My present object is to obtain the n integral conditions which constitute the equivalent of the single differential one.

Taking equation (2), we have

$$(dm-d\lambda_2)(m-\lambda_2+\lambda_2-\lambda_1)(m-\lambda_3)\dots(m-\lambda_n) = 0;$$

and, therefore,

$$(\lambda_1-\lambda_2)(dm-d\lambda_2)(m-\lambda_3)\dots(m-\lambda_n)$$
$$-(dm-d\lambda_2)(m-\lambda_2)(m-\lambda_3)\dots(m-\lambda_n) = 0.$$

* Vol. xxvii., pp. 35–38.

Making use of equation (1), this reduces to

$$(dm-d\lambda_2)(m-\lambda_3)\dots(m-\lambda_n)$$
$$-(m-\lambda_2)(m-\lambda_3)\dots(m-\lambda_n)\frac{d\lambda_1-d\lambda_2}{\lambda_1-\lambda_2}=0.$$

Similarly, we should obtain

$$(dm-d\lambda_3)(m-\lambda_2)(m-\lambda_4)\dots(m-\lambda_n)$$
$$-(m-\lambda_2)(m-\lambda_3)\dots(m-\lambda_n)\frac{d\lambda_1-d\lambda_3}{\lambda_1-\lambda_3}=0,$$

and so on. Adding together the $n-1$ equations so obtained, we have

$$d\{(m-\lambda_2)(m-\lambda_3)\dots(m-\lambda_n)\}$$
$$-(m-\lambda_2)(m-\lambda_3)\dots(m-\lambda_n)\left\{\frac{d\lambda_1-d\lambda_2}{\lambda_1-\lambda_2}+\frac{d\lambda_1-d\lambda_3}{\lambda_1-\lambda_3}+\&c.\right\}=0.$$

This may be written

$$\frac{d\{(m-\lambda_2)(m-\lambda_3)\dots(m-\lambda_n)\}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\dots(\lambda_1-\lambda_n)}$$

$$-(m-\lambda_2)(m-\lambda_3)\dots(m-\lambda_n)\sum_s\frac{d\lambda_1-d\lambda_s}{(\lambda_1-\lambda_2)\dots(\lambda_1-\lambda_s)^2\dots(\lambda_1-\lambda_n)}=0,$$

which immediately integrates in the form

$$\frac{(m-\lambda_2)(m-\lambda_3)\dots(m-\lambda_n)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\dots(\lambda_1-\lambda_n)}=\text{const.}$$

Similarly, we should obtain

$$\frac{(m-\lambda_1)(m-\lambda_3)\dots(m-\lambda_n)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)\dots(\lambda_2-\lambda_n)}=\text{const.},$$

and so on. Writing $e_1, e_2, \dots e_n$ for the n constants so obtained, we find that they obey the laws

$$e_r^2=e_r,\quad e_re_s=e_se_r=0,\text{*}$$

and also that we have

$$e_1+e_2+\dots+e_n=1.$$

Thus we see that the differential of the matrix is expressed in the form

$$dm=e_1d\lambda_1+e_2d\lambda_2+\dots+e_nd\lambda_n.$$

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* This has been shown by Study. See "Ueber Systeme von complexen Zahlen." *Gött. Nach.*, 1889, pp. 237-268. Also "Ueber Systeme complexer Zahlen, &c.," *Monatshefte für Math. u. Phys.*, I. (1890), pp. 283-355.

In fact, since we have

$$m = \Sigma \frac{(m-\lambda_2)(m-\lambda_3)\dots(m-\lambda_n)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\dots(\lambda_1-\lambda_n)}\lambda_1,$$

we see that the matrix is expressible linearly in terms of its latent roots as follows,

$$m = e_1\lambda_1 + e_2\lambda_2 + \dots + e_n\lambda_n,$$

and the above expression for the differential follows immediately from this.

In the paper* referred to in the postscript to my former note, I was dealing with binary matrices, and took the equation

$$\frac{2m-\lambda_1-\lambda_2}{\lambda_1-\lambda_2} = \text{const.}$$

as the required condition. This, of course, is a consequence of the two conditions that should hold, but it is not sufficient in itself. It will, however, be seen that, in the application made, the two conditions are separately satisfied, so that the work is quite valid.

Note on some Properties of Bessel's Functions. By E. W. Hobson.

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An integral theorem involving Bessel's functions is here given, which contains some special cases of interest. It is well known that, for integral values of m, there are an odd number of positive roots of the equation

$$J_{m+1}(x) = 0$$

lying between consecutive positive roots of the equation

$$J_m(x) = 0,$$

and this is easily seen to be true for fractional values of m; it does

* "On the Application of the Theory of Matrices to the Discussion of Linear Differential Equations with Constant Coefficients," *Proc. Camb. Phil. Soc.*, Vol. VIII., pp. 201–210.